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Integration of Fokker-Planck's equation in a compact Riemannian space

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1. The result. Let R be an n-dimensional compact Riemannian space with the metric $ds^2 = g_{ij}(x)\,dx^i\,dx^j$, and consider a *temporally homogeneous* MARKOFF *process* on R for which $P(t, x, y)$, $t > 0$, is the *transition probability* that the point x be transferred to y after the elapse of t units of time. We assume that $P(t, x, y)$ is continuous in (t, x, y) and hence satisfies SMOLUCHOVSKI's equation

$$(1.1) \qquad P(t+s, x, y) = \int_R P(t, x, z)\, P(s, z, y)\, dz \quad (t, s > 0),$$

where the volume measure

$$dz = \sqrt{g(z)}\, dz^1 \ldots dz^n, \quad g(z) = \det\,[g_{ij}(z)],$$

and the *probability hypothesis*

$$(1.2) \qquad P(t, x, y) \geqq 0, \quad \int_R P(t, x, y)\, dy = 1.$$

The "continuity" of the transition process $P(t, x, y)$ may be defined as follows.[1] Let $L^1(R)$ be the Banach space of functions $f(x)$ integrable with respect to dx over R. There exist functions $f(x)$ dense in $L^1(R)$ for which the so called FOKKER-PLANCK *equation* holds:

$$(1.3) \qquad \frac{\partial}{\partial t} f(t, x) = A \cdot f(t, x)\ (t \geqq 0), \qquad \begin{aligned} f(t, x) &= \int_R f(y)\, P(t, y, x)\, dy \ \ (t > 0), \\ f(0, x) &= f(x), \end{aligned}$$

where the operator A is defined by

[1] A. KOLMOGOROFF: Zur Theorie der stetigen zufälligen Prozesse, Math. Ann. 108 (1933). W. FELLER: Zur Theorie der stochastischen Prozesse, Math. Ann., 113 (1937).

$$(1.4)\quad (Af)(x) = \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}[-\sqrt{g(x)}\,a^i(x)\,f(x)] + \\ + \frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i \partial x^j}[\sqrt{g(x)}\,b^{ij}(x)\,f(x)]$$

with a positive definite quadratic form $b^{ij}(x)\,\xi_i\,\xi_j$.

The purpose of the present note is to show that, under certain continuity assumptions on $a(x)$ and $b(x)$, we may integrate (1.3) in the following manner: *there exists one and only one one-parameter semi-group* $\{U_t\}$, $0 \leqq t < \infty$, *of linear operators on* $L^1(R)$ *to* $L^1(R)$ *such that:*

(1.5) $U_t U_s = U_{t+s}$ $(t, s \geqq 0)$, $U_0 = I =$ *the identity;*

(1.6) strong $\lim_{t \to t_0} U_t f = U_{t_0} f$ *for* $f \in L^1(R)$*;*

(1.7) strong $\lim_{\delta \to 0} \frac{1}{\delta}[U_{t+\delta} - U_t] f = \tilde{A}\, U_t f$ *for* f *in a dense set in* $L^1(R)$, $\tilde{A}$ *denoting the closed extension of the operator* A *;*

(1.8) *if* $f(x) \in L^1(R)$ *is non-negative, then* $f(t, x) = (U_t f)(x)$ *is also non-negative and* $\| U_t f \|_1 = \int_R | f(t, x) | \, dx = \int_R | f(x) | \, dx = \| f \|_1$ (U_t may be called a *transition operator* on $L^1(R)$ to $L^1(R)$).

The method of proof is based upon the theory of semi-groups of linear operators due to E. HILLE[1] and the author[2] according to which *the operator* U_t *satisfying* (1.5)—(1.7) *is unique and may be given by*

$$(1.9)\quad U_t f = \text{strong}\lim_{n\to\infty} \exp\,[t\tilde{A}\,(I - n^{-1}\tilde{A})^{-1}]\,f = \\ = \text{strong}\lim_{n\to\infty} \exp\,[t\,n\,(I - n^{-1}\tilde{A})^{-1} - I]\,f,$$

if $I_n = (I - n^{-1}\tilde{A})^{-1}$ *exists and is of norm* $\leqq 1$ $(n = 1, 2, \ldots)$.[3] The condition (1.8) is implied by the fact that the I_n are transition operators since

$$\exp\,(t A I_n)\,f = \exp\,[n t\,(I_n - I)]\,f = \exp\,(-n t)\,\exp\,(n t I_n)\,f.$$

These results may be considered as an extension of the case in which R is the surface of the three-sphere.[4]

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[1] Functional analysis and semi-groups, New York (1948) Chap. 12.

[2] On the differentiability and the representation of one-parameter semi-groups of linear operators, Journ. Math. Soc. Japan, Vol. 1, No. 1 (1948), 15—21.

[3] Cf. E. HILLE, loc. cit., pp. 403—407, where the case of the n-dimensional euclidean space R and of constant $a(x)$, $b(x)$ is treated. Cf. also K. YOSIDA: An operator theoretical treatment of temporally homogeneous Markoff process, to appear in the Journ. Math. Soc. Japan, where the case of one-dimensional euclidean space R and non-constant $a(x)$, $b(x)$ is treated.

[4] K. YOSIDA: Brownian motion on the surface of the 3-sphere, to appear in the Ann. of Math.

2. Hypotheses and proof. The operator A may be written as

$$(2.1)\qquad (Af)(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\left[\sqrt{g(x)}\,b^{ij}(x)\frac{\partial f}{\partial x^j}\right]+c^i(x)\frac{\partial f}{\partial x^i}+e(x)f(x).$$

We assume that

(2.2) *$g(x)$, $b(x)$ and their first derivatives, $c(x)$ and $e(x)$ are all continuous in R;*

(2.3) *$b^{ij}(x)\,\xi_i\,\xi_j \geqq \alpha^{-1}\sum_i \xi_i^2$ with a positive constant α.*

Then the formally self-adjoint operator

$$(2.4)\qquad H=\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\left[\sqrt{g(x)}\,b^{ij}(x)\frac{\partial}{\partial x^j}\right]$$

has a *hypermaximal extension* $\tilde{H}$ and, since R is without boundary,

$$(2.5)\qquad (Hf,f)=-\int_R b^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\,dx\leqq 0.$$

Our process of integration may be carried out in two steps.

The first step: *I_n exists as an operator on $L^2(R)$ to $L^2(R)$ for large n.*

Proof. Since H satisfies (2.5), $(I-n^{-1}\tilde{H})^{-1}$ exists with the norm

$$\|(I-n^{-1}\tilde{H})^{-1}\|_2\leqq 1$$

as operator on $L^2(R)$ to $L^2(R)$ $(n=1,2,3,\ldots)$. Hence the range

$$\{h;\ h=(I-n^{-1}H)f,\ f\in \text{domain } D(H) \text{ of } H\}$$

is dense in $L^2(R)$. When $h=(I-n^{-1}H)f$, we have, by (2.4) since

$$\|(I-n^{-1}\tilde{H})^{-1}\|_2\leqq 1,$$

$$\left\|n^{-1}c^i(x)\frac{\partial f}{\partial x^i}\right\|_2\leqq\left\{n^{-2}\int_R\sum_i[c^i(x)]^2dx\cdot\int_R\sum_i\left(\frac{\partial f}{\partial x^i}\right)^2dx\right\}^{\frac12}$$

$$\leqq\left\{\alpha n^{-2}\int_R\sum_i[c^i(x)]^2dx\cdot\int_R b^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}dx\right\}^{\frac12}$$

$$=\left\{\alpha n^{-1}\int_R\sum_i[c^i(x)]^2dx\right\}^{\frac12}\cdot[-n^{-1}(Hf,f)]^{\frac12},$$

where

$$0 \leqq -n^{-1}(Hf, f) = -(f, f) + (h, f) \leqq \| I - n^{-1}\tilde{H})^{-1} h \|_2^2 + \| h \|_2 \| (I - n^{-1}\tilde{H})^{-1} h \|_2 \leqq 2 \| h \|_2^2$$

so that

$$\left\| n^{-1} c^i(x) \frac{\partial}{\partial x^i} (I - n^{-1}\tilde{H})^{-1} h \right\|_2 \leqq \left[2\, a\, n^{-1} \int_R \sum_i [c^i(x)]^2 \, dx \right]^{\frac{1}{2}} \| h \|_2 .$$

We have also

$$\| n^{-1} e (I - n^{-1}\tilde{H})^{-1} h \|_2 \leqq n^{-1} \sup | e(x) | \cdot \| h \|_2 .$$

Therefore, since $\{h;\, h = (I - n^{-1}H) f,\, f \in D(H)\}$ is dense in $L^2(R)$,

$$(2.6) \quad \| n^{-1}\tilde{K} \|_2 \leqq 1, \; (Kh)(x) = c^i(x) \frac{\partial}{\partial x^i} (I - n^{-1}\tilde{H})^{-1} h(x) + e(x)(I - n^{-1}\tilde{H})^{-1} h(x),$$

provided

$$(2.7) \qquad \left\{ 2\, a\, n^{-1} \int_R \sum_i [c^i(x)]^2 \, dx \right\}^{\frac{1}{2}} + n^{-1} \sup | e(x) | < 1 .$$

Hence $(I - n^{-1}\tilde{K})^{-1}$ exists and the range $\{g;\, g = (I - n^{-1}K) h,\, h \in D(K)\}$ is dense in $L^2(R)$ if (2.7) is satisfied. Therefore

$$\left\{ g;\, g = (I - n^{-1}H) f - n^{-1} c^i \frac{\partial f}{\partial x^i} - n^{-1} e f,\, f \in D(H) \right\} = \{g;\, g = (I - n^{-1}A) f,\, f \in D(H)\}$$

is dense in $L^2(R)$. Since $\| n^{-1}\tilde{K} \|_2 \leqq 1$ and $\| (I - n^{-1}\tilde{H})^{-1} \|_2 \leqq 1$, the solution f of

$$g = (I - n^{-1}A) f \text{ with } (I - n^{-1}H) f = h,\; (I - n^{-1}K) h = g$$

satisfies

$$\| f \|_2 \leqq \| (I - n^{-1}\tilde{H})^{-1} h \|_2 \leqq \| h \|_2 \leqq \| (I - n^{-1}\tilde{K})^{-1} g \|_2 \leqq \leqq \| (I - n^{-1}\tilde{K})^{-1} \|_2 \cdot \| g \|_2 .$$

Thus $(I - n^{-1}\tilde{A})^{-1}$ exists as a bounded operator on $L^2(R)$ to $L^2(R)$.

The second step: *Let $f(x)$ have continuous second order derivatives and put $f(x) - n^{-1}(Af)(x) = g(x)$, then, if $n > \sup | e(x) |$, we have*

$$f(x_0) \geqq [1 - n^{-1} e(x_0)]^{-1} g(x_0) \text{ for some } x_0 .$$

Proof. Let $f(x)$ reach its minimum at $x = x_0$. Since $\frac{\partial f}{\partial x^i}(x_0) = 0$, we have

$$f(x_0) - n^{-1} b^{ij}(x_0) \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) - n^{-1} e(x_0) f(x_0) = g(x_0),$$

that is,

$$f(x_0)[1 - n^{-1} e(x_0)] \geqq g(x_0).$$

Moreover we have

$$\int_R f(x)\, dx = \int_R f(x)\, dx - n^{-1} \int_R (Af)(x)\, dx = \int_R g(x)\, dx$$

since R is without boundary. Thus if the sequence $\{g_n(x)\}$ tends L^2-strongly to a non-negative function $g_\infty(x) \in L^2(R) < L^1(R)$, the corresponding functions $f_n(x)$ must tend to an almost everywhere non-negative function

$$f_\infty(x) \in L^2(R) < L^1(R) \text{ and } \| f_\infty \|_1 = \| g_\infty \|_1.$$

Since $L^2(R)$ is L^1-dense in $L^1(R)$. I_n exists as a transition operator on $L^1(R)$ to $L^1(R)$ if n is sufficiently large. By formula (1.9), U_t exists as a transition operator satisfying (1.5)—(1.8). Thus the integration process has been carried through.

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