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Deterministic and non-deterministic stationary random processes

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1. Let $x(t)$ be a complex-valued random process depending on a real continuous parameter t, which may be regarded as representing time. We will assume that, for every t, the mean value $E\{x(t)\}=0$ and the variance $E\{|x(t)|^2\}$ is finite. Then, in accordance with KHINTCHINE [3], we say that $x(t)$ is a stationary random process if the function

$$r(t)=E\{x(s+t)\overline{x(s)}\}$$

is independent of s. Then $r(t)$ has the properties

$$r(s-t)=E\{x(s)\overline{x(t)}\}$$
$$r(0)=E\{|x(t)|^2\}>0.$$

We shall also assume that $r(t)$ is continuous for $t=0$. Then

$$E\{|x(t+h)-x(t)|^2\}=2r(0)-r(h)-r(-h)\to 0$$

when $h\to 0$, and

$$|r(t)-r(s)|^2=|E\{(x(t)-x(s))\overline{x(0)}\}|^2\leqq E\{|x(t)-x(s)|^2\}E\{|x(0)|^2\}\to 0$$

when $s\to t$, so that $r(t)$ is continuous for every t. The process will then be called a continuous stationary random process (KHINTCHINE [3]).

We shall in this paper study such processes and prove a decomposition theorem, which says that an arbitrary process of this type is the sum of two other processes of the same type, where one is deterministic and the other is completely non-deterministic (Theorem 1), and where the completely non-deterministic part can be expressed in terms of a random spectral function (Theorem 2).

The corresponding decomposition theorem for a stationary process depending on an integral parameter or, in other words, for a stationary sequence, has been stated by WOLD [5] in 1938. It has later been simplified and completed by KOLMOGOROFF [4] using the technique of Hilbert space. In a more systematic way KARHUNEN [2] introduced Hilbert space methods into the theory of probability. Using his results we shall be able to prove our theorems.

2. Consider a continuous stationary process $x(t)$. It may be interpreted as a curve in the Hilbert space $L_2(x)$ which consists of random variables of the type

$$\sum_{\nu=1}^{n} c_\nu x(t_\nu)$$

c_ν being constants, and of random variables which are limits of sequences of such sums in the sense of mean convergence. The scalar product of two random variables x and y is defined as $E\{x\bar{y}\}$ and the norm $\|x\|$ is defined by

$$\|x\|^2 = E\{|x|^2\}.$$

Hence, two random variables in $L_2(x)$ are orthogonal if and only if they are uncorrelated.

That the process is continuous means that the curve is continuous, that is

$$\|x(t) - x(s)\| \to 0$$

when $s \to t$. As a consequence of the continuity we get that $L_2(x)$ is separable. For the set $\{x(r);\ r\ \textit{rational}\}$ is a countable complete set in $L_2(x)$.

If A and B are closed linear subspaces of $L_2(x)$, orthogonal to each other, we denote by $A \oplus B$ the direct sum of A and B, and, if $A \supset B$, we denote by $A \ominus B$ the orthogonal complement of B with respect to A. Further let P_A be the projection of $L_2(x)$ onto A.

Let T_h be the unitary linear transformation defined by

$$T_h x(t) = x(t+h)$$

(KARHUNEN [2], p. 55). The transformations T_h constitute an abelian group: $T_h T_k = T_{h+k}$. The existence of such a unitary transformation is equivalent to the process being stationary.

We denote by $L_2(x;a)$ the closed linear manifold in $L_2(x)$ determined by the set $\{x(t);\ t \leqq a\}$, that is the least closed linear manifold containing all the $x(t)$ when $t \leqq a$. If $z \in L_2(x;a)$ then $T_h z \in L_2(x;a+h)$ and conversely, so that we write

$$T_h L_2(x;a) = L_2(x;a+h). \tag{2.1}$$

From the definition of $L_2(x;a)$ follows

$$L_2(x;a) \subset L_2(x;b) \qquad a < b. \tag{2.2}$$

Since $x(t)$ is continuous, $L_2(x;a)$ is continuous in a, that is

$$\lim_{b \to a+0} L_2(x;b) = \lim_{b \to a-0} L_2(x;b) = L_2(x;a).$$

Then also the projection $P_{L_2(x;a)}$ is continuous in a, that is for every element z, $P_{L_2(x;a)} z$ is continuous in a.

Now let us take a fixed t and an arbitrary a and consider

$$X_t(a) = P_{L_2(x;\,a)}\, x(t)$$

that is the projection of $x(t)$ in $L_2(x;\,a)$. $X_t(a)$ may be considered as that part of $x(t)$ that is determined by the process at the time a. We want to study $\|X_t(a)\|$. This is a continuous function of a, which in consequence of (2.2) is non decreasing and constant for $a \geqq t$. Since

$$T_h X_t(a) = X_{t+h}(a+h) \tag{2.3}$$

we have

$$\|X_t(a)\| = \|X_{t+h}(a+h)\|.$$

Thus it will be sufficient to take $t = 0$. Instead of $X_0(a)$ we then write $X(a)$.

There will be two extreme cases.

1) $\|X(a)\|$ is constant. Then $\|X(a)\| = \|X(0)\| = \|x(0)\|$ for every a so that $X(a) = x(0)$. This may be written

$$x(0) \in L_2(x;\,a) \tag{2.4}$$

for every a. Then also

$$x(h) = T_h\, x(0) \in T_h\, L_2(x;\,a-h) = L_2(x;\,a) \tag{2.5}$$

for every h so that

$$L_2(x) = L_2(x;\,a).$$

Definition. A stationary process for which $L_2(x;\,a) = L_2(x)$ will be called deterministic.

Thus we have proved

Proposition A. If $\|X(a)\|$ is constant, then $x(t)$ is deterministic.

The converse is obvious.

Remark. It would be sufficient to know that (2.4) holds for some negative number a. For then, in consequence of (2.2), (2.5) holds for every $h \leqq 0$, so that $L_2(x;\,0) = L_2(x;\,a)$. Hence

$$L_2(x;\,n\,a) = L_2(x;\,(n-1)\,a) = \cdots = L_2(x;\,0)$$

for every integer n, positive or negative, so that the process is deterministic.

2) $\|X(a)\| \to 0$ when $a \to -\infty$. Put

$$M = \prod_a L_2(x;\,a) = \lim_{a \to -\infty} L_2(x;\,a).$$

Then

$$P_M\, x(0) = 0$$

and, since (2.1) implies $T_h M = M$, we have for every t

$$P_M x(t) = 0$$

so that $M \perp L_2(x)$. But $M \subset L_2(x)$, hence $M = 0$, that is M contains only the zero element.

DEFINITION. A stationary process for which $\prod_a L_2(x; a) = 0$ will be called completely non-deterministic.

Thus we have proved

PROPOSITION B. If $\|X(a)\| \to 0$ when $a \to -\infty$, then the process is completely non-deterministic.

Conversely if the process is completely non-deterministic, then $\|X(a)\| \to 0$, for if $X(a) \to z$, $\|z\| \neq 0$, then $z \in L_2(x; a)$ for every a, and thus $z \in M$, which is a contradiction.

We now shall prove that an arbitrary stationary process is the sum of two uncorrelated components, one deterministic and the other completely non-deterministic, or more precisely

THEOREM 1. $x(t)$ is a stationary process. Then there exist two other stationary processes $y(t)$ and $z(t)$ such that

a) $x(t) = y(t) + z(t)$

b) $y(t) \in L_2(x)$, $\quad z(t) \in L_2(x)$

c) $y(s) \perp z(t)$ for every s and t

d) $y(t)$ is deterministic

e) $z(t)$ is completely non-deterministic.

PROOF. Take

$$M = \prod_a L_2(x; a) \quad \text{and} \quad N = L_2(x) \ominus M.$$

From (2.1), we have $T_h M = M$, and hence, since T_h is unitary, $T_h N = N$. Put

$$y(t) = P_M x(t) \quad \text{and} \quad z(t) = P_N x(t).$$

Then a), b) and c) are satisfied. From

$$y(t+h) + z(t+h) = x(t+h) = T_h x(t) = T_h [y(t) + z(t)] = T_h y(t) + T_h z(t)$$

we obtain

$$y(t+h) = T_h y(t) \quad \text{and} \quad z(t+h) = T_h z(t).$$

Thus $y(t)$ and $z(t)$ are stationary processes. It remains to prove d) and e). From a) and from

$$L_2(y) \subset M \quad \text{and} \quad L_2(z) \subset N$$

we have

$$L_2(x) \subset L_2(y) \oplus L_2(z) \subset M \oplus N = L_2(x)$$

so that

$$L_2(y) = M \quad \text{and} \quad L_2(z) = N. \tag{2.6}$$

When $t \leqq a$ we have

$$y(t) \in M \subset L_2(x;\, a) \quad \text{and} \quad z(t) = x(t) - y(t) \in L_2(x;\, a)$$

and hence

$$L_2(y;\, a) \subset L_2(x;\, a) \quad \text{and} \quad L_2(z;\, a) \subset L_2(x;\, a).$$

But from a)

$$L_2(x;\, a) \subset L_2(y;\, a) \oplus L_2(z;\, a)$$

so that

$$L_2(x;\, a) = L_2(y;\, a) \oplus L_2(z;\, a)$$

or, with the aid of (2.6),

$$L_2(y;\, a) = M\, L_2(x;\, a) \quad \text{and} \quad L_2(z;\, a) = N\, L_2(x;\, a).$$

Now we get

$$L_2(y;\, a) = M\, L_2(x;\, a) = M = L_2(y).$$

Hence $y(t)$ is deterministic. Further

$$M_z = \prod_a L_2(z;\, a) = \prod_a N\, L_2(x;\, a) = N \prod_a L_2(x;\, a) = N\, M = 0.$$

Hence $z(t)$ is completely non-deterministic. This completes the proof of the theorem.

3. Now we will study the completely non-deterministic processes. Therefore we assume that $y(t) = 0$, so that $x(t) = z(t)$ is a completely non-deterministic process.

For every pair (a, b) of real numbers, $a < b$, we construct

$$L_2(x;\, a,\, b) = L_2(x;\, b) \ominus L_2(x;\, a)$$

and

$$x(a,\, b) = P_{L_2(x;\, a,\, b)}\, x(b).$$

Then $x(a, b) \perp L_2(x;\, a)$, and $x(a, b)$ may be interpreted as that part of $x(b)$ that is not determined of the process at the time a.

From the Remark to Proposition A, we conclude

$$x(b) \notin L_2(x;\, a)$$

or

$$x(a,\, b) \neq 0. \tag{3.1}$$

Hence $L_2(x; a, b)$ always contains elements with positive norm. It is clear that

$$T_h x(a, b) = x(a + h, b + h).$$

The element $x(a, b)$ is continuous in (a, b), for take, for instance, $a_1 < a$ and $b_1 > b$, then

$$\begin{aligned} x(a_1, b_1) - x(a, b) &= x(a_1, b_1) - x(a_1, b) + x(a_1, b) - x(a, b) \\ &= P_{L_2(x; a_1, b_1)}[x(b_1) - x(b)] + P_{L_2(x; a_1, a)}\, x(b). \end{aligned}$$

Here the last two terms tend to zero when $a_1 \to a$ and $b_1 \to b$ because

$$\|P_{L_2(x; a_1, b_1)}[x(b_1) - x(b)]\| \leqq \|x(b_1) - x(b)\| \to 0$$

independently of a_1, when $b_1 \to b$ and

$$P_{L_2(x; a_1, a)}\, x(b) \to 0$$

when $a_1 \to a$.

We are now going to define a random spectral function $Z(S)$ (KARHUNEN [2], p. 36). This means in this case, that for every measurable set S on the t-axis, with finite measure $m(S)$, there shall be a random variable $Z(S) \in L_2(x)$ satisfying

1) If S_1 and S_2 are disjoint sets

$$Z(S_1 + S_2) = Z(S_1) + Z(S_2).$$

2) If S_1 and S_2 are disjoint sets

$$Z(S_1) \perp Z(S_2).$$

3) $\|Z(S)\|^2 = m(S)$.

2) and 3) may be combined in

$$E\{Z(S_1)\, Z(S_2)\} = m(S_1 S_2)$$

for arbitrary S_1 and S_2.

If $Z(S)$ is defined and satisfies 1), 2) and 3), when S is an interval, then there is a unique extension to all measurable sets. Thus we have to define $Z(I_a^b)$ for every interval $I_a^b = (a, b)$.

Let u be a fixed positive number and take a $z \in L_2(x; 0, u)$. For every interval I_a^b we take

$$Z(I_a^b) = P_{L_2(x; a, b)} \int_A^B T_h z\, dh \tag{3.2}$$

where $A < a - u$ and $B > b$. (The integral is defined as a Riemann integral, CRAMÉR [1], p. 219.) $Z(I_a^b)$ is independent of A and B, since the variations of the integral, when A and B vary, are orthogonal to $L_2(x; a, b)$.

When $a < b < c$ we have

$$Z(I_a^b) + Z(I_b^c) = Z(I_a^c) \tag{3.3}$$

$$Z(I_a^b) \perp Z(I_b^c) \tag{3.4}$$

and for arbitrary h

$$T_h Z(I_a^b) = Z(I_{a+h}^{b+h}). \tag{3.5}$$

From these three properties we get

$$\|Z(I_a^b)\|^2 = \tau(b-a)$$

where $\tau \geqq 0$. τ depends on z, and we shall prove

PROPOSITION C. One can find a z, such that $\tau = 1$.

PROOF. It will be sufficient to find z so that $\tau > 0$, for then to $z/\sqrt{\tau}$ corresponds $\tau = 1$.

Suppose the contrary so that for every $z \in L_2(x; 0, u)$

$$\|Z(I_a^b)\|^2 = 0.$$

Then for every $z' \in L_2(x)$

$$E\{Z(I_a^b)\overline{z'}\} = 0. \tag{3.6}$$

We take

$$z = y(s_1, t_1) \quad \text{and} \quad z' = y(s_2, t_2)$$

where

$$0 \leqq s_1 < t_1 \leqq u \quad \text{and} \quad 0 \leqq s_2 < t_2 \leqq u \tag{3.7}$$

and where we have written

$$y(s, t) = x(s, u) - x(t, u) = P_{L_2(x; s, t)}\, x(u).$$

Thus z and z' are certain projections of $x(u)$, and they satisfy (3.6) for arbitrary (s_1, t_1) and (s_2, t_2) satisfying (3.7). But we know from (3.1) that

$$y(0, u) = x(0, u) = P_{L_2(x; 0, u)}\, x(u) \neq 0$$

and we shall show that this leads to a contradiction.

First we have $z \in L_2(x; 0, u)$ and $z' \in L_2(x; 0, u)$. We then obtain

$$\begin{aligned} 0 = E\{Z(I_0^u)\overline{z'}\} &= E\left\{P_{L_2(x; 0, u)} \int_{-u}^{u} T_h z\, dh\, \overline{z'}\right\} \\ &= E\left\{\int_{-u}^{u} T_h z\, dh\, \overline{z'}\right\} \\ &= \int_{-u}^{u} E\{T_h z\, \overline{z'}\}\, dh \end{aligned}$$

that is

$$\int_{-u}^{u} E\{T_h y(s_1, t_1)\overline{y(s_2, t_2)}\} = 0. \tag{3.8}$$

Let δ be a positive number $< \frac{1}{2}u$, and consider

$$E\{T_h y(0, u)\,\overline{y(\delta, u-\delta)}\}.$$

This expression is a continuous function of h, which when $h = 0$ takes the value $\|y(\delta, u-\delta)\|^2$, which is continuous in δ, and when $\delta \to 0$ we obtain

$$\|y(\delta, u-\delta)\|^2 \to \|y(0, u)\|^2 = \|x(0, u)\|^2 > 0.$$

Then it is possible to choose δ, such that $\|y(\delta, u-\delta)\|^2 > 0$ and $\gamma < \delta$ such that

$$L = \int_{-\gamma}^{\gamma} E\{T_h y(0, u)\,\overline{y(\delta, u-\delta)}\}\,dh \neq 0.$$

If we make a subdivision $t_0 = \delta, t_1, t_2, \ldots t_n = u - \delta$ of the interval $(\delta, u-\delta)$ by a finite number of points, then we have

$$\begin{aligned} L &= \sum_{i=1}^{n} \int_{-\gamma}^{\gamma} E\{T_h y(0, u)\,\overline{y(t_{i-1}, t_i)}\}\,dh \\ &= \sum_{i=1}^{n} \int_{-\gamma}^{\gamma} E\{T_h y(t_{i-1}-\gamma, t_i+\gamma)\,\overline{y(t_{i-1}, t_i)}\}\,dh. \end{aligned}$$

Let us compare this with

$$\begin{aligned} M &= \sum_{i=1}^{n} \int_{-\gamma-(t_i-t_{i-1})}^{\gamma+(t_i-t_{i-1})} E\{T_h y(t_{i-1}-\gamma, t_i+\gamma)\,\overline{y(t_{i-1}, t_i)}\}\,dh \\ &= \sum_{i=1}^{n} \int_{-u}^{u} E\{T_h y(t_{i-1}-\gamma, t_i+\gamma)\,\overline{y(t_{i-1}, t_i)}\}\,dh \\ &= 0 \end{aligned}$$

as a consequence of (3.8).

M and L are independent of the division of $(\delta, u-\delta)$. Since $L \neq 0$ and $M = 0$, if we can show $M = L$, we have the contradiction.

$$M - L = \sum_{i=1}^{n} \left(\int_{-\gamma-(t_i-t_{i-1})}^{-\gamma} + \int_{\gamma}^{\gamma+(t_i-t_{i-1})} \right) E\{T_h y(t_{i-1}-\gamma, t_i+\gamma)\,\overline{y(t_{i-1}, t_i)}\}\,dh.$$

$$\begin{aligned} |M - L| &\leq 2\sum_{i=1}^{n} (t_i - t_{i-1})\|y(0, u)\|\cdot\|y(t_{i-1}, t_i)\| \\ &\leq 2u\|y(0, u)\|\cdot\sup\|y(t_{i-1}, t_i)\|. \end{aligned} \tag{3.9}$$

But

$$y(t_{i-1}, t_i) + x(t_i, u) = x(t_{i-1}, u)$$

and

$$y(t_{i-1}, t_i) \perp x(t_i, u).$$

Hence

$$\|y(t_{i-1}, t_i)\|^2 = \|x(t_{i-1}, u)\|^2 - \|x(t_i, u)\|^2.$$

Since $\|x(t, u)\|^2$ is a continuous function of t, we can make $\sup \|y(t_{i-1}, t_i)\|$ as small as we please, by making the division fine enough. Hence (3.9) yields

$$|M - L| = 0$$

and we have got the contradiction.

Then we can take some z for which $\tau = 1$. The corresponding $Z(I_a^b)$ defined by (3.2) will be the sought-for random spectral function.

4. Let $L_2(Z)$ be the closed linear manifold determined by the set $\{Z(I_a^b)\}$. That $L_2(Z) = L_2(x)$ will be shown in Proposition D.

Let us write

$$\begin{aligned} Z(a) &= -Z(I_a^0) \qquad && a < 0 \\ Z(0) &= 0 \\ Z(a) &= Z(I_0^a) && a > 0. \end{aligned}$$

Then

$$Z(I_a^b) = Z(b) - Z(a).$$

This may be written

$$Z(I_a^b) = \int_a^b dZ(u).$$

Thus we have defined

$$\zeta = \int_{-\infty}^{\infty} g(u)\, dZ(u) \tag{4.1}$$

when $g(u) = 1$ in a finite interval and $= 0$ elsewhere. We will define it for every complex-valued function $g(u)$ such that

$$\int_{-\infty}^{\infty} |g(u)|^2\, du$$

is finite. If

$$\begin{aligned} g(u) &= c_\nu \quad \text{in} \quad (a_\nu, b_\nu) \\ &= 0 \ \text{elsewhere} \end{aligned}$$

where (a_ν, b_ν) are a finite number of finite intervals, we define

$$\int_{-\infty}^{\infty} g(u)\, dZ(u) = \Sigma c_\nu [Z(b_\nu) - Z(a_\nu)].$$

In the general case the integral is defined as the limit of a sequence of integrals of functions $g_n(u)$, such that

$$\int_{-\infty}^{\infty} |g(u) - g_n(u)|^2 \, du \to 0$$

that is

$$g(u) = \underset{n\to\infty}{\text{l. i. m.}}\; g_n(u)$$

where $g_n(u)$ are functions taking a constant value in each of a finite number of finite intervals. For a complete discussion of this kind of integral see KARHUNEN [2], p. 37.

We have

$$E\left\{ \int_{-\infty}^{\infty} g_1(u)\, dZ(u) \overline{\int_{-\infty}^{\infty} g_2(u)\, dZ(u)} \right\} = \int_{-\infty}^{\infty} g_1(u)\, \overline{g_2(u)}\, du \tag{4.2}$$

and hence

$$\left\| \int_{-\infty}^{\infty} g(u)\, dZ(u) \right\|^2 = \int_{-\infty}^{\infty} |g(u)|^2\, du.$$

It is clear that ζ in (4.1) is in $L_2(Z)$. Conversely, for every $\zeta \in L_2(Z)$, there is a function $g(u)$ such that (4.1) holds. For this is obvious if $\zeta = Z(I_a^b)$ and then also if $\zeta = \Sigma c_\nu Z(I_{a_\nu}^{b_\nu})$. To prove the general case, we only have to take a sequence ζ_n, $\zeta_n \to \zeta$, such that the ζ_n are sums of this type. Then the corresponding functions $g_n(u)$ converge in the mean to a function $g(u)$. And such a sequence always exists since $\zeta \in L_2(Z)$. The function $g(u)$ is uniquely determined almost everywhere, for suppose

$$\int_{-\infty}^{\infty} g(u)\, dZ(u) = \int_{-\infty}^{\infty} g_1(u)\, dZ(u).$$

Then

$$0 = \left\| \int_{-\infty}^{\infty} [g(u) - g_1(u)]\, dZ(u) \right\|^2 = \int_{-\infty}^{\infty} |g(u) - g_1(u)|^2\, du \tag{4.3}$$

so that $g(u) = g_1(u)$ almost everywhere.

Now put

$$x_1(t) = P_{L_2(Z)}\, x(t).$$

Then $T_h x_1(t) = x_1(t+h)$, and hence $x_1(t)$ is a stationary process. For every t

$$L_2(Z) = L_2[Z(I_a^b);\quad a < b \leqq t] \oplus L_2[Z(I_a^b);\quad t \leqq a < b].$$

From the definition of $Z(I_a^b)$ we conclude

$$x(t) \perp L_2[Z(I_a^b);\quad t \leqq a < b]$$

$$Z(I_a^b) \in L_2(x;\, a, b) \subset L_2(x;\, b).$$

Hence

$$x_1(t) \in L_2[Z(I_a^b);\ a < b \leqq t] \subset L_2(x;\, t) \tag{4.4}$$

and hence $L_2(x_1;\, a) \subset L_2(x;\, a)$, so that

$$\prod_a L_2(x_1;\, a) \subset \prod_a L_2(x;\, a) = 0$$

that is, $x_1(t)$ is completely non-deterministic.

$\|x_1(t)\|$ is independent of t. We must have

$$\|x_1(t)\| \neq 0. \tag{4.5}$$

For if $x_1(t) = 0$ then $x(t) \perp L_2(Z)$, and hence $L_2(x) \perp L_2(Z)$. But as $L_2(Z) \subset L_2(x)$ and contains elements with positive norm, this is a contradiction.

Proposition D. $L_2(Z) = L_2(x)$, or what is equivalent, $x_1(t) = x(t)$.

Proof. Let

$$y(t) = x(t) - x_1(t).$$

Then

$$T_h\, y(t) = y(t+h)$$

so that $y(t)$ is a stationary process. From (4.4) we have

$$y(t) \in L_2(x;\, t).$$

Hence

$$L_2(y;\, a) \subset L_2(x;\, a) \tag{4.6}$$

so that $y(t)$ is also completely non-deterministic. From

$$y(t) = x(t) - x_1(t) = P_{L_2(x) \ominus L_2(Z)}\, x(t)$$

we conclude

$$y(t) \perp L_2(Z). \tag{4.7}$$

We now have proved that

$$x(t) = x_1(t) + y(t)$$

is a decomposition of $x(t)$ into two orthogonal completely non-deterministic processes, where $x_1(t) \neq 0$ (4.5). We shall prove that then we have $y(t) = 0$.

Suppose the contrary. Then we construct $y_1(t) \neq 0$ from $y(t)$ in the same manner as $x_1(t)$ from $x(t)$:

$$y(t) = y_1(t) + z(t).$$

Here $z(t) \perp y_1(t)$ may be $= 0$ or $\neq 0$.

$$x(t) = x_1(t) + y_1(t) + z(t).$$

Now $x_1(0) \in L_2(Z)$ so that

$$x_1(0) = \int_{-\infty}^{\infty} g(u)\, d\, Z(u)$$

for some function $g(u)$, for which

(4.8) $$\int_{-\infty}^{\infty} |g(u)|^2 d\, u \text{ is finite.}$$

From (4.4) we conclude that

(4.9) $$g(u) = 0 \text{ for almost every } u > 0,$$

that is

$$x_1(0) = \int_{-\infty}^{0} g(u)\, d\, Z(u).$$

Then we get from (3.5)

(4.10) $$x_1(t) = \int_{-\infty}^{t} g(u-t)\, d\, Z(u).$$

Similarly there is a random spectral function $Z'(I_a^b) \in L_2(y)$ and a complex-valued function $g'(u)$, such that

$$y_1(t) = \int_{-\infty}^{t} g'(u-t)\, d\, Z'(u).$$

From (4.7) and from

$$L_2(Z') \subset L_2(y)$$

it follows that

$$L_2(Z') \perp L_2(Z).$$

We also have

$$z(t) \in L_2(y) \perp L_2(Z)$$

and analogously to (4.7)

$$z(t) \perp L_2(Z').$$

From (4.6) we conclude

$$Z'(I_a^b) \subset L_2(y;\, b) \subset L_2(x;\, b)$$

so that we have

(4.11) $$L_2[Z'(I_a^b);\; a < b \leqq t] \subset L_2(x;\, t).$$

Now take the element

$$\eta = \int_{s}^{0} \overline{g'(s-u)}\, d\, Z(u) - \int_{s}^{0} \overline{g(s-u)}\, d\, Z'(u)$$

where s is some negative number. We assert that for a convenient s

a) $\|\eta\| \neq 0$

b) $\eta \in L_2(x; 0)$

c) $\eta \perp x(t), \quad t \leqq 0.$

In fact

$$\|\eta\|^2 = \int_s^0 |g'(s-u)|^2 du + \int_s^0 |g(s-u)|^2 du > 0$$

for some s since

$$\lim_{s \to -\infty} \int_s^0 |g(s-u)|^2 du = \int_{-\infty}^0 |g(u)|^2 du = \|x_1(0)\|^2 > 0.$$

Thus a) holds. b) is an immediate consequence of (4.4) and (4.11), and to prove c) we write

$$x(t) = \int_{-\infty}^t g(u-t)\, dZ(u) + \int_{-\infty}^t g'(u-t)\, dZ'(u) + z(t).$$

Then c) is trivial when $t \leqq s$, and when $s < t \leqq 0$, it follows from

$$E\{x(t)\,\bar{\eta}\} = \int_s^t g(u-t)\, g'(s-u)\, du - \int_s^t g'(u-t)\, g(s-u)\, du = 0.$$

But it is a contradiction that a), b), and c) all hold. Hence $y(t) = 0$. This completes the proof of Proposition D.

5. We denote by $L^2(a, b)$ the class of complex-valued functions $f(u)$ defined for $a \leqq u \leqq b$, for which

$$\int_a^b |f(u)|^2 du$$

is finite. Let us make the convention that, for every $f(u) \in L^2(a, b)$, we write $f(u) = 0$ when $u < a$ or $u > b$. Then

$$f(u) \in L^2(-\infty, \infty) \text{ and } L^2(a, b) \subset L^2(-\infty, \infty).$$

Take the function $g(u)$ defined in the previous section. (4.8) and (4.9) yield that

$$g(u) \in L^2(-\infty, 0). \tag{5.1}$$

Hence $g(u-t) \in L^2(-\infty, 0)$ for every $t \leqq 0$. Denote by $L_2\{g(u-t);\ t \leqq 0\}$ the closed linear manifold in $L^2(-\infty, 0)$ determined by these elements.

Now we may sum up the results in

THEOREM 2. For every completely non-deterministic stationary process $x(t)$ we have an integral representation

$$x(t) = \int_{-\infty}^{t} g(u-t)\, dZ(u) \tag{5.2}$$

where

a) $Z(I_a^b) = Z(b) - Z(a)$ is a random spectral function satisfying

$$T_h Z(I_a^b) = Z(I_{a+h}^{b+h}) \tag{5.3}$$

b) $g(u) \in L^2(-\infty, 0)$

c) $L_2\{g(u-t);\ t \leqq 0\} = L^2(-\infty, 0)$.

If

$$x(t) = \int_{-\infty}^{t} g_1(u-t)\, dZ_1(u) \tag{5.4}$$

is another representation, $g_1(u)$ and $Z_1(I_a^b)$ satisfying a), b), and c), then there is a complex number ω, $|\omega| = 1$, such that

$$g_1(u) = \bar{\omega}\, g(u) \text{ for almost every } a$$

$$Z_1(I_a^b) = \omega\, Z(I_a^b),$$

and conversely, for every ω, $|\omega| = 1$, $g_1(u)$ and $Z_1(I_a^b)$ determined by these formulas satisfy a), b), c), and (5.4).

PROOF. The existence of the integral representation is an immediate consequence of Proposition D and (4.10). a) is (3.5) and b) is (5.1). We obtain from (5.2)

$$L_2(x;\ 0) \subset L_2[Z(I_a^b);\ a < b \leqq 0].$$

But (4.4) yields

$$L_2[Z(I_a^b);\ a < b \leqq 0] \subset L_2(x;\ 0)$$

so that

$$L_2(x;\ 0) = L_2[Z(I_a^b);\ a < b \leqq 0]. \tag{5.5}$$

Now let us prove c). Suppose that c) is false. Then there is a function $f(u) \in L^2(-\infty, 0)$, such that

$$\int_{-\infty}^{0} |f(u)|^2\, du > 0$$

$$\int_{-\infty}^{0} g(u-t)\, \overline{f(u)}\, du = 0 \quad t \leqq 0. \tag{5.6}$$

Take the corresponding element in $L_2[Z(I_a^b);\ a < b \leqq 0]$

$$\zeta = \int_{-\infty}^{0} f(u)\, d\, Z(u).$$

Then

$$\|\zeta\|^2 = \int_{-\infty}^{0} |f(u)|^2\, d\, u > 0$$

and from (5.6)

$$\zeta \perp L_2(x;\ 0) \tag{5.7}$$

in contradiction to (5.5).

Conversely c) implies (5.5). For the falseness of (5.5) yields the existence of a $\zeta \neq 0$ satisfying (5.7), and then we obtain from (5.6) the falseness of c).

From (5.5) we get

$$L_2(x;\ a, b) = L_2[Z(I_c^d);\ a \leqq c < d \leqq b]. \tag{5.8}$$

Now let $g_1(u)$ and $Z_1(I_a^b)$ satisfy the conditions in the theorem. Then $Z_1(I_a^b)$ satisfies (5.5) and then (5.8). Hence we obtain

$$L_2[Z(I_c^d);\ a \leqq c < d \leqq b] = L_2[Z_1(I_c^d);\ a \leqq c < d \leqq b].$$

In particular for every finite interval (a, b) there is a function $f(u) \in L^2(a, b)$, such that

$$Z_1(I_a^b) = \int_a^b f(u)\, d\, Z(u). \tag{5.9}$$

By virtue of (4.3) $f(u)$ is uniquely determined almost everywhere. This function $f(u)$, defined by (5.9) for the interval (a, b), will satisfy the same relation for every sub-interval. For since $Z(I_a^b)$ and $Z_1(I_a^b)$ satisfy (3.3), we have $(a < c < b)$

$$Z_1(I_a^c) + Z_1(I_c^b) = \int_a^c f(u)\, d\, Z(u) + \int_c^b f(u)\, d\, Z(u)$$

and from (3.4) we then get

$$Z_1(I_a^c) = \int_a^c f(u)\, d\, Z(u) \quad \text{and} \quad Z_1(I_c^b) = \int_c^b f(u)\, d\, Z(u). \tag{5.10}$$

Conversely, if $f(u)$ satisfies (5.10) then also (5.9) holds for this function. Therefore, if we define a function $f(u)$ for every real u, such that (5.9) holds for every interval of the type $(n, n+1)$ where n is an arbitrary integer, then this function satisfies (5.9) for every interval (a, b).

Now we shall prove that this function $f(u)$ is constant almost everywhere. Since $Z(I_a^b)$ and $Z_1(I_a^b)$ satisfy (5.3), we have from (5.9)

$$Z_1(I_a^b) = T_h\, Z_1(I_{a-h}^{b-h}) = T_h \int_{a-h}^{b-h} f(u)\, d\, Z(u) = \int_a^b f(u-h)\, d\, Z(u).$$

Hence for every h

$$f(u) = f(u-h)$$

for almost every u. Then it is a simple consequence that $f(u)$ is constant almost everywhere, that is there exists a number ω such that

$$f(u) = \omega$$

almost everywhere. Then (5.9) yields

$$Z_1(I_a^b) = \omega Z(I_a^b)$$

and from $\|Z_1(I_a^b)\|^2 = \|Z(I_a^b)\|^2$ we conclude that $|\omega| = 1$. Hence

$$x(t) = \int_{-\infty}^{t} g(u-t)\, d\, Z(u) = \int_{-\infty}^{t} \bar{\omega}\, g(u-t)\, d\, Z_1(u)$$

and from (5.4) we obtain

$$g_1(u-t) = \bar{\omega}\, g(u-t)$$

for almost every u.

Finally, if $|\omega| = 1$, then it is evident that $g_1(u)$ and $Z_1(I_a^b)$ satisfy a), b), c), and (5.4). This completes the proof of Theorem 2.

6. We may express Theorem 2 in another way. We obtain from (4.2)

$$r(s-t) = E\{x(s)\overline{x(t)}\} = \int_{-\infty}^{\infty} g(u-s)\overline{g(u-t)}\, d\, u. \tag{6.1}$$

The integral formula in Theorem 2 then establishes an isomorphism between $L_2(x)$ and $L^2(-\infty, \infty)$, where the element $x(t) \in L_2(x)$ corresponds to $g(u-t) \in \in L^2(-\infty, \infty)$. $\{x(t)\}$ is a complete set in $L_2(x)$ on account of the definition of $L_2(x)$, and $\{g(u-t);\, t \text{ arbitrary}\}$ is a complete set in $L_2(-\infty, \infty)$, which follows from c) in Theorem 2. We also conclude from c) that $L_2(x; a)$ corresponds to $L^2(-\infty, a)$ and hence that $L_2(x; a, b)$ corresponds to $L^2(a, b)$.

To prove Theorem 2 we have constructed $Z(I_a^b)$ and then obtained $g(u)$ with the properties b) and c). Now we see, that if we have found $g(u)$ satisfying b), c) and (6.1), then we are able to establish the isomorphism and denote by $Z(I_a^b)$ the element in $L_2(x)$ that corresponds to the function

$$\begin{aligned} f(u) &= 1 \quad a < u < b \\ &= 0 \quad \text{elsewhere} \end{aligned}$$

in $L^2(-\infty, \infty)$.

Remark. This construction of $Z(I_a^b)$ is possible even if $g(u)$ does not satisfy c). Then the following example shows that the uniqueness pronounced at the end of Theorem 2, is false if we omit c). For take

$$g(u) = 0 \quad u \geqq 0 \qquad \text{and} \qquad g_1(u) = 0 \quad u \geqq 0$$
$$\quad = e^{u} \quad u < 0 \qquad\qquad\qquad = e^{2[u]-u+1} \quad u < 0.$$

Then

$$\int_{-\infty}^{\infty} g(u-s)\, g(u-t)\, d\,u = \int_{-\infty}^{\infty} g_1(u-s)\, g_1(u-t)\, d\,u.$$

Here c) holds for $g(u)$ but is false for $g_1(u)$.

7. A simple consequence of Theorem 2 is

THEOREM 3. Let $y(t) \neq 0$ and $z(t) \neq 0$ be two uncorrelated completely non-deterministic processes and $x(t) = y(t) + z(t)$. Then

$$L_2(x) \neq L_2(y) \oplus L_2(z).$$

PROOF. From Theorem 2 we have

$$y(t) = \int_{-\infty}^{t} g(u-t)\, d\,Z(u) \qquad \text{and} \qquad z(t) = \int_{-\infty}^{t} g'(u-t)\, d\,Z'(u).$$

Put

$$\eta = \int_{0}^{\infty} \overline{g'(-u)}\, d\,Z(u) - \int_{0}^{\infty} \overline{g(-u)}\, d\,Z'(u).$$

Obviously $E\{x(t)\,\bar{\eta}\} = 0$ for every t, so that $\eta \perp L_2(x)$. Since $\eta > 0$ this proves the theorem.

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