ARKIV FÖR MATEMATIK Band 1 nr 22

Communicated 26 April 1950 by MARCEL RIESZ and ARNE BEURLING

On null-sets for continuous analytic functions

By LENNART CARLESON

1. Let E be a compact set with a connected complement Ω. If Γ is the class of functions $f(z)$ which are analytic in Ω and have a certain property P, then a set E is said to be a "null-set" with respect to Γ, if this class consists entirely of constants. An investigation of these null-sets for certain properties P has recently been published by AHLFORS and BEURLING [1]. For example they consider the PAINLEVÉ problem where P is boundedness. In this paper P is a continuity property, and our aim is to give metrical conditions on the corresponding null-sets.

We denote by $L_\alpha(E)$ and $C_\alpha(E)$ HAUSDORFF measure and capacity of order α, $0 < \alpha < 2$; for their definitions we refer to [2] A function $f(z)$ (not necessarily single valued) is said to belong to Lip α, $0 < \alpha < 1$, if for every circular arc γ of length $|\gamma| < 1$ and for every branch of $f(z)$,

$$\left|\int_\gamma f'(z)\, dz\right| \leq M\, |\gamma|^\alpha,$$

where M is a constant independent of γ.

2. Our first theorem is concerned with multiple valued functions $f(z)$.

Theorem: *Let Γ be the class of functions belonging to* Lip α *and having single valued real part. Then E is a null-set if and only if*

$$L_\alpha(E) = 0.$$

If $L_\alpha(E) > 0$, then there exists a real, completely additive set function μ vanishing outside E such that

(a) $\mu(E) = 0$,

(b) $\int_E |d\mu| = 1$,

(c) $|\mu(C)| \leq M r^\alpha$ for every circle C of radius r.

The function

$$f(z) = \int_E \log(z - \zeta)\, d\mu(\zeta)$$

is non-constant and belongs to Γ; the continuity of $\mathbf{Re}\, f(z)$ is proved in [2], page 16, and the continuity of $\mathbf{Im}\, f(z)$ can be proved similarly.

If, on the other hand, $L_\alpha(E) = 0$, suppose that $f(z) = u(z) + i\,v(z)$ belongs to Γ. We cover E by a family of disjoint circles $\{C_\nu\}$ with radii $\{r_\nu\}$ such that

$$\sum r_\nu^\alpha \leq \varepsilon.$$

This is always possible since $L_\alpha(E) = 0$ and $\alpha < 1$. Let γ by any closed, smooth curve, not meeting any circle C_ν. Then

$$\int_\gamma \frac{\partial u}{\partial n}\, d s = \sum \int_{C_\nu} \frac{\partial u}{\partial n}\, d s = \sum \int_{C_\nu} d v = \sum O(r_\nu^\alpha),$$

where the summation runs over those ν which correspond to circles interior to γ. Hence, letting $\varepsilon \to 0$, we obtain

$$\int_\gamma \frac{\partial u}{\partial n}\, d s = 0.$$

$f(z)$ is thus single valued and bounded whence (see [1], page 121)

$$f(z) \equiv \text{constant},$$

and the theorem is proved.

3. If we suppose furthermore that the imaginary part of $f(z)$ is single valued, the dimension of the null-sets is increased by 1.

Theorem: *Let Γ_α be the class of single valued functions belonging to* Lip α. *Then E is a null-set if $L_{1+\alpha}(E) = 0$. If $C_{1+\alpha}(E) > 0$, E is no null-set.*

The second part of the theorem is proved in [2]. We suppose that $L_{1+\alpha}(E) = 0$ and $f(z) \in \text{Lip}\ \alpha$. Let $\{R_\nu\}$ as in the sequel denote a covering of E by a finite number of squares with sides $\{\delta_\nu\}$ such that R_ν and R_μ, $\nu \neq \mu$, have parallel sides and no interior points in common. We also suppose that the set on the boundary of R_ν which belongs to E has measure zero. We here suppose that

$$\sum \delta_\nu^{1+\alpha} \leq \varepsilon.$$

If

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots,$$

it is sufficient to prove that

$$(1) \qquad c_1 = 0,$$

as the argument then can be repeated on the function $z(f(z) - c_0)$ etc. By CAUCHY's formula we have

$$c_1 = \frac{1}{2\pi i} \sum \int\limits_{\overset{R_\nu}{\circlearrowleft}} f(z)\, dz.$$

Let $z_\nu \in R_\nu$. Then

$$|c_1| \leq \frac{1}{2\pi} \sum \int\limits_{\dot{R}_\nu} |f(z) - f(z_\nu)|\,|dz| = \sum O(\delta_\nu^{1+\alpha}).$$

Thus (1) holds and the theorem is established.

4. The limit case $\alpha = 1$ is particularly interesting and gives rise to functions with bounded derivatives. In order to characterize the null-sets of this class from a metrical point of view, we need to devide the family of sets with *positive* Lebesgue measure into classes of null-sets. It is remarkable that the generalized capacities can also serve for this purpose. We shall for the sake of simplicity only consider sets E interior to the closed unit circle ω.

We define the classes N_α, $0 \leq \alpha \leq 2$, of null-sets: E belongs to N_α if

$$C_\alpha(\omega - E) = C_\alpha(\omega), \qquad 0 \leq \alpha < 2,$$

$$m E = 0, \qquad \alpha = 2,$$

where C_0 denotes the logarithmic capacity.[1] Every set E belongs to N_0, since the mass of the equilibrium distribution is situated on the boundary of ω. Furthermore, every set in N_2 belongs to N_α, $\alpha < 2$. We shall actually prove that the set N_α increases as α decreases:

$$N_\alpha < N_\beta, \quad \text{if } \alpha > \beta. \tag{2}$$

Every set E thus defines a cut α' in the sense that $E \in N_\alpha$ if $\alpha < \alpha'$, but $E \notin N_\alpha$ if $\alpha > \alpha'$.

To prove (2), let $E \in N_\alpha$ and O_n be an open set consisting of n circles C_i^n with radii $\leq \delta_n$, such that $\omega > O_n > E$, $\lim\limits_{n=\infty} O_n = E$, and put $F_n = \omega - O_n$. Let μ_n and μ be the equilibrium distributions corresponding to F_n and ω and the kernel $r^{-\alpha}$. If ν is the distribution corresponding to ω and $r^{-\beta}$, we define the completely additive set functions ν_n as follows:

$$\nu_n(e) = \begin{cases} \mu_m(e) \cdot \dfrac{\nu(C_i^n)}{\mu_m(C_i^n)}, & e < C_i^n, \quad i = 1, 2, \ldots n, \\ \nu(e), & e \text{ outside all } C_i^n. \end{cases}$$

Since

$$\lim_{m=\infty} \mu_m(e) = \mu(e),$$

we can choose $m > n$ so that

$$\nu(C_i^n) \leq K \mu_m(C_i^n), \qquad i = 1, 2, \ldots, n,$$

[1] A condition of this kind is given in [1].

where K is a constant independent of n. Furthermore we have

$$\nu_n(\omega - E) = 1.$$

If u_n and u are the β-potentials generated by ν_n and ν, and $\varepsilon > 0$, we find

$$u(z) - u_n(z) = \int_{|z-\zeta| \le \varepsilon} \left\{\frac{d\nu(\zeta)}{|z-\zeta|^\beta} - \frac{d\nu_n(\zeta)}{|z-\zeta|^\beta}\right\} + \int_{|z-\zeta| > \varepsilon} \frac{d(\nu - \nu_n)}{|z-\zeta|^\beta},$$

whence

$$|u(z) - u_n(z)| \le \delta(\varepsilon) + K\varepsilon^{\alpha-\beta} \int_\omega \frac{d\mu_n(\zeta)}{|z-\zeta|^\alpha} + \varepsilon^{-\beta} - (\varepsilon + \delta_n)^{-\beta},$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We conclude

$$\limsup_{n=\infty}\ \sup_z |u(z) - u_n(z)| = 0$$

and

$$C_\beta(\omega - E) = C_\beta(\omega),$$

which was our assertion.

5. We shall now prove that the cut defined by the null-sets for the class Γ_1 of functions with bounded derivatives is $\alpha' = 2$. More precisely we have the following

Theorem: *A sufficient condition that E is a null-set for the class Γ_1 is that $E \in N_2$. A necessary condition is that $E \in N_p$ for every $p < 2$.*

Suppose that $mE = 0$ and let $\{R_\nu\}$ be a covering as above. We have, $z_\nu \in R_\nu$,

$$\int_{R_\nu \circlearrowleft} f(z)\,dz = -\int_{R_\nu \circlearrowleft} (z - z_\nu) f'(z)\,dz = O(\delta_\nu^2).$$

Thus

$$c_1 = \frac{1}{2\pi i} \sum \int_{R_\nu \circlearrowleft} f(z)\,dz = \sum O(\delta_\nu^2)$$

for all coverings of this kind, whence

$$c_1 = 0,$$

and the first part of the theorem follows as above.

Suppose, on the other hand, that $C_p(\omega - E) < C_p(\omega)$, $1 < p < 2$. Let F be a closed subset of $\omega - E$, bounded by a finite number of circles. Let μ be the corresponding equilibrium distribution such that

$$V - u(\zeta) = V - \int_F \frac{d\mu(t)}{|\zeta - t|^p} = 0 \text{ on } F.$$

We define

$$(3) \qquad f(z) = \iint_{\omega} \frac{V - u(\zeta)}{\zeta - z} \, d\xi \, d\eta, \qquad \zeta = \xi + i\eta.$$

$f(z)$ is holomorphic outside $\omega - F$, and we shall prove that

$$(4) \qquad \lim_{z \to \infty} |z f(z)| \geq \delta > 0$$

and

$$(5) \qquad |f'(z)| \leq M,$$

where δ and M are independent of F.

To prove (4), let ν be the equilibrium distribution for ω and the kernel r^{-p}. We obtain

$$\int_{\omega} [V - u(\zeta)] \, d\nu(\zeta) = \frac{1}{C_p(F)} - \int_F d\mu(t) \int_{\omega} \frac{d\nu(\zeta)}{|\zeta - t|^p} = \frac{1}{C_p(F)} - \frac{1}{C_p(\omega)} \geq \delta' > 0$$

according to our hypothesis. As furthermore $V - u(\zeta) \geq 0$ and ν has a continuous density, (4) follows.

As to (5), we suppose for the sake of simplicity that the origin belongs to F and consider $f'(0)$. Setting $\zeta = r e^{i\theta}$, we obtain

$$f'(0) = \iint_{\omega} \frac{u(0) - u(\zeta)}{\zeta^2} \, d\xi \, d\eta = \int_F d\mu(t) \int_0^{2\pi} e^{-2i\theta} \, d\theta \int_0^1 \frac{dr}{r} \left\{ \frac{1}{|t|^p} - \frac{1}{|t - \zeta|^p} \right\}.$$

We devide the last integral into three parts where the integration is taken over the intervals $\left(0, \frac{|t|}{2}\right)$, $\left(\frac{|t|}{2}, 2|t|\right)$ and $(2|t|, 1)$ respectively. Denoting by A certain absolute constants we obtain the following estimations of the integrals:

$$|I_1| \leq A \int_F d\mu(t) \frac{1}{|t|^{p+1}} \int_0^{\frac{|t|}{2}} \frac{r \, dr}{r} = A \int_F \frac{d\mu(t)}{|t|^p} = A \, u(0).$$

$$|I_2| \leq \int_F d\mu(t) \int_{\frac{|t|}{2}}^{2|t|} \frac{dr}{r} \int_0^{2\pi} \frac{d\theta}{|t - \zeta|^p} \leq A \int_F d\mu(t) \int_{\frac{|t|}{2}}^{2|t|} \frac{dr}{r} \frac{1}{|t|^p} \frac{r^{p-1}}{|r - |t||^{p-1}} \leq A \, u(0).$$

$$|I_3| \leq A \int_F d\mu(t) \int_{2|t|}^{1} \frac{dr}{r^{1+p}} \leq A \, u(0).$$

We thus find

$$|f'(0)| \leq A\, u(0),$$

and since the argument works for all points on F and $f'(z)$ is evidently bounded outside ω, (5) is proved.

We now choose a sequence of sets F_n such that

$$\lim_{n=\infty} F_n = \omega - E;$$

the corresponding functions $f_n(z)$ then satisfy (4) and (5). We can choose a subsequence n_i such that

$$\lim_{i=\infty} f_{n_i}(z) = f(z)$$

exists, where $f(z)$ is holomorphic outside E and satisfies (4) and (5). From (4) it follows that $f(z) \not\equiv 0$, and the theorem is proved.

If we suppose that $f'(z)$ is uniformly continuous outside E, the picture is completely changed as shown by the following theorem.

Theorem: *A necessary and sufficient condition that E is a null-set for the class of functions with a uniformly continuous derivative outside E, is that E has no inner points.*

The necessity is evident. Suppose that E has no interior point and let $\{R_\nu\}$ be a covering. We find

$$\int_{R_\nu} f(z)\, dz = -\int_{R_\nu} (f'(z) - f'(z_\nu))\,(z - z_\nu)\, dz.$$

If

$$\omega_1(\delta) = \sup_{|h| \leq \delta} \sup_{z} |f'(z+h) - f'(z)|$$

we get

$$\left| \int_{R_\nu} f(z)\, dz \right| \leq 4\, \delta_\nu^2\, \omega_1(\delta_\nu),$$

whence

$$|c_1| \leq \frac{2}{\pi} \sum \delta_\nu^2\, \omega_1(\delta_\nu).$$

But this last sum is as small as we please and so

$$c_1 = 0.$$

The theorem follows as before.

6. The linear sets are particularly simple for the PAINLEVÉ problem. For the class Γ_1 certain product sets of a simple kind have a similar position. If E_x and E_y are two sets on the x- resp. y-axis, the set E of points $z = x + iy$ with $x \in E_x$ and $y \in E_y$ is denoted by

$$E = E_x \times E_y.$$

Theorem: *If E_x is the interval* (0, 1), *then $E = E_x \times E_y$ is a null-set for the class Γ_1 if and only if $E \in N_2$.*[1]

We suppose $E \notin N_2$. Then $m E_y > 0$, and there exists a function $\varphi(z)$ which is bounded and holomorphic outside E_y. The function

$$f(z) = \int_0^1 \varphi(z - \xi)\, d\xi$$

is non-constant and has a bounded derivative, for

$$f'(z) = \int_0^1 \varphi'(z - \xi)\, d\xi = \varphi(z) - \varphi(z - 1).$$

These product sets E also give us imformation about the properties of the null-sets of the class Γ_0 of functions which are holomorphic and uniformly continuous outside E. A necessary condition on the null-sets is $C_1(E) = 0$. This condition is not sufficient and there is no equivalent condition which is expressible in purely metrical terms as shown by the following

Theorem: $E = E_x \times E_y$, $m E_y > 0$, *is a null-set for Γ_0 if and only if E_x is countable.*

Suppose that E_x is countable and that $f(z) \in \Gamma_0$. The modulus of continuity

$$\omega(\delta) = \sup_{|h| \leq \delta} \sup_{z} |f(z + h) - f(z)|$$

then tends to zero. If $\{R_\nu\}$ is a covering of E, we obtain as before

$$|c_1| \leq \frac{2}{\pi} \sum \delta_\nu\, \omega(\delta_\nu).$$

It is easily seen that, under our assumptions, this last sum can be made as small as we please, whence

$$c_1 = 0$$

and $f(z) \equiv$ constant, which was our assertion.

On the other hand, if E_x is not countable, there is a distribution μ of the unit mass on E_x which is continuous. If $\varphi(z)$ is bounded and holomorphic outside E_y,

$$f(z) = \int_{E_x} \varphi(z - \xi)\, d\mu(\xi)$$

is an example of the desired kind.

[1] See DENJOY [3].

To see that the condition $C_1(E) = 0$ is not sufficient, we need only choose a set E_ε which is not countable but has logarithmic capacity zero, for, as is easily shown, a necessary and sufficient condition that $C_1(E) = 0$ is that E_ε has vanishing logarithmic capacity.

REFERENCES. [1] **Ahlfors, L.** and **Beurling, A.**, Conformal invariants and function-theoretic null-sets. Acta Math. 83 (1950). — [2] **Carleson, L.**, On a class of meromorphic functions and its associated exceptional sets. Uppsala 1950. — [3] **Denjoy, A.**, Sur les singularités discontinues des fonctions analytiques uniformes. Comptes rendus 149 (1909).

Tryckt den 25 juli 1950

Uppsala 1950. Almqvist & Wiksells Boktryckeri AB