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Solid spaces and absolute retracts

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1. The well-known TIETZE extension theorem says that a bounded real-valued continuous function defined on a closed subset of a normal space can be extended to a function defined on the whole space and there having the same lower and upper bounds as the original function. This theorem, which is of great importance in the theory of normal spaces, can also be looked upon as giving a property of the closed interval: Any mapping into a closed interval of a closed subset of a normal space has an extension to the whole space. STEENROD [7] has suggested the name "solid" for spaces having this property.

Definition. A space X is called *solid,* if for any normal space Y, any closed subset B of Y, and any mapping $f:B \to X$ there exists an extension $F: Y \to X$ of f.

TIETZE's extension theorem then simply asserts that a closed interval is solid.

Lemma 1.1. *Any topological product of solid spaces is solid.*

Proof. For if X is the topological product of the spaces X_α, then a mapping $f:B \to X$ of a closed subset B of a normal space Y is equivalent to a collection of mappings $f_\alpha:B \to X_\alpha$, obtained from f by projection onto each X_α. X_α being solid, f_α can be extended to Y. These extensions together define an extension of f.

Since a closed interval is a solid space, so also is any cube, i.e. a product of closed intervals. In particular the Hilbert cube is solid.

2. There is a strong connection between the concept of a solid space and of an absolute retract. We shall in this paper study this connection.

Using KURATOWSKI's extension ([5] p. 270) of BORSUK's original definition, we mean by an absolute retract (abbreviated AR) a separable metric space X such that, whenever X is imbedded as a closed subset of a separable metric space Z, X is a retract of Z.

Similarly we mean by an absolute neighborhood retract (abbreviated ANR) a separable metric space X such that, whenever X is imbedded as a closed subset of a separable metric space Z, X is a retract of some neighborhood of X in Z.

Lemma 2.1. *A retract of a solid space is solid.*

Proof. Assume X is a retract of Z. Denote the retraction by $r:Z \to X$. Let $f:B \to X$ be a given mapping of a closed subset B of a normal space Y. Considering f as a mapping into Z we have an extension $F: Y \to Z$. Then $rF: Y \to X$ is an extension of f to Y relative to X.

Any compact AR is a retract of the Hilbert cube. So we deduce from this lemma that all compact AR's are solid. Later, however, we shall see by an example (example 5.1) that there are non-compact AR's which are not solid.

3. We now introduce the concepts that correspond to AR and ANR, if in the definition "separable metric" is replaced by "normal" (cf. [3], [6]).

Definition. A normal space X is called an *absolute retract relative to normal spaces* (abbreviated ARN) if, whenever X is imbedded as a closed subset of a normal space Z, X is a retract of Z.

Definition. A normal space X is called an *absolute neighborhood retract relative to normal spaces* (abbreviated ANRN) if, whenever X is imbedded as a closed subset of a normal space Z, X is a retract of a neighborhood of X in Z.

The relation between these concepts and the concepts of AR and ANR will be discussed in the next section. In this section we prove the following two theorems.

Theorem 3.1. *A normal space is an ARN if and only if it is solid.*

Theorem 3.2. *A normal space is an ANRN if and only if any mapping $f:B \to X$ of a closed subset B of a normal space Y can be extended to some neighborhood U of B in Y.*

The corresponding theorems for AR's and ANR's with the normal pair (Y, B) replaced by a separable metric pair are well-known (cf. [2]).

The proofs of theorems 3.1 and 3.2 depend upon lemma 3.3 below. Assume that there is given a mapping $f:B \to X$. where B is a closed subset of Y, and where the spaces X and Y are normal. Then we construct a new topological space Z as follows. In the free union $X \cup Y$ of X and Y, i.e. the space in which X and Y are complementary disjoint open sets, we identify every point $y \in B$ with $f(y) \in X$. The identification space (cf. [1] p. 64) is denoted by Z. The natural mapping of $X \cup Y$ onto Z, restricted to X and to Y, yields two mappings $j:X \to Z$ and $k:Y \to Z$. A set O in Z is open if and only if $j^{-1}(O)$ and $k^{-1}(O)$ are open. Z is clearly a T_1-space.

The mapping j is a homeomorphism into Z. Therefore we can identify X with $j(X) \subset Z$, so that X is a subset of Z, in fact a closed subset. Note that $k(y) = f(y)$ for $y \in B$, and that $k \mid Y - B$ is a homeomorphism onto $Z - X$.

Lemma 3.3. *The space Z just defined is normal.*

Proof. Let F_1 and F_2 be two disjoint closed sets in Z. We have to find two disjoint open sets G_1 and G_2 for which

$$G_1 \supset F_1, \quad G_2 \supset F_2. \tag{1}$$

First, we use the normality of X to find two open subsets U_1 and U_2 of X such that

$$\overline{U}_1 \cap \overline{U}_2 = \emptyset \ (= \text{the empty set}), \tag{2}$$

$$U_1 \supset F_1 \cap X, \quad U_2 \supset F_2 \cap X. \tag{3}$$

Note that since X is closed in Z,

$$\bar{U}_1 \subset X, \quad \bar{U}_2 \subset X. \tag{4}$$

Secondly, we see that the two sets

$$F_1 \cup \bar{U}_1, \quad F_2 \cup \bar{U}_2,$$

are disjoint closed sets in Z, so by the normality of Y we have two open subsets V_1 and V_2 of Y such that

$$V_1 \cap V_2 = O, \tag{5}$$

$$V_1 \supset k^{-1}(F_1 \cup \bar{U}_1), \quad V_2 \supset k^{-1}(F_2 \cup \bar{U}_2). \tag{6}$$

Now set

$$G_1 = k(V_1 - B) \cup U_1, \quad G_2 = k(V_2 - B) \cup U_2.$$

We recall that $k \mid Y - B$ is a homeomorphism between $Y - B$ and $Z - X$. Then we see from (3) and (6) that (1) is true and from (2), (4), and (5) that G_1 and G_2 are disjoint.

Finally, to prove that G_i, $i = 1, 2$, are open, we have to show that $j^{-1}(G_i)$ and $k^{-1}(G_i)$ are open in X and Y respectively. Now

$$j^{-1}(G_i) = G_i \cap X = U_i,$$

which is open in X, and

$$k^{-1}(G_i) = (V_i - B) \cup k^{-1}(U_i).$$

Since $k^{-1}(U_i)$ is open in B and by (6) contained in V_i, we can write

$$k^{-1}(U_i) = B \cap H_i = B \cap H_i \cap V_i,$$

where H_i is an open subset of Y. From

$$k^{-1}(G_i) = (V_i - B) \cup [B \cap (H_i \cap V_i)]$$
$$= (V_i - B) \cup (H_i \cap V_i)$$

we then conclude, that $k^{-1}(G_i)$ is open in Y.

This proves lemma 3.3.

We now give the proof of theorem 3.2, theorem 3.1 being proved in an analogous way.

Proof of theorem 3.2. To prove the necessity, let X be an ANRN. Suppose Y is a given normal space, B a closed subset, and $f: B \to X$ a mapping. Construct as above the normal space Z. X is a closed subset of Z. Since X is an ANRN, X is therefore a retract of some open neighborhood U of X. Denote the retraction ay $r: U \to X$. Let $k: Y \to Z$ be the same as above. Then $V = k^{-1}(U)$ is an open set in Y containing B, and the function $F: V \to X$ defined by

$$F(v) = rk(v) \text{ for } v \in V$$

is an extension of f.

The sufficiency is clear from the fact that if X is a closed subset of a normal space Z, the condition in the theorem yields, that the identity mapping $i: X \to X$ has an extension to some neighborhood of X in Z.

This proves theorem 3.2.

4. We now want to study the connection between the concepts AR and ARN (and between ANR and ANRN). Since AR (ANR) is defined only for separable metric spaces, we then have to assume that the space X is separable metric.

The difference between the definition of a separable metric ARN and of an AR is as follows. Let X be a closed subset of any space Z. Then if X is a separable metric ARN, X has to be a retract of Z, whenever Z is normal. If X is an AR, however, X is only required to be a retract of Z, when Z is separable metric. It is therefore clear that a separable metric ARN is an AR. Conversely, we have seen earlier that every compact AR is solid or, as we have shown to be the same, is an ARN. But this is not true in general for non-compact AR's. A characterization of the AR's that are also ARN's is given in theorem 4.1.

The corresponding distinction exists between separable metric ANRN's and ANR's.

Theorem 4.1. *A separable metric space is an ARN if and only if it is an AR and an absolute G_δ.*

Theorem 4.2. *A separable metric space is an ANRN if and only if it is an ANR and an absolute G_δ.*

By an absolute G_δ we mean a metric space which, whenever imbedded in a metric space, is a G_δ, i.e. a countable intersection of open sets. Since a closed set in a metric space is a G_δ, all compact metric spaces are absolute G_δ's. Also all locally compact metric spaces can be shown to be absolute G_δ's.

The class of all absolute G_δ's is known to be the same as the class of all topologically complete spaces, i.e. spaces which can be given a complete metric. For further information about these spaces, see for instance KURATOWSKI [4], Chapter 3.

It is known that any subset of the Hilbert cube which is a G_δ in the Hilbert cube, is an absolute G_δ. Thus the separable metric absolute G_δ's are the spaces homeomorphic to a G_δ in the Hilbert cube.

We now prove theorem 4.2, theorem 4.1 being proved similarly.

Proof of theorem 4.2. We have to show that the condition given in the theorem is both necessary and sufficient.

a) Necessity. Suppose that X is a separable metric ANRN. Then X is an ANR. Let us prove that X is an absolute G_δ.

Let X be imbedded in the Hilbert cube I_ω. We construct a new space Z. The points of Z shall be in $1-1$-correspondence with the points of I_ω. Let $h(z) \in I_\omega$ be the point corresponding to $z \in Z$ under this $1-1$-correspondence. Let $X' = h^{-1}(X)$, i.e. the subset of Z corresponding to X. The topology of Z is determined by taking as open sets all sets of the form

$$h^{-1}(O) \cup A, \tag{1}$$

where O is any open subset of I_ω, and A is any subset of $Z - X'$. Z is readily seen to be a Hausdorff space. Let us show that Z is normal.

Suppose F_1 and F_2 are two disjoint closed subsets of Z. Let the distance between two points of Z be the same as between the corresponding points of I_ω. (This metric is not in general a metrization of the *topological* space Z.) Consider a point $x_1 \in F_1 \cap X'$. By the ε-sphere $S(x_1, \varepsilon)$, $\varepsilon > 0$, we mean the set of all points having a distance less than ε to x_1. Since $x_1 \in X'$, the collection of all $S(x_1, \varepsilon)$ make up a base for neighborhoods of x_1. Therefore, since F_2 is closed, for some ε_1

$$S(x_1, \varepsilon_1) \cap F_2 = \emptyset.$$

Choose $\varepsilon_1 = \varepsilon_1(x_1)$ in this way for each $x_1 \in F_1 \cap X'$. Then the set

$$G_1 = F_1 \cup \bigcup_{x_1 \in F_1 \cap X'} S\left(x_1, \frac{\varepsilon_1}{2}\right)$$

is open and contains F_1. Similarly, choose $\varepsilon_2 = \varepsilon_2(x_2)$ for each $x_2 \in F_2 \cap X'$ such that

$$S(x_2, \varepsilon_2) \cap F_1 = \emptyset,$$

and take

$$G_2 = F_2 \cup \bigcup_{x_2 \in F_2 \cap X'} S\left(x_2, \frac{\varepsilon_2}{2}\right).$$

Then G_2 is open and contains F_2. But G_1 and G_2 are disjoint, which proves that Z is normal.

The $1-1$-mapping $h: Z \to I_\omega$ is continuous, but not topological. However, $h \mid X'$ is topological, showing that X and X' are homeomorphic. Then X' is an ANRN, and since X' is closed in Z, there exists a retraction $r: U \to X'$ of an open neighborhood U of X'. Since U is open in Z, U can be written in the form (1). Then X' and A are disjoint, and we have

$$X' \subset h^{-1}(O).$$

Therefore we may assume $h(U)$ to be open in X, otherwise replacing U by $h^{-1}(O)$.

In order to show that X is a G_δ in I_ω we consider for each $n = 1, 2, \ldots$ the set $A_n \subset Z$ of all points $z \in U$ such that the distance

$$d(z, r(z)) \geq \frac{1}{n}.$$

We see that

$$A_n \cap X' = \emptyset$$

$$X' = U - \bigcup_{n=1}^{\infty} A_n. \tag{2}$$

We assert that

$$\overline{h(A_n)} \cap X = \emptyset. \tag{3}$$

For suppose there did exist a point $x \in \overline{h(A_n)} \cap X$. Take $S = S\left(x, \frac{1}{2n}\right)$, the $\frac{1}{2n}$-sphere with center x. S is an open neighborhood of x. Therefore

$$(4) \qquad x \in \overline{h(A_n) \cap S}.$$

Under the 1—1-correspondence $h: Z \to I_\omega$ any two points $x \in X$ and $x' = h^{-1}(x) \in X'$ have corresponding sets as neighborhoods. Hence (4) implies

$$x' = h^{-1}(x) \in \overline{A_n \cap h^{-1}(S)}.$$

Since r is a continuous function, we obtain

$$(5) \qquad x' = r(x') \in \overline{r(A_n \cap h^{-1}(S))}.$$

But any two points of $h^{-1}(S)$ have a distance from each other less than $\frac{1}{n}$. So by the definition of A_n

$$(6) \qquad r(A_n \cap h^{-1}S) \cap h^{-1}(S) = \emptyset.$$

Now (5) and (6) contradict each other, since $h^{-1}(S)$ is a neighborhood of x'. This contradiction shows that (3) is true.

Finally, since $h(U)$ is open in X, (2) and (3) show that X is a G_δ in I_ω:

$$X = \bigcap_{n=1}^{\infty} (h(U) - \overline{h(A_n)}).$$

b) Sufficiency. The space X is an ANR and an absolute G_δ. We shall show that X is an ANRN.

Let X' be a subset of the Hilbert cube I_ω homeomorphic to X. Denote the homeomorphism by $h: X \to X'$.

Take the product space $I_\omega \times I$, where I is the closed interval $0 \leq t \leq 1$. We identify I_ω with $I_\omega \times \{0\}$ and X' with $X' \times \{0\}$, so that I_ω and X' are subsets of $I_\omega \times I$.

Let us now, using an idea of Fox [2], consider the set

$$T = X' \cup (I_\omega \times (0, 1]),$$

where $(0, 1]$ stands for the half-open interval $0 < t \leq 1$. X' is closed in T, hence there is a retraction $r: U \to X'$ of an open subset U of T containing X'.

In order to show that X is an ANRN, let X be imbedded as a closed subset of a normal space Z. The mapping $h: X \to X'$ is a mapping onto the subset X' of I_ω. Since X is a closed subset of the normal space Z and since I_ω is solid, there exists a mapping $H: Z \to I_\omega$ having the same values as h at all points of X.

X' is an absolute G_δ, so we can write

$$X' = I_\omega - \bigcup_{n=1}^{\infty} A_n,$$

where the sets A_n are closed subsets of I_ω. Then the sets

$$B_n = H^{-1}(A_n)$$

are closed in Z and disjoint from X. Thus there exist mappings $e_n : Z \to I$ such that

$$e_n(z) = 1 \quad \text{for } z \in B_n,$$
$$e_n(z) = 0 \quad \text{for } z \in X.$$

The mapping $e : Z \to I$ defined by

$$e(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n(z)$$

has the following properties:

$$e(z) > 0 \quad \text{for } z \in \bigcup_{n=1}^{\infty} B_n = H^{-1}(I_\omega - X')$$

$$e(z) = 0 \quad \text{for } z \in X.$$

Therefore the mapping $g : Z \to I_\omega \times I$ defined by

$$g(z) = (H(z), e(z)) \quad \text{for } z \in Z$$

is into T, and we have

$$g(x) = h(x) \quad \text{for } x \in X.$$

Put $V = g^{-1}(U)$, and define $f : V \to X$ by

$$f(v) = h^{-1} rg(v) \quad \text{for } v \in V.$$

Then f is a retraction of V onto X. But V is open in Z. Hence X is an ANRN.

This completes the proof of theorem 4.2.

5. If we combine theorem 3.1 and theorem 4.1, we see that the separable metric spaces that are solid are those which are AR's and absolute G_δ's. Examples of such spaces are all compact AR's. That they are solid was proved directly in section 2. Further examples are the real line and the product of a countable number of real lines. The last space is not locally compact.

We want to show by an example that not all AR's are absolute G_δ's.

Example 5.1. Let X be the set in the xy-plane consisting of all points for which $x^2 + y^2 < 1$ and all points on $x^2 + y^2 = 1$ with rational x. This space

X is known to be an AR ([2] p. 273), but it is not an absolute G_δ. Hence it is not an ARN or, what is the same, not solid. That X is not an ARN, can be proved directly by considering it as a subset of the normal (but not metric) space, obtained from the set $x^2 + y^2 \leqq 1$ in the same way as Z is derived from I_ω in the proof of theorem 4.2.

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