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On equivalent analytic functions

By Bengt Andersson

With 1 figure in the text

1. We denote by $\mathcal{R}$ the class of functions $f(z)$ that are analytic in a circle $|z| \leq R$. Two functions $f(z)$ and $g(z)$ of $\mathcal{R}$ are called *equivalent* if $f(z)$ is transformed into $g(z)$ by

(i) multiplication with a constant of modulus 1,
(ii) a transformation $z' = z e^{i\alpha}$ (α real),
(iii) replacing of all coefficients in the power series of $f(z)$ by their conjugate values.

Thus

$$g(z) = e^{i\beta} f(z e^{i\alpha}) \quad \text{or} \quad g(z) = e^{i\beta} \overline{f(\bar{z} e^{i\alpha})}.$$

We also call two harmonic functions $u(z)$ and $u_1(z)$ or two curves c and c_1 equivalent if one is transformed into the other by

(i) rotating the z-plane an angle α about $z = 0$,
(ii) reflection in a straight line through $z = 0$.

Thus

$$u_1(z) = u(z e^{i\alpha}) \quad \text{or} \quad u_1(z) = u(\bar{z} e^{i\alpha}).$$

We obtain immediately that if $f(z)$ and $g(z)$ of $\mathcal{R}$ are equivalent, then the harmonic functions $\log |f|$ and $\log |g|$ are equivalent.

Let $f(z)$ belong to $\mathcal{R}$. Given $r \leq R$, we put $z = r e^{i\varphi}$ and define $e_f(r, a)$ as the set of φ, $0 \leq \varphi \leq 2\pi$, such that $|f(r e^{i\varphi})| \leq a$ in e_f. Denoting by $\Phi_f(r, a)$ the measure of e_f we will call Φ_f the M-function of $f(z)$.

According to the definition, Φ_f is a non-decreasing function of a. If $M(r)$ and $m(r)$ denote as usual the maximum and minimum of $|f(z)|$ for $|z| = r$, then $\Phi_f = 0$ for $a < m(r)$ and $\Phi_f = 2\pi$ for $a > M(r)$. It is easily seen that if $f(z)$ and $g(z)$ are equivalent, then Φ_f and Φ_g are identical for all $r \leq R$.

In the following we always exclude the case that $f(z)$ is a power of z, $f(z) = a z^m$. In this case the obtained results are trivial. Therefore we assume that $m(r) < M(r)$,[1] and that $\Phi_f(r, a)$ is increasing in the interval $m(r) \leq a \leq M(r)$.

[1] There is at most one value of r for which $m(r) = M(r)$. This special value is of no interest here. See Blumenthal (1), Valiron (2).

The function $\Psi_f(r, \theta)$ is defined in the interval $0 \leq \theta \leq 2\pi$ as the measure of the set $E(\Phi_f(r, a) \leq \theta)$, $a > 0$. Then $\Phi(a)$ and $\Psi(\theta)$ are inverse functions, and from the definition it follows that

$$m E(\Psi_f(r, \theta) \leq a) = \Phi_f(r, a) = m e_f(r, a).$$

This equality gives the following lemma, which in this case, according to the simple character of the function $|f(re^{i\varphi})|$, nearly seems to be trivial.[1]

Lemma 1. *$G(\sigma)$ is a function, defined for $m(r) \leq \sigma \leq M(r)$. Then we have*

$$\int_0^{2\pi} G[\Psi_f(r, \theta)]\, d\theta = \int_0^{2\pi} G[|f(re^{i\varphi})|]\, d\varphi$$

whenever one of the integrals exists.

Hence

Cor. *If $f(z)$ and $g(z)$ have identical M-functions for $|z| = r$, $\Phi_f(r, a) = \Phi_g(r, a)$, then*

$$\int_0^{2\pi} G[|f(re^{i\varphi})|]\, d\varphi = \int_0^{2\pi} G[|g(re^{i\varphi})|]\, d\varphi.$$

It is now convenient to study the distribution of values of an analytic function in connexion with the functions Φ and Ψ.
We have the following theorem:

Theorem 1. *Let $f(z)$ and $g(z)$ be functions of $\mathcal{R}$ and have identical M-functions in an interval $0 < r \leq r_1$. Then the functions are equivalent.*

Before we give the proof, we require some preliminary studies and remarks. Put

$$f(z) = k z^q \sum_{n=0}^{\infty} A_n z^n, \qquad A_0 = 1$$

$$g(z) = k_1 z^{q_1} \sum_{n=0}^{\infty} B_n z^n. \qquad B_0 = 1$$

We apply lemma 1 for $G(\sigma) = \sigma^2$. Then for all $r \leq r_1$,

$$|k|^2 r^{2q} \sum_{n=0}^{\infty} |A_n|^2 r^{2n} = |k_1|^2 r^{2q_1} \sum_{n=0}^{\infty} |B_n|^2 r^{2n}.$$

Hence

(1) $$|k| = |k_1|, \quad q = q_1, \quad |A_n| = |B_n|, \qquad n = 0, 1, 2, \ldots$$

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[1] J. v. NEUMANN (3) states a similar lemma for more general real functions.

We denote by $\mathcal{N}_s$ the class of functions of $\mathcal{R}$ with power series of the form

$$1 + \sum_{n=s}^{\infty} a_n z^n$$

and satisfying the following conditions,

(i) $a_s = \frac{1}{s}$

(ii) the highest common divisor of the indices n for which $a_n \neq 0$ is 1.

Then $f(z)$ and $g(z)$ can be expressed

$$f(z) = k z^q f_1(c z^m), \quad g(z) = k_1 z^{q_1} g_1(c_1 z^{m_1})$$

where $f_1(z) \in \mathcal{N}_s$, $g_1(z) \in \mathcal{N}_{s_1}$. (1) gives immediately

$$s = s_1, \quad |c| = |c_1|, \quad m = m_1.$$

Further it is easily seen that $f_1(z)$ and $g_1(z)$ have identical M-functions for $0 < r \leq \varrho$, $\varrho = |c| r_1^m$. It is therefore sufficient to prove the theorem for functions of the same class $\mathcal{N}_s$.

2. Consider the harmonic function

$$u(z) = \log |f(z)|$$

where $f(z) \in \mathcal{N}_s$. $u(z)$ is regular in the circle $|z| \leq R$, where $f(z)$ is holomorphic, with the exception only of the finite number of zeros of $f(z)$. On the circle $|z| = r$, $|f(r e^{i\varphi})|$ is a continuous function of φ and attains its extreme values in those points on the circle where $\frac{\partial u}{\partial \varphi} = 0$. When r varies, the loci of these points are the level curves $\frac{\partial u}{\partial \varphi} = 0$, and they are in the following called extreme value curves (e. c.). These curves and the values of $|f|$ attained on them have been examined by BLUMENTHAL (1), who shows their simple analytic character.

Let us write

$$u + iv = \log f$$

u and v are harmonic functions, regular in the neighbourhood of $z = 0$. Consider the function

$$w = \frac{1}{i}\left(\frac{\partial u}{\partial \varphi} + i \frac{\partial v}{\partial \varphi}\right) = \frac{1}{i}\left(\frac{\partial u}{\partial \varphi} + i r \frac{\partial u}{\partial r}\right) = z \frac{f'(z)}{f(z)} \tag{2}$$

or

$$w = z^s \mathcal{P}(z),[1] \qquad \mathcal{P}(0) = 1$$

[1] We denote by $\mathcal{P}(z)$ a general power series of z with positive radius of convergence.

$w(z)$ is meromorphic in $|z| \leq R$, and the e. c.'s of $f(z)$ are determined by

$$\frac{\partial u}{\partial \varphi} = -\mathcal{J}\{w\} = 0. \tag{3}$$

It is possible to divide the circle $|z| \leq R$ in a finite number of annular regions Γ_ν,

$$r_\nu < |z| < r_{\nu+1}, \quad r_0 = 0, \quad r_{n+1} = R$$

so that in each annular Γ_ν we have an even number $2\,n_\nu$ of connected e. c.'s and each of them can be expressed in polar coordinates $\varphi = \varphi(r)$, where $\varphi(r)$ is analytic in the interval $r_\nu < r < r_{\nu+1}$. On a circle $|z| = r$ in Γ_ν the modulus $|f(z)|$ attains its maximum and minimum values in the points where the circle intersects the e. c.'s. The value of $|f(z)|$ on an e. c., expressed as a function of r, is called an extreme value function (e. f.). This function is analytic in r.

Consider an e. c. $\varphi = \varphi_0(r)$; the e. f. obtained on $\varphi_0(r)$ is $\mu(r)$. Then we have on the e. c.

$$\frac{\partial u}{\partial \varphi} = 0, \quad \frac{d u}{d r} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \varphi}\frac{d \varphi_0}{d r} = \frac{d \log \mu(r)}{d r}$$

$$\frac{d}{d r}\frac{\partial u}{\partial \varphi} = \frac{\partial^2 u}{\partial r\, \partial \varphi} + \frac{\partial^2 u}{\partial \varphi^2}\frac{d \varphi_0}{d r} = 0$$

$$\frac{d^2 u}{d r^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial r\, \partial \varphi}\frac{d \varphi_0}{d r} = \frac{d^2 \log \mu(r)}{d r^2}.$$

Further, u is harmonic

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0.$$

From these conditions we obtain the following equation for the e. c.:

$$\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{\varphi_0(r)} \left\{\left(\frac{d \varphi_0}{d r}\right)^2 + \frac{1}{r^2}\right\} = -\frac{1}{r}\frac{d}{d r}\left(\frac{d \log \mu(r)}{d \log r}\right). \tag{4}$$

If $|f(z)|$ attains a maximum on the e. c., then $\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{\varphi_0(r)} < 0$. Thus

$$\frac{d}{d r}\left(\frac{d \log \mu(r)}{d \log r}\right) > 0.$$

$\log \mu(r)$ is therefore a convex function of $\log r$. In the same way we obtain that if $\mu(r)$ is a minimum e. f., then $\log \mu(r)$ is a concave function of $\log r$. According to their analytic properties, two e. f.'s are equal only for a finite number of values of r if they are not identical in an interval. Further, a minimum function cannot be identical with a maximum function.

In the proof of theorem 1 we use the following lemma:

Lemma 2. *Suppose that the function $f(z)$ attains on an e. c. $\varphi = \varphi_0(r)$ an e. f. $\mu(r)$, identical with an e. f. of $g(z)$, attained on an e. c. $\varphi = \varphi_1(r)$. Further, putting*

$$u = \log |f|, \qquad u_1 = \log |g|$$

if we have

$$\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{\varphi_0(r)} = \left(\frac{\partial^2 u_1}{\partial \varphi^2}\right)_{\varphi_1(r)}$$

then $f(z)$ and $g(z)$ are equivalent.

From (4) we see that in an interval $r' < r < r''$ we have

$$\frac{d\varphi_0}{dr} = \frac{d\varphi_1}{dr} \quad \text{or} \quad \frac{d\varphi_0}{dr} = -\frac{d\varphi_1}{dr}.$$

In both cases the e. c.'s are equivalent. Then there is a function $g_1(z)$ equivalent to $g(z)$ that attains the e. f. $\mu(r)$ on the e. c. $\varphi = \varphi_0(r)$. Put

$$\mathcal{U} = \log |f| - \log |g_1|.$$

Hence for $\varphi = \varphi_0(r)$ we have

$$\mathcal{U} = 0, \qquad \frac{\partial \mathcal{U}}{\partial \varphi} = 0. \qquad\qquad r' < r < r''$$

Then, from the well-known properties of harmonic functions it follows immediately that $\mathcal{U} \equiv 0$. Thus $f(z) = e^{i\beta} g_1(z)$ and $f(z)$ is therefore equivalent to $g(z)$.

3. We now pass to a detailed study of the function $\Phi_f(r, a)$. Here we shall suppose, for the sake of simplicity, that $f(z)$ belongs to a class $\mathcal{N}_s$, and that $0 < r \leq r_1$, where r_1 can be chosen sufficiently small for every circle $|z| = r \leq r_1$ to intersect only the e. c.'s ending at $z = 0$, and for each e. c. to be intersected only once. Two e. f.'s are equal for such a value of r only if they are identical in the whole interval. Further, $f(z) \neq 0$ in the circle $|z| \leq r_1$.

Studying the function $w(z)$ defined above, we see that there are $2s$ e. c.'s abutting at $z = 0$, s e. c.'s where $|f(z)|$ attains a relative maximum, and s e. c.'s where the extreme value is a relative minimum.

On a circle $|z| = r$, $u \doteq \log |f(re^{i\varphi})|$ is an analytic function of φ at every point $z_0 = re^{i\varphi_0}$. Thus, for small values of $|\varphi - \varphi_0|$

$$u(re^{i\varphi}) - u(re^{i\varphi_0}) = \sum_{n=1}^{\infty} \frac{1}{\underline{|n}} \left(\frac{\partial^n u}{\partial \varphi^n}\right)_{z_0} (\varphi - \varphi_0)^n. \tag{5}$$

Further

$$|f(re^{i\varphi})| - |f(re^{i\varphi_0})| = |f(re^{i\varphi_0})| \left[e^{u(re^{i\varphi}) - u(re^{i\varphi_0})} - 1\right]. \tag{6}$$

If z_0 is not a point on an e. c., we have $\left(\dfrac{\partial u}{\partial \varphi}\right)_{z_0} \neq 0$. Then from (5) and (6) we obtain

$$\varphi - \varphi_0 = \frac{a - a_0}{a_0} \frac{1}{\left(\frac{\partial u}{\partial \varphi}\right)_{z_0}} \mathcal{P}(a - a_0), \qquad \mathcal{P}(0) = 1 \tag{7}$$

where

$$a = |f(r e^{i\varphi})|, \quad a_0 = |f(r e^{i\varphi_0})|.$$

If z_0 is a point on an e. c., then $\left(\frac{\partial u}{\partial \varphi}\right)_{z_0} = 0$, and if r_1 is sufficiently small, we can assume that $\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{z_0} \neq 0$. Put $|f(z_0)| = \mu(r)$, where $\mu(r)$ is the corresponding e. f. Then, in the neighbourhood of $\varphi = \varphi_0$ we obtain the inverse function

$$\begin{aligned} &\varphi > \varphi_0; \quad \varphi - \varphi_0 = \sqrt{2 \frac{a - \mu(r)}{\mu(r) \left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{z_0}}} \mathcal{P}(\sqrt{|a - \mu(r)|}) \\ &\varphi < \varphi_0; \quad \varphi_0 - \varphi = \sqrt{2 \frac{a - \mu(r)}{\mu(r) \left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{z_0}}} \mathcal{P}(- \sqrt{|a - \mu(r)|}) \\ &\qquad\qquad \mathcal{P}(0) = 1 \end{aligned} \tag{8}$$

where the root is positive.

We have $\Phi_f(r, a) = 0$ for $a < m(r)$. If a_0 is not an extreme value on $|z| = r$, $m(r) < a_0 < M(r)$, then $|f(z)|$ attains the value a_0 in a finite number of points on the circle. If $a - a_0$ is positive and sufficiently small, then

$$\Phi_f(r, a) - \Phi_f(r, a_0) = m\, e_\varphi\, (a_0 < |f(r e^{i\varphi})| \leq a)$$

is the sum of a finite number of intervals of the form (7). Thus

$$\begin{cases} \Phi_f(r, a) = \Phi_f(r, a_0) + (a - a_0) \mathcal{P}_1(a - a_0) \\ \qquad\qquad \mathcal{P}_1(0) > 0 \end{cases} \tag{9}$$

Now a and a_0 can be permutated, and we have the same expansion for $a < a_0$. By power series of this form $\Phi_f(r, a)$ can be continued from a_0 to the nearest extreme values. The minimum e. f.'s attained on $|z| = r$ are $m_j(r)$, the maximum e. f.'s are $M_j(r)$. Then, by the choice of r it follows that

$$0 < m(r) = m_1(r) \leq m_2(r) \leq \cdots \leq m_s(r)$$

$$M_s(r) \leq M_{s-1}(r) \leq \cdots \leq M_1(r) = M(r).$$

There are h_j e. f.'s identical with $m_j(r)$ and h_j' e. f.'s identical with $M_j(r)$.

Putting $a_0 = m_j(r)$ we have for $a < m_j(r)$ an expansion of the form (9). To this expansion (regular in a_0), we must add, by analytic continuation (for $a > m_j(r)$), the contribution from the intervals containing the h_j points $z_\nu = r e^{i\varphi_\nu}$ where $|f(r e^{i\varphi_\nu})| = m_j(r)$. The lengths of these intervals are calculated from (8). We obtain for $a > m_j(r)$

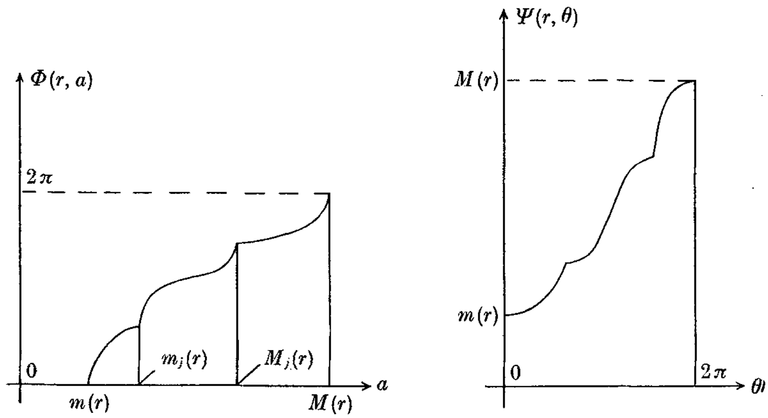

Fig. 1. The functions $\Phi(r, a)$ and $\Psi(r, \theta)$.

$$(10)\quad \Phi(r,a)=\Phi(r,m_j(r))+\frac{2\sqrt{2}}{\sqrt{m_j(r)}}\left(\sum_\nu \frac{1}{\sqrt{\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{z_\nu}}}\right)\sqrt{a-m_j(r)}\,\mathfrak{P}_1\left(\sqrt{a-m_j(r)}\right),$$

$$\mathfrak{P}_1(0)=1.$$

Similarly we obtain the behaviour of $\Phi(r, a)$ at a maximum value $M_j(r)$. For $a > M_j(r)$ we have a regular expression of the form (9) and for $a < M_j(r)$ we have

$$(11)\quad \Phi(r,a)=\Phi(r,M_j(r))-\frac{2\sqrt{2}}{\sqrt{M_j(r)}}\left(\sum_\nu \frac{1}{\sqrt{-\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{z_\nu}}}\right)\sqrt{M_j(r)-a}\,\mathfrak{P}_1\left(\sqrt{M_j(r)-a}\right),$$

$$\mathfrak{P}_1(0)=1$$

the sum being taken for the h_j' points $z_\nu = r e^{i\varphi_\nu}$ where $|f(re^{i\varphi})| = M_j(r)$.

It is clear that the function $\Phi(r, a)$ has this simple analytic character in the whole interval $0 < r \le R$. The expansions in the neighbourhood of extreme values may be somewhat altered, however, on a finite number of circles.

4. Consider the function

$$(12)\qquad w = z\frac{f'(z)}{f(z)} = z^s\,\mathfrak{P}(z); \qquad \mathfrak{P}(0) = 1$$

Then for small r we obtain for the e. c.'s

$$\frac{\partial u}{\partial \varphi} = -\mathfrak{J}\{w\} = -r^s \sin s\varphi\,(1 + O(r))$$

$$\frac{\partial^2 u}{\partial \varphi^2} = -s r^s \cos s\varphi\,(1 + O(r)).$$

Denote the e. c.'s ending at $z=0$ by c_ν, $(\nu = 0, 1, \ldots, 2s-1)$, where the index ν is subjected to the condition that the angle between the positive real axis and the tangent of c_ν at $z=0$ is $\nu \frac{\pi}{s}\cdot$

Then for c_ν we have

$$\text{(13)} \qquad \begin{cases} \varphi_\nu = \nu \frac{\pi}{s}(1 + O(r)) \\ \left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{c_\nu} = (-1)^{\nu+1} r^s (1 + O(r)) \end{cases}$$

Suppose $f(z)$ and $g(z)$ of $\mathfrak{N}_s$ have identical M-functions for small r, thus fulfilling the condition in theorem 1. Then $\Phi_f(r, a)$ and $\Phi_g(r, a)$ have the same singularities in their analytic character. Each e. f. $\mu(r)$ of $f(z)$ in the neighbourhood of $z=0$ is therefore an e. f. of $g(z)$. Suppose $\mu(r)$ is a h-tiple e. f. of $f(z)$ and a h'-tiple e. f. of $g(z)$. *Then* $h = h'$.

For if $\mu(r)$ is a minimum function (or a maximum function) the coefficient of $\sqrt{a - \mu(r)}$ (respectively $\sqrt{\mu(r) - a}$) in the developments of $\Phi_f(r, a)$ and $\Phi_g(r, a)$ to the right (left) of $\mu(r)$ are identical. Then if $u_1 = \log|g|$ we obtain from (10) and (11)

$$\text{(14)} \qquad \sum \left(\left| \frac{\partial^2 u}{\partial \varphi^2} \right|_{c_\nu}^{-\frac{1}{2}} \right)_r = \sum \left(\left| \frac{\partial^2 u_1}{\partial \varphi^2} \right|_{c'_\nu}^{-\frac{1}{2}} \right)_r$$

the sums being taken for the h e. c.'s c_ν respectively the h' e. c.'s c'_ν, where $f(z)$ and $g(z)$ attain the e. f. $\mu(r)$. By (13) we write this condition for small r:

$$h r^{-\frac{s}{2}} (1 + O(r)) = h' r^{-\frac{s}{2}} (1 + O(r))$$

and the result $h = h'$ follows immediately.

The proof of theorem 1 now follows from the following lemma.

Lemma 3. *Every function* $f(z)$ *of* $\mathfrak{N}_s$ *has in the neighbourhood of* $z=0$ *at least one e. f., non-identical with any other e. f.*

Suppose the lemma holds. Then $f(z)$ and $g(z)$ have an e. f. $\mu(r)$ with the multiplicity $h = h' = 1$. The corresponding e. c. of $f(z)$ is $\varphi = \varphi_0(r)$ and of $g(z)$, $\varphi = \varphi_1(r)$. Then from (14)

$$\left(\frac{\partial^2 u}{\partial \varphi^2}\right)_{\varphi_0(r)} = \left(\frac{\partial^2 u_1}{\partial \varphi^2}\right)_{\varphi_1(r)}.$$

Then from lemma 2 we obtain that $f(z)$ and $g(z)$ are equivalent, and this proves theorem 1.

Proof of lemma 3. Consider the function $w(z)$ defined by (12). Let the e. f. on the e. c. c_ν be $\mu_\nu(r)$. Then

$$\text{(15)} \qquad \frac{d \log \mu_\nu(r)}{d(\log r)} = \left(r \frac{\partial u}{\partial r}\right)_{c_\nu} = \mathcal{R}\{w\}_{c_\nu}.$$

In the neighbourhood of $z=0$, $w=0$ the inverse function $z(w)$ has an expansion of the form

$$\begin{cases} z = t\,\mathcal{P}_1(t) \\ w = t^s \end{cases} \qquad \mathcal{P}_1(0) = 1$$

Hence

$$\log z = \log r + i\varphi = \log t + Q(t) \tag{16}$$

where

$$Q(t) = \sum_{n=1}^{\infty} c_n t^n \tag{17}$$
$$0 \le \varphi < 2\pi, \quad 0 \le \arg t < 2\pi.$$

The e. c.'s c_ν, $(\nu = 0, 1, \ldots, 2s-1)$ correspond to the real axis in the w-plane, and by the suitable choice of the index ν (cf. p. 84) we obtain that the e. c. c_ν is represented by the straight line in the t-plane

$$\arg t = \nu \frac{\pi}{s} \qquad (\nu = 0, 1, \ldots, 2s-1)$$
$$0 \le |t| < \delta.$$

Putting $t = \varrho e^{i\nu\frac{\pi}{s}}$ we obtain the following equation for c_ν [from (15), (16), (17)]

$$\log r = \log \varrho + \mathcal{R}\{Q(\omega^\nu \varrho)\}$$
$$\varphi_\nu = \nu\frac{\pi}{s} + \mathcal{J}\{Q(\omega^\nu \varrho)\} \tag{18}$$
$$(-1)^\nu \varrho^s = \frac{d \log \mu_\nu(r)}{d(\log r)}$$

where $\omega = e^{i\frac{\pi}{s}}$. If ν is even, $\mu_\nu(r)$ is a maximum function; if ν is odd, the e. f. is a minimum function.

The function $Q(t)$ in (16) is regular at $t=0$. We write

$$Q(t) = \sum_{j=1}^{k} t^{n_j s + \sigma_j} Q_j(t^s)$$

where

$$0 < \sigma_1 < \sigma_2 \cdots < \sigma_k \le s, \quad n_j \ge 0, \quad Q_j(0) \ne 0.$$

Therefore in the power series of $Q(t)$ we have $c_{n_j s + \sigma_j} \ne 0$ and all coefficients $c_n \ne 0$ have indices of the form $n = Ns + \sigma_j$. Now *the highest common divisor* $(\sigma_1, \sigma_2, \ldots, \sigma_k, s) = 1$.

Let us assume that this divisor is $m > 1$. Then $Q(t)$ is a regular function of t^m and we obtain

$$z^m = t^m \mathcal{P}_2(t^m), \qquad \mathcal{P}_2(0) = 1$$

$$t^m = z^m \mathcal{P}_3(z^m)$$

$$w = t^s = z^{m\frac{s}{m}} \mathcal{P}_4(z^m).$$

Here $m \mid s$ and w is a regular function of z^m. Then it is easily seen that $f(z)$ is a regular function of z^m, $m > 1$, which is impossible if $f(z)$ belongs to a class $\mathcal{N}_s$.

If $s > 1$, two e. f.'s in the neighbourhood of $z = 0$ may be identical. As maximum functions are increasing, and the minimum functions decreasing functions of r, the identical e. f.'s must be of the same kind. Suppose

$$\mu_\nu(r) \equiv \mu_{\nu_1}(r).$$

Then we can write

(19)
$$\nu_1 \equiv \nu + 2m \qquad (\text{mod } 2s)$$
$$0 < m \leq s - 1.$$

From the equations (18) follows that on c_ν and c_{ν_1}, ϱ is the same function of r and conversely $\log r$ must be the same function of ϱ. Hence

$$\mathcal{R}\{Q(\omega^\nu \varrho)\} \equiv \mathcal{R}\{Q(\omega^{\nu_1}\varrho)\}$$

or

$$\mathcal{R}\left\{\sum_1^\infty c_n \omega^{n\nu} \varrho^n\right\} \equiv \mathcal{R}\left\{\sum_1^\infty c_n \omega^{n\nu_1} \varrho^n\right\}.$$

If $c_n = |c_{Ns+\sigma_j}| e^{i\beta_{Ns+\sigma_j}} \neq 0$ we have

$$\cos\left(\sigma_j \nu \frac{\pi}{s} + \beta_{Ns+\sigma_j}\right) = \cos\left(\sigma_j \nu_1 \frac{\pi}{s} + \beta_{Ns+\sigma_j}\right)$$

or from (19)

$$\cos\left(\sigma_j(\nu + 2m)\frac{\pi}{s} + \beta_{Ns+\sigma_j}\right) = \cos\left(\sigma_j \nu \frac{\pi}{s} + \beta_{Ns+\sigma_j}\right).$$

Therefore at least one of the following two conditions holds.

(A)
$$\sigma_j m \equiv 0 \qquad (\text{mod } s)$$

(B)
$$\sigma_j(\nu + m) + \frac{s}{\pi}\beta_{Ns+\sigma_j} \equiv 0. \qquad (\text{mod } s)$$

We express s in a standard form of primes

$$s = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_q^{\alpha_q}$$

where p_i are distinct primes > 1, $\alpha_i \geq 1$. The primes may be arranged as they appear in the following calculation.

Since $(\sigma_1, \sigma_2, \ldots, \sigma_k, s) = 1$ there is at least one σ_j not divisible by p_i $(i = 1, 2, \ldots, h)$. We denote by $\theta(p_a, p_b, \ldots, p_h)$ the subsequence of $\{\sigma_j\}$ with the property that each $\sigma_j \in \theta(p_a, \ldots, p_h)$ is not divisible by at least one of the primes $p_a, \ldots, p_h$.

Suppose that $f(z)$ has two identical e. f.'s corresponding to the couple (ν, ν_1) or (ν, m_0) by (19). We write

$$(m_0, s) = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_h^{\gamma_h} \cdot p_{h+1}^{\alpha_{h+1}} \ldots p_q^{\alpha_q}$$
$$0 \leq \gamma_i < \alpha_i, \quad 1 \leq h \leq q.$$

For no $\sigma_j \in \theta(p_1, \ldots, p_h)$ the condition (A) can hold. Thus

$$\sigma_j(\nu + m_0) + \frac{s}{\pi}\beta_{Ns+\sigma_j} \equiv 0 \qquad (\text{mod } s)$$
$$\sigma_j \in \theta(p_1, \ldots, p_h).$$

Now suppose that $\mu_{\nu'}(r)$, $\nu' \equiv \nu + m_0$ (mod s), is identical with another e. f. and that the corresponding number m determined by (19) is m_1. The tangent of $c_{\nu'}$ at the origin is a bisectrise to the tangents of c_ν and c_{ν_1}. We now study the conditions (A) and (B) for the couple (ν', m_1). (B) can be written

$$(\text{B})' \qquad \sigma_j(\nu + m_0 + m_1) + \frac{s}{\pi}\beta_{Ns+\sigma_j} \equiv 0. \qquad (\text{mod } s)$$

If $\sigma_j \in \theta(p_1, \ldots, p_h)$, the conditions (A) and (B) for m_1 are identical and we obtain $p_1^{\alpha_1} | m_1, \ldots, p_h^{\alpha_h} | m_1$ and thus $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_h^{\alpha_h} | m_1$. If $h = q$, we should have $s | m_1$, which is impossible, since $0 < m_1 \leq s - 1$. Then $\mu_{\nu'}(r)$ could not be identical with any other e. f., and lemma 3 holds.

If $h < q$ we put

$$(m_1, s) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_h^{\alpha_h} \cdot p_{h+1}^{\gamma_{h+1}} \ldots p_{h'}^{\gamma_{h'}} p_{h'+1}^{\alpha_{h'+1}} \ldots p_q^{\alpha_q}$$
$$0 \leq \gamma_i < \alpha_i, \qquad h < h' \leq q.$$

If $\sigma_j \in \theta(p_1, p_2, \ldots, p_{h'})$ the condition (B)$'$ holds. Now repeating the argument, suppose that $\mu_{\nu''}(r)$, $\nu'' \equiv \nu + m_0 + m_1$ (mod s), is identical with another e. f., corresponding to the number m_2. Then we must have $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{h'}^{\alpha_{h'}} | m_2$, and this is possible only if $h' < q$. Then we go on studying the e. f. $\mu_{\nu'''}(r)$, $\nu''' \equiv \nu + m_0 + m_1 + m_2$ (mod s). The corresponding number m_3 must be divisible by $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{h''}^{\alpha_{h''}}$, $h'' > h'$. After a finite number of such steps, we obtain an e. f. $\mu_\nu(r)$ which is identical with another e. f., only if the corresponding number m is divisible by s, which is impossible. This proves lemma 3 and the proof of theorem 1 is now complete.

5. We denote as usual the *mean values* of $|f(z)|$ on circles $|z| = r$ for real $p \neq 0$.

$$M_p(f, r) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi\right]^{1/p}.$$

We shall state the following theorem.

Theorem 2. *Let $f(z)$ and $g(z)$ be functions of $\mathcal{R}$ and let the mean values $M_p(f, r)$ and $M_p(g, r)$ on a circle $|z| = r \leq R$ be equal for an infinite number of p, $p = p_1, p_2, \ldots p_n, \ldots \lim |p_n| = \infty$.*

Then, for this r, the M-functions $\Phi_f(r, a)$ and $\Phi_g(r, a)$ are identical, and all mean values are therefore equal.

It is sufficient to prove that the functions $\Psi_f(r, \theta)$ and $\Psi_g(r, \theta)$ are identical for this r.

By lemma 1 we have

$$\int_0^{2\pi} \Psi_f(r, \theta)^{p_n} \, d\theta - \int_0^{2\pi} \Psi_g(r, \theta)^{p_n} \, d\theta = 0 \tag{20}$$

$$n = 1, 2, \ldots$$

From the simple analytic character of the functions Φ and Ψ we see that the values of θ for which the functions $\Psi_f(r, \theta)$ and $\Psi_g(r, \theta)$ may be distinct, form a finite number of intervals. Suppose $\theta_1 < \theta < \theta_2$ is the last of these intervals and suppose that the sequence $\{p_n\}$ has the limit point $+\infty$.

We can assume that $\Psi_f(r, \theta) > \Psi_g(r, \theta)$ for $\theta_1 < \theta < \theta_2$. It is evident that if (20) holds for one $p_n \neq 0$, then $\theta_1 > 0$. Put for $\theta_1 < \theta < \theta_2$

$$\Psi_f(r, \theta) = a(1 + \varphi(\theta)), \qquad \Psi_g(r, \theta) = a(1 + \psi(\theta))$$

$$a = \Psi_f(r, \theta_1) = \Psi_g(r, \theta_1). \qquad\qquad a > 0$$

We have $\varphi(\theta) - \psi(\theta) > 0$ for $\theta_1 < \theta < \theta_2$ and

$$\int_{\theta_1}^{\theta_2} (\varphi(\theta) - \psi(\theta)) \, d\theta = \omega$$

where $\omega > 0$.

Now from (20) we have for $p_n > 0$:

$$0 < a^{p_n} \int_{\theta_1}^{\theta_2} \{[1 + \varphi(\theta)]^{p_n} - [1 + \psi(\theta)]^{p_n}\} \, d\theta =$$

$$= \int_0^{\theta_1} [\Psi_g(r, \theta)^{p_n} - \Psi_f(r, \theta)^{p_n}] \, d\theta < a^{p_n} \theta_1.$$

Hence for $p_n > 0$:

$$0 < \int_{\theta_1}^{\theta_2} [1 + \psi(\theta)]^{p_n} \left\{ \left[1 + \frac{\varphi(\theta) - \psi(\theta)}{1 + \psi(\theta)} \right]^{p_n} - 1 \right\} d\theta < \theta_1. \tag{21}$$

For $x \geq 0$, $p \geq 1$ we have the inequality

$$(1 + x)^p - 1 \geq p x.$$

Using this inequality for $x = \dfrac{\varphi(\theta) - \psi(\theta)}{1 + \psi(\theta)}$ we obtain from (21) for $p_n \geq 1$,

$$0 < p_n \int\limits_{\theta_1}^{\theta_2} [1 + \psi(\theta)]^{p_n - 1} [\varphi(\theta) - \psi(\theta)]\, d\theta < \theta_1.$$

Hence

$$0 < \int\limits_{\theta_1}^{\theta_2} [\varphi(\theta) - \psi(\theta)]\, d\theta = \omega < \frac{\theta_1}{p_n}.$$

For $p_n \to \infty$ we obtain $\omega = 0$, which shows the impossibility of the existence of the interval (θ_1, θ_2), and the functions $\Psi_f(r, \theta)$ and $\Psi_g(r, \theta)$ must be identical.

If $+\infty$ is not a limit point of $\{p_n\}$, then $\lim p_n = -\infty$. For this case we prove similarly that there cannot be any *first* interval (nearest to $\theta = 0$) where $\Psi_f(r, \theta) \neq \Psi_g(r, \theta)$. This proves the theorem.

Theorem 3. *Let $f(z)$ and $g(z)$ be functions of $\mathcal{R}$, and let the M-functions $\Phi_f(r, a)$ and $\Phi_g(r, a)$ be identical for an infinite number of r, $r = r_i$, $r_i \leq R$ $(i = 1, 2, \ldots)$. Then the functions $f(z)$ and $g(z)$ are equivalent.*

The functions $(f(z))^m$, $(g(z))^m$, $(m = 1, 2, 3, \ldots)$ are all analytic in $|z| \leq R$. Put

$$(f(z))^m = \sum_{n=0}^{\infty} A_n^{(m)} z^n, \quad (g(z))^m = \sum_{n=0}^{\infty} B_n^{(m)} z^n.$$

Then for $r = r_1, r_2, \ldots, r_n, \ldots$

$$M_{2m}^{2m}(f, r) = M_{2m}^{2m}(g, r) \tag{22}$$

or

$$\sum_{n=0}^{\infty} |A_n^{(m)}|^2 r^{2n} = \sum_{n=0}^{\infty} |B_n^{(m)}|^2 r^{2n}. \tag{23}$$

These power series are convergent for $r < R + \delta$ if δ is a sufficiently small positive number, and we obtain immediately that $|A_n^{(m)}| = |B_n^{(m)}|$ for all m and n. The equality (22) therefore holds for all r in the interval $(0, R)$. By theorem 2 $f(z)$ and $g(z)$ have identical M-functions in the interval $(0, R)$. Then by theorem 1 the functions are equivalent.

6. The following lemma gives another proof of theorem 1 for functions that can be referred to the class $\mathcal{N}_1$.

Lemma 4. *Let $f(z)$ and $g(z)$ of $\mathcal{R}$ have power series of the form*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad g(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$$

where $a_0 a_1 \neq 0$. Put

$$(f(z))^p = \sum_{\nu=0}^{\infty} a_\nu^{(p)} z^\nu, \quad (g(z))^p = \sum_{\nu=0}^{\infty} b_\nu^{(p)} z^\nu$$

$$a_\nu^{(1)} = a_\nu, \quad b_\nu^{(1)} = b_\nu.$$

Suppose that

$$|b_\nu^{(p)}| = |a_\nu^{(p)}|$$

for $\nu = 0, 1, \ldots, n$ *and* $p = p_1, p_2, \ldots, p_n$, *where* p_k *are real and unequal.*

Then

$$b_\nu = a_\nu e^{i(\alpha + \nu\beta)}, \qquad \nu = 0, 1, \ldots, n$$

or

$$b_\nu = \bar{a}_\nu e^{i(\alpha + \nu\beta)}, \qquad \nu = 0, 1, \ldots, n$$

where α *and* β *are real.*

The lemma says that there is a function $g_1(z) = \sum b'_\nu z^\nu$ equivalent to $g(z)$, such that $b'_\nu = a_\nu$, $\nu = 0, 1, \ldots, n$. Therefore in the proof we can substitute $g(z)$ by a convenient equivalent function.

It is easily seen that the lemma is true for $n = 1$. We may suppose $a_0 = b_0 = 1$ and a_1 real and positive. We write

$$(f(z))^p = 1 + \sum_{\nu=1}^{\infty} a_\nu^{(p)} z^\nu, \qquad a_\nu^{(1)} = a_\nu$$

$$(g(z))^p = 1 + \sum_{\nu=1}^{\infty} b_\nu^{(p)} z^\nu, \qquad b_\nu^{(1)} = b_\nu$$

Putting

$$A_{m\mu} = \sum_{\mu_i} \frac{\underline{|\mu}}{\underline{|\mu_1}\,\underline{|\mu_2} \cdots \underline{|\mu_{m-\mu+1}}} \cdot a_1^{\mu_1} a_2^{\mu_2} \cdots a_{m-\mu+1}^{\mu_{m-\mu+1}}$$

the summation being over

$$\begin{cases} \mu_i \geq 0, \qquad \mu_1 + \mu_2 + \cdots + \mu_{m-\mu+1} = \mu \\ \mu_1 + 2\mu_2 + \cdots + (m - \mu + 1)\mu_{m-\mu+1} = m \end{cases}$$

we obtain

$$a_m^{(p)} = \sum_{\mu=1}^{m} \binom{p}{\mu} A_{m\mu}.$$

Similarly we write

$$b_m^{(p)} = \sum_{\mu=1}^{m} \binom{p}{\mu} B_{m\mu}.$$

We prove the lemma by induction. The lemma is true for $n = 1$, let it be true for $n - 1$. Then we may suppose

$$b_1 = a_1, \; b_2 = a_2, \ldots, b_{n-1} = a_{n-1}.$$

Then

$$B_{n\mu} = A_{n\mu}, \qquad \mu = 2, 3, \ldots, n$$

The equalities

$$|b_n^{(p)}| = |a_n^{(p)}|, \qquad p = p_1, p_2, \ldots, p_n$$

can be written

$$\left|\binom{p}{1} a_n + \sum_{\mu=2}^{n} \binom{p}{\mu} A_{n\mu}\right| = \left|\binom{p}{1} b_n + \sum_{\mu=2}^{n} \binom{p}{\mu} A_{n\mu}\right|$$

$$p = p_1, p_2, \ldots, p_n$$

or

$$\binom{p}{1}(|a_n|^2 - |b_n|^2) + \sum_{\mu=2}^{n} \binom{p}{\mu} \{(a_n \bar{A}_{n\mu} + \bar{a}_n A_{n\mu}) - (b_n \bar{A}_{n\mu} + \bar{b}_n A_{n\mu})\} = 0$$

$$p = p_1, p_2, \ldots, p_n.$$

Since p_k are unequal, the determinant

$$D = \begin{vmatrix} \binom{p_1}{1}, & \binom{p_1}{2}, & \ldots, & \binom{p_1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{p_n}{1}, & \binom{p_n}{2}, & \ldots, & \binom{p_n}{n} \end{vmatrix}$$

$$= \frac{p_1 \, p_2 \ldots p_n}{\lfloor 2 \, \lfloor 3 \ldots \lfloor n} \prod_{i>k} (p_i - p_k)$$

does not vanish. Therefore

$$|b_n| = |a_n|$$

and

$$a_n \bar{A}_{n\mu} + \bar{a}_n A_{n\mu} = b_n \bar{A}_{n\mu} + \bar{b}_n A_{n\mu} \qquad \mu = 2, 3, \ldots, n.$$

If $A_{n\mu} \neq 0$ these conditions give

$$\arg(a_n \bar{A}_{n\mu}) = \pm \arg(b_n \bar{A}_{n\mu}) \qquad \mu = 2, 3, \ldots, n. \tag{24}$$

Now we shall prove: If all the coefficients $a_1, a_2, \ldots, a_{n-1}$ are real, then we must have $b_n = a_n$ or $b_n = \bar{a}_n$. If at least one of the coefficients $a_2, \ldots, a_{n-1}$ is complex, then we must have $b_n = a_n$. If $a_n = 0$ there is nothing to prove. As $A_{nn} = a_1^n$ is real and positive the condition (24) gives that either $b_n = a_n$ or $b_n = \bar{a}_n$. Then the lemma is proved in the first case. If a_m, $m < n$, is the first complex coefficient, we see that

$$A_{n;n-m+1} = (n - m + 1)\, a_1^{n-m} a_m$$

$$+ \text{(a polynomial of } a_1, a_2, \ldots, a_{m-1})$$

cannot be real. Putting in (24) $\mu = n$ and then $\mu = n - m + 1$ we obtain $b_n = a_n$. This proves the lemma.

Now suppose the conditions of theorem 1 are fullfilled and $f(0)\, f'(0) \neq 0$. Then from the corollary of lemma 1 follows that

$$M_{2p}(f, r) = M_{2p}(g, r), \qquad p = 1, 2, \ldots, n, \ldots$$

in an interval $0 < r < r_1$. Then we have (compare the proof of theorem 3) $|b_n^{(p)}| = |a_n^{(p)}|$ for all integers p and n. Then from lemma 4 the functions $f(z)$ and $g(z)$ are equivalent.

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