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Retraction and extension of mappings of metric and non-metric spaces

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Introduction

1. The two kinds of topological spaces that are called absolute retracts and absolute neighborhood retracts, were originally defined by BORSUK ([5], [6]) for compact metric spaces. Later on these concepts were extended to several other classes of spaces.

A closed subset X of a space Z is called a retract of Z if there is a mapping $r: Z \to X$ such that $r(x) = x$ for each $x \in X$. The mapping r itself is called a retraction of Z onto X. By an absolute retract we mean a space X, such that whenever X is imbedded as a closed subset of a space Z, X is a retract of Z. However, if this definition is to have a meaning, we have to determine which spaces Z are allowed. There is, for instance, an example (example 17.7) of a separable metric space X, which is a retract of any separable metric space in which it is imbedded as a closed subset, but which can be imbedded as a closed subset of a normal space Z in such a way that it is not a retract of Z.

A closed subset X of a space Z is called a neighborhood retract of Z, if there is an open set O in Z, such that $X \subset O$, and a retraction $r: O \to X$. The mapping r itself is called a neighborhood retraction. By an absolute neighborhood retract we mean a space X such that whenever X is imbedded as a closed subset of a space Z, X is a neighborhood retract of Z. Again we must know which spaces Z are allowed. In order to give a simple example let X be the Hausdorff space consisting of only two points. This is a neighborhood retract of any Hausdorff space Z in which it is imbedded. However, it is not necessarily a neighborhood retract when imbedded in a T_1-space.

Thus when changing the class of spaces from which Z shall be taken, we get different concepts absolute retract and absolute neighborhood retract. The purpose of this paper is to study the properties of these concepts and the relationships between them.

We will mainly concentrate on some special classes of spaces. These classes are listed in § 2. In §§ 3–7 we have gathered together some facts about these spaces that will be useful in the sequel.

In § 2 we also define two other kinds of spaces, called extension spaces and neighborhood extension spaces. We will show in §§ 8–11 that they are closely related to absolute retracts and absolute neighborhood retracts. We study in § 12 our four concepts for contractible spaces.

In §§ 13–17 we take up the relationships between our concepts for different classes of spaces. The main results are collected in theorems 17.1 and 17.2. Under certain conditions the property of a space to be a neighborhood extension space (or an absolute neighborhood retract) is a local property. This is proved in §§ 18–23. In §§ 24–27 we prove that certain infinite polyhedra are neighborhood extension spaces. Finally in § 28 we give some homotopy theorems.

Definitions and general preliminaries

2. All topological spaces considered in this paper will be **Hausdorff** spaces. Let X be a topological space and A a subset. By a neighborhood of A in X we mean a set $U \subset X$ such that there is an open set O satisfying $A \subset O \subset U$. Thus a neighborhood is not necessarily open.

We leave the proof of the following lemma to the reader.

Lemma 2.1. *Let $A \subset U \subset V \subset X$. If U is a neighborhood of A in V and V is a neighborhood of A in X, then U is a neighborhood of A in X.*

A space X is normal if any two disjoint closed subsets have disjoint neighborhoods. This can also be formulated thus: A space X is normal if for any closed set $A \subset X$ and any neighborhood U of A, there is a closed neighborhood V of A contained in U. By a pair (Y, B) we mean a space Y and a closed subset B of Y. If (Y, B) is a pair and $F: Y \to X$ and $f: B \to X$ are two mappings into a space X such that $F(y) = f(y)$ for $y \in B$, we call F an extension of f to Y and f the restriction of F to B, denoted $f = F \mid B$. If F is only defined on some neighborhood of B in Y, F is called a neighborhood extension of f in Y.

Let Q be a class of topological spaces. We require:

2.2. *If $X \in Q$ and if A is a closed subset of X, then $A \in Q$.*

A space that belongs to Q will often be called a Q-space, and a pair (Y, B) for which $Y \in Q$ will be called a Q-pair.

The classes Q, in which we shall be principally interested, are the following:

α) Tychonoff spaces (= completely regular spaces),

β) normal spaces,

γ) collectionwise normal spaces (see § 4),

δ) fully normal spaces (see § 5),

ε) Lindelöf spaces (see § 6),

ζ) compact spaces,

η) metric spaces,

ϑ) separable metric spaces,

ι) compact metric spaces.

These classes of spaces will often be referred to by Greek letters given in this list. It may be natural to order them in a diagram.

$$\begin{tikzcd}
 & \text{Tychonoff} \arrow[d] & & \\
 & \text{normal} \arrow[d] & & \\
 & \text{coll. normal} \arrow[d] & & \\
 & \text{fully normal} \arrow[dl] \arrow[dr] & & \\
\text{metric} \arrow[dr] & & \text{Lindelöf} \arrow[dl] \arrow[dr] & \\
 & \text{sep. metr.} \arrow[dr] & & \text{compact} \arrow[dl] \\
 & & \text{comp. metr.} &
\end{tikzcd}$$

In this diagram each arrow goes from a class to a subclass. We have to verify this fact and the fact that all classes α–ι satisfy 2.2. This will be done in §§ 4–6 for some cases. The remaining cases are all well known.

Definition 2.3. A space X is called an *absolute retract relative to the class Q* (abbreviated AR(Q)) if

a) $X \in Q$,

b) whenever X is topologically imbedded as a closed subset of a Q-space Z, then X is a retract of Z.

Definition 2.4. A space X is called an *absolute neighborhood retract relative to the class Q* (abbreviated ANR(Q)) if

a) $X \in Q$,

b) whenever X is topologically imbedded as a closed subset of a Q-space Z, then X is a retract of some neighborhood U of X in Z.

Definition 2.5. A space X is called an *extension space for the class Q* (abbreviated ES(Q)) if, for any Q-pair (Y, B) and any mapping $f: B \to X$, there exists an extension $F: Y \to X$ of f to Y.

Definition 2.6. A space X is called a *neighborhood extension space for the class Q* (abbreviated NES(Q)) if, for any Q-pair and any mapping $f: B \to X$, there exists an extension $F: U \to X$ of f to a neighborhood U of B in Y.

Let us point out that in the last two definitions we do not assume that X belongs to the class Q.

If all spaces in Q are normal, we can in the definitions 2.4 and 2.6 let the neighborhood U be closed in Z and Y respectively. Then U is a Q-space.

Lemma 2.7. *Let Q contain a space which is not normal. Then an $NES(Q)$ (or an $ES(Q)$) is never Hausdorff unless it consists merely of one single point.*

Proof. Let X be an NES(Q) with more than one point. Take two different points $x_1, x_2 \in X$. If X is Hausdorff there are disjoint neighborhoods U_1 and U_2 of x_1 and x_2 respectively. There is a space $Y \in Q$, which is not normal. Take in this space Y two disjoint closed sets B_1 and B_2 which do not have disjoint neighborhoods. Define $f: B_1 \cup B_2 \to X$ by

$$f(B_1) = x_1 \text{ and } f(B_1) = x_2.$$

Clearly f is continuous. Now $B_1 \cup B_2$ is closed in Y, $Y \in Q$, and X is an NES(Q). Hence there exists an extension $F: U \to X$, U being a neighborhood of $B_1 \cup B_2$

in Y. But then $F^{-1}(U_1)$ and $F^{-1}(U_2)$ are disjoint neighborhoods of B_1 and B_2 respectively. This is a contradiction, which proves lemma 2.7.

To avoid this case we shall always assume that Q in definitions 2.5 and 2.6 only consists of normal spaces. We mainly use the classes $\beta - \iota$.

There are several immediate relationships between the four concepts AR(Q), ANR(Q), ES(Q), and NES(Q). We give them without proofs.

2.8. *Any $AR(Q)$ is an $ANR(Q)$.*

2.9. *Any $ES(Q)$ is an $NES(Q)$.*

2.10. *Any $ES(Q)$ belonging to Q is an $AR(Q)$.*

2.11. *Any $NES(Q)$ belonging to Q is an $ANR(Q)$.*

If Q and Q_1 are two classes both satisfying condition 2.2, then

2.12. $Q \subset Q_1$ *implies that any $ES(Q_1)$ is an $ES(Q)$ and that any $NES(Q_1)$ is an $NES(Q)$.*

2.13. $Q \subset Q_1$ *implies that any $AR(Q_1)$ belonging to Q is an $AR(Q)$ and that any $ANR(Q_1)$ belonging to Q is an $ANR(Q)$.*

Easy to prove are the following statements:

2.14. *A retract of an $ES(Q)$ is an $ES(Q)$* (cf. [14] p. 375).

2.15. *A neighborhood retract of an $NES(Q)$ (or an $ES(Q)$) is an $NES(Q)$.*

2.16. *Any open subset of an $NES(Q)$ is an $NES(Q)$* (cf. [15] p. 391).

2.17. *Any topological product of $ES(Q)$'s is an $ES(Q)$* (cf. [14] p. 375).

2.18. *Any topological product of a finite number of $NES(Q)$'s is an $NES(Q)$.*

Example 2.19. In our terminology TIETZE's extension theorem says that a closed interval is an ES(normal). Hence by 2.17 any cube, i.e. a product of closed intervals, is an ES(normal). Such a space is compact.

Example 2.20. It is known that TIETZE's extension theorem is true even if the closed interval is replaced by a real line, i.e. a real line is an ES(normal). (Cf. also 2.16 and theorem 12.3 below.) Hence also the product of any number of real lines is an ES(normal). The real line itself is locally compact. The product of a countable number of real lines is a metric space which is not locally compact. The product of uncountably many real lines is a Tychonoff space which is not normal (cf. [29] p. 981). The last space is an example of an ES(normal) which is not normal. Hence it is not an AR(normal). We return to this example in example 17.8.

Let us compare our notations with the notations used earlier in the literature on this subject. This we will do first for the concept ANR. All notations for the concept AR are similar.

The original concept ANR defined by BORSUK [6] is in our notation ANR (comp. metr.). KURATOWSKI generalized this to separable metric spaces thus introducing ANR(sep. metr.). In a recent paper DUGUNDJI [12] considered arbitrary metric spaces and obtained what is here called ANR(metric).

Other generalizations of BORSUK's original concept are obtained by considering non-metric spaces. Thus SAALFRANK [26] considered ANR(compact) and HU [17] considered ANR (Tychonoff), by him called ANR*.

The author considered ANR(normal) ([14], there called ANRN), and in a recent paper C. H. DOWKER considered ANR(coll. normal) ([11], by him called ANR_{cn}).

Thus all classes of spaces α–ι have been used except the two classes of fully normal spaces and of Lindelöf spaces. The class of fully normal spaces seems natural to introduce when analyzing the concept local ANR(Q) (and local NES(Q)) and the class of Lindelöf spaces is proposed by theorems 14.5 and 19.4.

We shall see below (§§ 8–10) that for any of the classes β–ι an ANR(Q) is an NES(Q) belonging to Q, and conversely. Thus if we only considered Q-spaces the special notation NES(Q) would be superfluous. However, in §§ 25 and 28 we have to consider some NES(Q)'s which do not belong to Q, and in example 2.20 we already saw an ES(Q) not belonging to Q.

Steenrod ([28] p. 54) and the author [14] have considered ES(normal) under the name solid space.

3. We shall often have to consider collections of subsets of a space X. Let $\alpha=\{U_\lambda\}$ be such a collection, indexed by a set $\Lambda=\{\lambda\}$. To avoid some trivial exceptions we assume that the index set Λ is never void. If all sets U_λ are open we call α an open collection; if all sets U_λ are closed we call α a closed collection.

A collection $\alpha=\{U_\lambda\}$ is called a covering of X if the union of the sets U_λ is X. We shall often have to do with open coverings and sometimes with closed coverings. In § 6 we need some coverings which are neither open nor closed. Notice that if A is a subset of X and $\alpha=\{U_\lambda\}$ is an open covering of A then the sets U_λ are subsets of A which are open in A but in general not in X.

Let $\alpha=\{U_\lambda\}$ be a covering of X and A any subset of X. By the star of A with respect to α we mean the union of all sets U_λ intersecting A, i.e.

$$\operatorname{St}(A,\alpha)=\bigcup_{U_\lambda\cap A\neq\emptyset} U_\lambda$$

($\emptyset$ denotes the void set). By the star of the covering α we mean the covering

$$\operatorname{St}\alpha=\{\operatorname{St}(U_\lambda,\alpha)\}.$$

If α is open, the star of any set is open and the covering $\operatorname{St}\alpha$ is open.

Let $\alpha=\{U_\lambda\}$ and $\beta=\{V_\mu\}$ be two open coverings of X. Then β is called a refinement of α if for each μ there is a λ such that $V_\mu\subset U_\lambda$. Thus e.g. β is a refinement of $\operatorname{St}\beta$. If $\operatorname{St}\beta$ is a refinement of α, β is called a star-refinement of α.

A collection α is locally finite, if for each point $x\in X$ there is a neighborhood of x meeting U_λ for at most a finite number of λ's. (Two sets are said to meet if they have a non-void intersection.) When we want to point out the space X in this definition we say that α is locally finite in X. Let A be a closed subset of X and α a collection of subsets of A such that α is locally finite in A. Then α is also locally finite in X. This is not necessarily true if A is not closed in X.

The proofs of the following two lemmas are omitted.

Lemma 3.1. *Let $\{U_\lambda\}$ be a locally finite collection. Then $\{\overline{U}_\lambda\}$ is also locally finite.*

Lemma 3.2. *Let* $\{U_\lambda\}$ *be a locally finite closed collection. Then the set*

$$\bigcup_{\lambda \in \Lambda} U_\lambda$$

is closed.

Using the last lemma we can prove

Lemma 3.3. *Let* $\{U_\lambda\}$ *be a locally finite closed covering of* Y *and let*

$$f: Y \to X$$

be a function such that each $f \mid U_\lambda$ *is continuous. Then* f *is continuous.*

Proof. Let A be any closed set in X. We have to prove that $f^{-1}(A)$ is closed in Y. Since $f_\lambda = f \mid U_\lambda$ is continuous the set

$$f_\lambda^{-1}(A) = U_\lambda \cap f^{-1}(A)$$

is closed in U_λ and hence closed in Y, so that

$$f^{-1}(A) = \bigcup_{\lambda \in \Lambda} (U_\lambda \cap f^{-1}(A))$$

is a union of a locally finite collection of closed sets. Hence, by lemma 3.2, $f^{-1}(A)$ is closed. This proves that f is continuous.

Lemma 3.4. *Let* $\{U_\lambda\}$ *be a locally finite open covering of a normal space* X. *Then there is an open covering* $\{V_\lambda\}$, *such that for each* λ *we have* $\overline{V}_\lambda \subset U_\lambda$.

For the proof see LEFSCHETZ ([23] p. 26) or DIEUDONNÉ ([8] p. 71).

Lemma 3.5. *Let* $\{U_\lambda\}$ *be a locally finite closed covering of a space* X. *Then for each point* $x \in X$ *there is a neighborhood* V *such that*

$$U_\lambda \cap V \neq \emptyset$$

is true only for those λ (*finite in number*) *for which* $x \in U_\lambda$.

Proof. Since $\{U_\lambda\}$ is locally finite there is a neighborhood U of x which meets U_λ for only a finite number of indices λ, say $\lambda_1, \ldots, \lambda_n$. Suppose $x \in U_\lambda$ for $\lambda = \lambda_1, \ldots, \lambda_m$ and $x \notin U_\lambda$ for $\lambda = \lambda_{m+1}, \ldots, \lambda_n$. Then, since each U_λ is closed,

$$V = U - \bigcup_{i=m+1}^{n} U_{\lambda_i}$$

is a neighborhood of x. Let U_λ meet V. Then, since $V \subset U$, λ is one of $\lambda_1, \ldots, \lambda_n$. But V is disjoint to U_λ for $\lambda = \lambda_{m+1}, \ldots, \lambda_n$. Hence λ is one of $\lambda_1, \ldots, \lambda_m$ so that $x \in U_\lambda$. This proves the lemma.

We call a covering $\alpha = \{U_\lambda\}$ of X star-finite if, for fixed λ_1, U_λ meets U_{λ_1} for at most a finite number of λ's.

A star-finite open covering is certainly locally finite.

Lemma 3.6. *Let* $\{U_\lambda\}$ *be a star-finite open covering of* X. *Then* $\{\overline{U}_\lambda\}$ *is star-finite.*

Proof. Suppose $\overline{U}_\lambda$ meets $\overline{U}_{\lambda_1}$. Choose a point

$$x \in \overline{U}_\lambda \cap \overline{U}_{\lambda_1}.$$

$\{U_\lambda\}$ is a covering of X, hence $x \in U_{\lambda_2}$ for some λ_2. Since U_{λ_2} is open this implies

$$U_{\lambda_2} \cap U_{\lambda_1} \neq \emptyset, \tag{1}$$

$$U_{\lambda_2} \cap U_\lambda \neq \emptyset. \tag{2}$$

But, for fixed λ_1, (1) is true only for a finite number of λ_2's and, for each of these λ_2's, (2) is true only for a finite number of λ's. Hence, for fixed λ_1, $\overline{U}_\lambda$ meets $\overline{U}_{\lambda_1}$ only for a finite number of λ's, i.e. $\{\overline{U}_\lambda\}$ is star-finite.

4. Definition 4.1. A Hausdorff space X is called *collectionwise normal* if, for every locally finite collection $\{A_\lambda\}$ of mutually disjoint closed subsets of X, there is a collection $\{U_\lambda\}$ of mutually disjoint open sets such that $A_\lambda \subset U_\lambda$ for each λ. (BING [4] p. 176.)

Every collectionwise normal space is certainly normal. BING ([4] p. 184) has given an example of a normal space which is not collectionwise normal.

A closed subset A of a collectionwise normal space X is collectionwise normal. For if $\{A_\lambda\}$ is a locally finite closed collection in A it is also a locally finite closed collection in X.

Lemma 4.2 (C. H. DOWKER). *Let X be a collectionwise normal space and $\{A_\lambda\}$ a locally finite collection of mutually disjoint closed subsets of X. Then there are open subsets U_λ of X such that $A_\lambda \subset U_\lambda$ for each λ and such that $\{\overline{U}_\lambda\}$ is a locally finite collection of mutually disjoint sets.*

For the proof see [11].

5. Definition 5.1. A Hausdorff space is called *fully normal* if every open covering has an open star-refinement (TUKEY [30] p. 53).

Definition 5.2. A Hausdorff space is called *paracompact* if every open covering has a locally finite open refinement (DIEUDONNÉ [8]).

Lemma 5.3 (A. H. STONE). *Every fully normal space is paracompact and every paracompact space is fully normal.*

For the proof see [29].

Hence the two concepts are equivalent and the class of fully normal spaces is the same as the class of paracompact spaces. We will in the sequel only use the name fully normal spaces for these spaces.

We see immediately that a closed subset of a fully normal space is fully normal.

It is known that every fully normal space is normal. We also have

Lemma 5.4. *Every fully normal space is collectionwise normal.*

For the proof see [4] p. 183. As was pointed out by BING, an example given by DIEUDONNÉ of a normal space which is not fully normal is collectionwise normal.

Lemma 5.5 (TUKEY). *Every metric space is fully normal.*

For the proof see [30] p. 53.

Lemma 5.6. *Every compact space is fully normal.*

Proof. A compact space is obviously paracompact.

Let α be an open covering of a fully normal space X. Then we can obtain some new coverings by the following methods:

a) We can take an open star-refinement of α. This is the definition of full normality.

b) We can take a locally finite open refinement of α. For a fully normal space is paracompact.

c) If $\alpha = \{U_\lambda\}$ is locally finite we can take an open covering $\alpha' = \{U'_\mu\}$ such that each U'_μ meets U_λ for only a finite number of λ's. This is essentially the definition of local finiteness. Since any refinement of α' has the same property as α', we can assume α' to be locally finite. Then we can repeat the process and take an open covering α'' such that each element of α'' meets only a finite number of elements of α'.

d) If $\alpha = \{U_\lambda\}$ is locally finite we can take a covering $\beta = \{V_\lambda\}$ such that for each λ we have $\bar{V}_\lambda \subset U_\lambda$. For, since a fully normal space is normal, we can apply lemma 3.4.

6. Definition 6.1. A space is said to have the *Lindelöf property* if from each open covering there can be selected a countable covering. A regular space having the Lindelöf property is called a *Lindelöf space.*

The name is suggested by the Lindelöf covering theorem, which in our terminology says that a separable metric space has the Lindelöf property (cf. [23] p. 6).

An immediate consequence is:

Lemma 6.2. *A regular space is a Lindelöf space if and only if every open covering has a countable refinement.*

Any Lindelöf space is normal. For we can apply TYCHONOFF's well-known proof of the fact that a regular space with a countable base is normal ([31], cf. [34] p. 6).

S. KAPLAN proved ([19] p. 249) that any open covering of a separable metric space has a star-finite open refinement. A slightly different proof of this was used by the author in [15] p. 393. Our main interest in Lindelöf spaces in this paper depends upon the fact that KAPLAN's result extends to Lindelöf spaces.

Lemma 6.3. *Any open covering of a Lindelöf space has a countable star-finite open refinement.*

This lemma has been proved by MORITA ([25] p. 66). Note that any star-finite (or locally finite) open covering of a Lindelöf space is countable.

Lemma 6.4. *Any Lindelöf space is fully normal.*

Proof. This follows from lemma 6.3, since a star-finite open refinement is locally finite.

Since obviously all compact spaces are Lindelöf spaces, the class of Lindelöf spaces is a class between the class of compact spaces and the class of fully normal spaces.

Lemma 6.5. *A metric space is a Lindelöf space if and only if it is separable.*

Proof. The sufficiency is just the Lindelöf covering theorem. To prove the necessity we select for each integer $n > 0$ a countable covering out of the covering by $\frac{1}{n}$-spheres. This gives a countable base for the space. Hence the space is separable. This proves lemma 6.5.

Thus any non-separable metric space gives an example of a fully normal space which is not a Lindelöf space, and any non-compact separable metric space gives an example of a Lindelöf space which is not compact.

We see easily that a closed subset of a Lindelöf space is a Lindelöf space. However an arbitrary subset of a Lindelöf space need not be a Lindelöf space, since any Tychonoff space can be imbedded as a subset of a compact space.

We leave the proof of the following lemma to the reader.

Lemma 6.6. *A regular space which is the union of a countable number of subsets with the Lindelöf property is a Lindelöf space.*

Thus any regular space which is the union of a countable number of compact sets is a Lindelöf space. Hence such a space is also fully normal. That this is true when the space is also locally compact was proved by DIEUDONNÉ ([8] p. 68).

Exemple 6.7. Let X be the set of real numbers, $-\infty < x < \infty$. Define a topology on X by taking as a base for open sets all intervals of the form $a \leqq x < b$. This gives us a Lindelöf space, which is not metrizable. That each open covering of this space has a star-finite open refinement follows from lemma 6.3. See also BEGLE ([3] p. 579). SORGENFREY [27] proved that the topological product of this space with itself is not normal.

7. In this paragraph we give some lemmas which will be used in the sequel.

Lemma 7.1. *Let (Y, B) be a pair and let U be an open subset of B. If O is an open set in Y and $U \subset O$ then*

$$V = U \cup (O - B)$$

is open in Y.

Proof. U is open in B. Hence there is an open subset W of Y such that

$$U = W \cap B.$$

Then

$$\begin{aligned} V &= (U \cap O) \cup (O - B) \\ &= (W \cap B \cap O) \cup (O - B) \\ &= (W \cap O) \cup (O - B) \end{aligned}$$

and V, being the union of two open sets, is open. This proves lemma 7.1.

Lemma 7.2. *Let (Y, B) be a normal pair and $\alpha = \{U_n\}$ a countable star-finite open covering of B. Then there is an open covering $\beta = \{V_n\}$ of Y such that*

a) *$U_n = V_n \cap B$ for each n,*

b) *β is locally finite.*

Proof. By lemma 3.6 $\{\overline{U}_n\}$ is star-finite. Therefore, for fixed m

$$\overline{U}_m \cap \overline{U}_n = \emptyset \tag{1}$$

for all indices n except a finite number. We want to define for each integer n

an open set W_n in Y such that $W_n \supset U_n$ and such that for each pair (m, n) satisfying (1) we have

$$\overline{W}_m \cap \overline{W}_n = \emptyset. \tag{2}$$

The definition of the sets W_n will proceed by induction. Let $m \geqq 1$. If $m > 1$ suppose that W_n is already defined for $n < m$. Let $W_m \supset U_m$ be chosen such that for (m, n) satisfying (1) we have

$$\begin{aligned} \overline{W}_m \cap \overline{W}_n &= \emptyset \quad \text{for } n<m, \\ \overline{W}_m \cap \overline{U}_n &= \emptyset \quad \text{for } n>m. \end{aligned} \tag{3}$$

Since Y is normal this is possible, because the union of all the sets that $\overline{W}_m$ have, by (3), to avoid is a closed set (cf. lemma 3.2) which is disjoint to $\overline{U}_m$. By (3) the sets W_n satisfy (2).

Now put

$$W = \bigcup_n W_n.$$

Then, since $\{U_n\}$ covers B, W is an open neighborhood of B. By (2), $\{W_n\}$ is star-finite, hence locally finite in W. Choose an open set V such that

$$B \subset V \text{ and } \overline{V} \subset W$$

and put

$$V_1 = U_1 \cup (Y - B),$$

$$V_n = U_n \cup [(W_n \cap V) - B] \quad \text{for } n > 1.$$

Then $\beta = \{V_n\}$ is a covering of Y. By lemma 7.1 each V_n is open. Since $V_n \subset W_n$ for $n > 1$, $\{V_n, n > 1\}$ is locally finite in W, and since $V_n \subset \overline{V}$, which is closed in Y, $\{V_n, n > 1\}$ is locally finite in Y. Thus β is locally finite.

Lemma 7.3 (C. H. Dowker). *Let (Y, B) be a collectionwise normal pair and $\{U_\lambda\}$ a locally finite open covering of B. Then there is an open covering $\beta = \{V_\lambda\}$ of Y such that*

a) $V_\lambda \cap B \subset U_\lambda$ *for each* λ,
b) β *is locally finite.*

For the proof see [11].

Lemma 7.4. *Let (Y, B) be a fully normal pair and $\alpha = \{U_\lambda\}$ a locally finite open covering of B. Then there is an open covering $\beta = \{V_\lambda\}$ of Y such that*

a) $U_\lambda = V_\lambda \cap B$ *for each* λ,
b) β *is locally finite.*

Proof. Since α is locally finite in B it is locally finite in Y. Hence there is an open covering $\beta' = \{V'_\mu\}$ of Y such that each V'_μ meets only a finite number of elements of α. Since Y is fully normal we can take an open star-refinement $\gamma = \{W_\nu\}$ of β'.

Choose some $\lambda_0 \in \{\lambda\}$ and define

$$V_{\lambda_0} = U_{\lambda_0} \cup (Y - B),$$

$$V_\lambda = U_\lambda \cup [\mathrm{St}(U_\lambda, \gamma) - B] \quad \text{for } \lambda \neq \lambda_0.$$

Then $\beta = \{V_\lambda\}$ is a covering of Y. Clearly a) is satisfied. Since each $\mathrm{St}(U_\lambda, \gamma)$ is open we obtain from lemma 7.1 that V_λ is open. To prove b) it will be sufficient to show, that each $W_\nu \in \gamma$ meets only a finite number of sets V_λ. But if $\lambda \neq \lambda_0$,

$$W_\nu \cap V_\lambda \neq \emptyset$$

implies

$$W_\nu \cap \mathrm{St}(U_\lambda, \gamma) \neq \emptyset,$$

or, what is the same,

$$\mathrm{St}(W_\nu, \gamma) \cap U_\lambda \neq \emptyset. \tag{4}$$

Since $\gamma = \{W_\nu\}$ is a star-refinement of $\{V'_\mu\}$, there is a V'_μ such that

$$\mathrm{St}(W_\nu, \gamma) \subset V'_\mu. \tag{5}$$

By (4) and (5):

$$V'_\mu \cap U_\lambda \neq \emptyset,$$

which for fixed μ is true for at most a finite number of λ's.

This proves lemma 7.4.

For the case of a metric space Y, another proof has been given by C. H. Dowker ([10] p. 643).

Retraction and extension of mappings

8. We shall show in §§ 8–10 that if Q is any one of the classes β–ι the only difference between the two concepts ANR(Q) and NES(Q) is that an ANR(Q) belongs to Q. Among Q-spaces the two concepts are identical.

By definition any ANR(Q) belongs to Q. Therefore let X be a Q-space. Then if X is an NES(Q) it is an ANR(Q) (cf. 2.11). We want to prove the converse.

Theorem 8.1. *Let Q be any of the classes β–ι. Then any ANR(Q) is an NES(Q).*

The analogous theorem on AR(Q) and ES(Q) is also true.

Theorem 8.2. *Let Q be any of the classes β–ι. Then any AR(Q) is an ES(Q).*

We only prove theorem 8.1. The proof of theorem 8.2 is similar.

Proof of theorem 8.1. We have to consider all classes β–ι. For many of these classes the theorem has already been proved by other authors.

The theorem was first proved by Borsuk ([6] p. 224) in the compact metric case. The separable metric case was proved by Kuratowski ([21] p. 276). For a simple proof of this case see [13] p. 273. The case of all metric spaces is proved in a recent paper by Dugundji ([12] p. 363).

Consider now the non-metric cases. The compact case can be proved by Borsuk's method for the compact metric case (cf. Saalfrank [26] p. 97). However all our non-metric cases can be proved by one method. This method was used by the author ([14] p. 376) to prove the normal case. The method is as follows.

Let X be an ANR(Q), (Y, B) any Q-pair and $f: B \to X$ any mapping. We want to find a neighborhood extension of f. In order to use the retract property of X we construct a new space Z containing X as a closed subset.

The space Z is the identification space (cf. [1] p. 64) obtained from the free union $X \cup Y$ of X and Y by identifying each $y \in B$ with $f(y) \in X$. There are two natural mappings $j: X \to Z$ and $k: Y \to Z$. A set O is open in Z if and only if $j^{-1}(O)$ and $k^{-1}(O)$ are open. Since j is a homeomorphism into Z, we can identify X with $j(X) \subset Z$ so that X is a subset of Z. The mapping $k \mid Y - B$ is a homeomorphism onto $Z - X$. X is closed in Z.

The mapping $k: Y \to Z$ is an extension of $f: B \to X$ to Y relative to Z. If we prove that Z is a Q-space, it would follow, since X is an ANR(Q), that X is a neighborhood retract of Z. Let $r: U \to X$ be the neighborhood retraction. Then the function $F: k^{-1}(U) \to X$, defined by

$$F(y) = rk(y) \quad \text{for } y \in k^{-1}(U),$$

is a neighborhood extension of f.

Hence, in order to complete the proof of theorem 8.1, we shall show that the space Z constructed above is a Q-space. This will be done in the following two paragraphs.

9. We need some open sets in Z. They will be constructed in the following way.

Lemma 9.1. *Let U be an open set in X. Hence $k^{-1}(U)$ is open in B. Let V be some open set in Y such that $V \cap B = k^{-1}(U)$. Then*

$$W = U \cup k(V)$$

is open in Z.

Proof. Since

$$k(V \cap B) = k\,k^{-1}(U) \subset U,$$

we have

$$W = U \cup k(V - B).$$

Therefore, since $k \mid Y - B$ is a $1-1$-mapping,

$$k^{-1}(W) = k^{-1}(U) \cup (V - B) = V, \tag{1}$$

and, since $k(V - B) \subset Z - X$,

$$j^{-1}(W) = W \cap X = U \cap X = U. \tag{2}$$

But U is open in X and V is open in Y. Hence (1) and (2) imply that W is open. This proves lemma 9.1.

Now let us start with the case of all normal spaces. The proof of this case was made by the author in [14] p. 376. (We do not need to repeat the proof here since the methods used in it will be found below in the proof of the collectionwise normal case.) Hence theorem 8.1 is proved for normal spaces. This also shows that, in the remaining cases, Z is at least normal.

Let us use this last remark. Suppose X and Y are Lindelöf spaces. Then $X \cup Y$ is also a Lindelöf space. Z is the image of $X \cup Y$ under a continuous mapping. Hence, as is easily shown, Z has the Lindelöf property. But we know that Z is normal, hence also regular. Thus Z is a Lindelöf space.

Similarly, when X and Y are compact, Z is compact.

There now remains two cases: the collectionwise normal case and the fully normal case.

Let X and Y be collectionwise normal and let $\{A_\lambda\}$ be any locally finite collection of mutually disjoint closed sets in Z. Then using lemma 4.2 we can find open subsets U_λ of X such that

$$A_\lambda \cap X \subset U_\lambda,$$

$$\{\overline{U}_\lambda\} \text{ is locally finite,}$$

$$\text{the sets } \overline{U}_\lambda \text{ are mutually disjoint.}$$

Then $\{k^{-1}(A_\lambda \cup \overline{U}_\lambda)\}$ is a locally finite collection of mutually disjoint closed sets in Y. Hence we can find mutually disjoint open sets O_λ in Y such that

$$O_\lambda \supset k^{-1}(A_\lambda \cup \overline{U}_\lambda).$$

By lemma 7.1 we have, since $O_\lambda \supset k^{-1}(U_\lambda)$, that the set

$$V_\lambda = k^{-1}(U_\lambda) \cup (O_\lambda - B)$$

is open in Y. We have

$$k^{-1}(A_\lambda) \subset k^{-1}(U_\lambda) \cup (k^{-1}(A_\lambda) - \dot{B}) \subset$$

$$\subset k^{-1}(U_\lambda) \cup (O_\lambda - B) = V_\lambda.$$

Hence

$$W_\lambda = U_\lambda \cup k(V_\lambda)$$

contains A_λ. By lemma 9.1, W_λ is open. Finally the sets W_λ are mutually disjoint. For let $\lambda \neq \lambda_1$. Then if

$$W_\lambda \cap W_{\lambda_1} \cap X \neq \emptyset,$$

we have

$$U_\lambda \cap U_{\lambda_1} \neq \emptyset,$$

which is impossible, since the sets U_λ are mutually disjoint, and if

$$W_\lambda \cap W_{\lambda_1} \cap (Z - X) \neq \emptyset,$$

we have

$$(k(V_\lambda) - X) \cap (k(V_{\lambda_1}) - X) \neq \emptyset$$

and therefore

$$V_\lambda \cap V_{\lambda_1} \neq \emptyset$$

which is impossible, since the sets V_λ are mutually disjoint.

Thus the collection $\{W_\lambda\}$ shows that Z is collectionwise normal.

10. Finally we consider the fully normal case. We need the following lemma.

Lemma 10.1. *Let X and Y be fully normal spaces and let $\{U_\lambda\}$ be any locally finite open covering of X. Then there is in Z a locally finite open collection $\{W_\lambda\}$ such that*

$$U_\lambda = W_\lambda \cap X.$$

If we already knew that Z is fully normal this would be a consequence of lemma 7.4.

Proof. The given covering $\alpha = \{U_\lambda\}$ is locally finite in X. Hence there is an open covering $\alpha' = \{U'_\mu\}$ of X such that each U'_μ meets only a finite number of elements of α. We may assume that α' is locally finite.

Once more, since α' is locally finite, there is an open covering $\alpha'' = \{U''_\nu\}$ of X such that each U''_ν meets only a finite number of elements of α'.

Since $B = k^{-1}(X)$, we have that $\{k^{-1}(U_\lambda)\}$, $\{k^{-1}(U'_\mu)\}$, and $\{k^{-1}(U''_\nu)\}$ are open coverings of B. Each set $k^{-1}(U'_\mu)$ meets only a finite number of sets $k^{-1}(U_\lambda)$, and each set $k^{-1}(U''_\nu)$ meets only a finite number of sets $k^{-1}(U'_\mu)$.

Each $k^{-1}(U'_\mu)$ is open in B. Put

$$V'_\mu = k^{-1}(U'_\mu) \cup (Y - B).$$

Then $\beta = \{V'_\mu\}$ is an open covering of the fully normal space Y. Let us take an open star-refinement $\gamma = \{G_\varkappa\}$ of β. Set

$$V_\lambda = k^{-1}(U_\lambda) \cup [\mathrm{St}(k^{-1}(U_\lambda), \gamma) - B].$$

Since $\mathrm{St}(k^{-1}(U_\lambda), \gamma)$ is open in Y and contains $k^{-1}(U_\lambda)$, we obtain from lemma 7.1 that V_λ is open. Hence by lemma 9.1, the set

$$W_\lambda = U_\lambda \cup k(V_\lambda)$$

is open in Z. Since $U_\lambda = W_\lambda \cap X$, our lemma is proved when we have shown that $\{W_\lambda\}$ is locally finite. Hence for each $z \in Z$ we want to find a neighborhood meeting only a finite number of sets W_λ. There will be two cases.

If $z \in Z - X$, it will be sufficient to prove that $\{V_\lambda\}$ is locally finite in Y. Take some $G_\varkappa \in \gamma$. Then

$$G_\varkappa \cap V_\lambda \neq \emptyset \tag{1}$$

implies

$$G_\varkappa \cap \mathrm{St}(k^{-1}(U_\lambda), \gamma) \neq \emptyset,$$

so that

$$\mathrm{St}(G_\varkappa, \gamma) \cap k^{-1}(U_\lambda) \neq \emptyset. \tag{2}$$

But γ is a star-refinement of β. Thus

$$\mathrm{St}(G_\varkappa, \gamma) \subset V'_\mu$$

for some μ. Then (2) implies

$$[k^{-1}(U'_\mu) \cup (Y-B)] \cap k^{-1}(U_\lambda) \neq \emptyset$$

and, since $k^{-1}(U_\lambda) \subset B$,

$$k^{-1}(U'_\mu) \cap k^{-1}(U_\lambda) \neq \emptyset.$$

But for fixed μ this is possible only for a finite number of λ's. Hence for fixed $\varkappa$, (1) is true for at most a finite number of λ's. Hence, since $\{G_\varkappa\}$ covers Y, $\{V_\lambda\}$ is locally finite in Y.

If $z \in X$, we construct a neighborhood of z in Z as follows. Starting with some U''_ν containing z we put

$$V''_\nu = k^{-1}(U''_\nu) \cup [\mathrm{St}(k^{-1}(U''_\nu),\ \gamma) - B]$$

and

$$W''_\nu = k(V''_\nu) \cup U''_\nu.$$

Then W''_ν is a neighborhood of z (by lemmas 7.1 and 9.1). Now let

$$W''_\nu \cap W_\lambda \neq \emptyset. \tag{3}$$

If

$$W''_\nu \cap W_\lambda \subset X,$$

we have

$$U''_\nu \cap U_\lambda \neq \emptyset.$$

But U''_ν only meets a finite number of sets U'_μ, each meeting only a finite number of sets U_λ. Hence, since ν is fixed, λ is by (3) restricted to a finite number of values.

Therefore suppose that there is a point z_1 such that

$$z_1 \in W''_\nu \cap W_\lambda \quad \text{and} \quad z_1 \in Z - X.$$

Then

$$y_1 = k^{-1}(z_1) \in k^{-1}(W''_\nu) \cap k^{-1}(W_\lambda) = V''_\nu \cap V_\lambda \subset \mathrm{St}(k^{-1}(U''_\nu), \gamma) \cap \mathrm{St}(k^{-1}(U_\lambda), \gamma).$$

This implies

$$\mathrm{St}(\{y_1\}, \gamma) \cap k^{-1}(U''_\nu) \neq \emptyset,$$

$$\mathrm{St}(\{y_1\}, \gamma) \cap k^{-1}(U_\lambda) \neq \emptyset.$$

But γ is a star-refinement of β. Hence

$$\mathrm{St}(\{y_1\}, \gamma) \subset V'_\mu = k^{-1}(U'_\mu) \cup (Y-B)$$

for some μ. We obtain

$$k^{-1}(U'_\mu) \cap k^{-1}(U''_\nu) \neq \emptyset,$$

$$k^{-1}(U'_\mu) \cap k^{-1}(U_\lambda) \neq \emptyset.$$

Again we see that since ν is fixed, λ is by (3) restricted to a finite number of values.

Hence $\{W_\lambda\}$ is locally finite. This proves lemma 10.1.

Now we use lemma 10.1 to prove the fully normal case of theorem 8.1.

Suppose X and Y are fully normal. Let $\alpha = \{O_\varkappa\}$ be any open covering of Z. We want to show that there is a locally finite open refinement of α.

Consider the open covering $\{O_\varkappa \cap X\}$ of X. Since X is fully normal there is a locally finite open refinement $\{U_\lambda\}$ of $\{O_\varkappa \cap X\}$. For each U_λ choose some $O_{\varkappa_\lambda}$ such that $U_\lambda \subset O_{\varkappa_\lambda}$. By lemma 10.1 there is a locally finite open collection $\{W_\lambda\}$ in Z such that $U_\lambda = W_\lambda \cap X$. We may assume that $W_\lambda \subset O_{\varkappa_\lambda}$ otherwise replacing W_λ by $W_\lambda \cap O_{\varkappa_\lambda}$.

Put

$$W = \bigcup_\lambda W_\lambda.$$

This is an open set in Z containing X. Hence $k^{-1}(W)$ is open in Y and contains B. Y is normal. Hence we can take an open set V^* in Y such that

$$V^* \supset Y - k^{-1}(W),$$
$$\overline{V}^* \cap B = \emptyset.$$

Since $\overline{V}^*$ is a closed subset of Y, $\overline{V}^*$ is fully normal. Hence there is a locally finite open refinement $\{V_\mu^*\}$ of the open covering

$$\{k^{-1}(O_\varkappa) \cap \overline{V}^*\}$$

of $\overline{V}^*$. V_μ^* is open in $\overline{V}^*$, hence $V_\mu^* \cap V^*$ is open in $Y - B$. Put

$$W_\mu^* = k(V_\mu^* \cap V^*).$$

Since $k \mid Y - B$ is topological onto the open subset $Z - X$ of Z, W_μ^* is open in Z. The collection $\{V_\mu^* \cap V^*\}$ is locally finite in $\overline{V}^*$. Hence $\{W_\mu^*\}$ is locally finite in $k(\overline{V}^*)$, which is closed in Z. Thus $\{W_\mu^*\}$ is locally finite in Z.

Now consider the collection β consisting of all the sets W_λ and all the sets W_μ^*. β is a covering of Z. For

$$\bigcup_\mu W_\mu^* = k(\bigcup_\mu (V_\mu^* \cap V^*)) = k(V^*) \supset k(Y - k^{-1}(W)) = Z - W.$$

Since all the sets W_λ and all the sets W_μ^* are open in Z, β is an open covering. Finally β is locally finite, for $\{W_\lambda\}$ and $\{W_\mu^*\}$ are locally finite collections in Z.

As is easily verified, β is a refinement of α. Hence β is a locally finite refinement of α. This proves that Z is fully normal.

This completes the proof of theorem 8.1.

Remark 10.2. In the metric (but not compact) cases the above method cannot be used since in general the space Z will not be metrizable. However, HAUSDORFF ([16], cf. also [2] p. 16) has shown that there is a metric on Z, giving Z another topology, and a mapping $k: Y \to Z$ such that:

a) X is imbedded as a closed subset of Z.
b) k is an extension of $f: B \to X$ relative to Z.
c) $k \mid Y - B$ is a topological mapping onto $Z - X$.

If we use this metrizable topology on Z our method works also in the metric cases.

A simple proof of this theorem of HAUSDORFF for the separable metric case was given by KURATOWSKI [22].

11. For Tychonoff spaces theorems 8.1 and 8.2 cannot be true, since there is no NES(Tychonoff) (or ES(Tychonoff)) which is Hausdorff, except the space consisting of a single point (cf. lemma 2.7). However we have the following theorems, proved by Hu ([17] p. 1052).

Theorem 11.1. *Any ANR(Tychonoff) is an NES(normal).*

Theorem 11.2. *Any AR(Tychonoff) is an ES(normal).*

These theorems can also be proved by the method used above. For if X is an ANR(Tychonoff) and (Y, B) is a normal pair we can define the space Z as in § 8. Then we only have to prove that Z is a Tychonoff space. But this is easily done.

Contractibility

12. For compact spaces BORSUK ([6] p. 229) proved that any AR is contractible and that conversely any contractible ANR is an AR. In this paragraph we shall take up the study of the corresponding relationships between AR(Q), ANR(Q), and contractibility for other classes Q.

We need the following lemma. Let I denote the closed interval $0 \leqq t \leqq 1$, and denote by $X \times I$ the topological product of a space X and I.

Lemma 12.1. *Let Q be any of the classes δ–ι. Then, if X is a Q-space, $X \times I$ is a Q-space.*

In fact, this is well-known if Q is any of the classes ζ–ι. It was proved for fully normal spaces by DIEUDONNÉ ([8] p. 70) and is proved for Lindelöf spaces in an analogous way.

Remark 12.2. This lemma also holds for Tychonoff spaces. Whether it holds for normal spaces or for collectionwise normal spaces is still an open question.

Theorem 12.3. *Let all Q-spaces be normal. Then any contractible NES(Q) is an ES(Q).*

Proof. Let X be an NES(Q) which is contractible. Then there is a homotopy

$$h: X \times I \to X$$

such that for any $x \in X$

$$h(x, 0) = x,$$

$$h(x, 1) = x_0,$$

where x_0 is some point in X. Let (Y, B) be any Q-pair and $f: B \to X$ any given mapping. Since X is an NES(Q) we have a neighborhood extension $g: O \to X$ of f to some open set $O \supset B$. Since Y is normal we can take a mapping $e: Y \to I$ such that

$$e(y) = 0 \text{ for } y \in B,$$

$$e(y) = 1 \text{ for } y \in \text{ some open set containing } Y - O.$$

Define $F: Y \to X$ by

$$F(y) = h(g(y), e(y)) \text{ for } y \in O,$$

$$F(y) = x_0 \qquad\qquad \text{for } y \in Y - O.$$

Then F is easily proved to be continuous. Since $F|B = f$, we conclude that X is an ES(Q).

Theorem 12.4. *Let Q be any of the classes $\delta-\iota$. Then any AR(Q) is contractible.*

Proof. Let X be an AR(Q). By lemma 12.1, $X \times I$ is a Q-space. Consider the closed set $B = X \times \{0\} \cup X \times \{1\}$ of $X \times I$. Define $f: B \to X$ by

$$f(x, 0) = x,$$

$$f(x, 1) = x_0,$$

where x_0 is some point of X. Since, by theorem 8.2, X is an ES(Q) there is an extension $h: X \times I \to X$ of f. The existence of h shows that X is contractible.

Theorem 12.5. *Let Q be the class of normal spaces or the class of collectionwise normal spaces. Then any fully normal AR(Q) is contractible.*

Proof. For such a space is an AR(fully normal) and hence contractible.

Let us sum up the main results in this paragraph in

Theorem 12.6. *Let Q be any of the classes $\beta-\iota$. Then a fully normal space is an AR(Q) if and only if it is a contractible ANR(Q).*

Remark 12.7. This is not true for the class of Tychonoff spaces. For we shall see in example 17.4 that a real line is a contractible ANR(Tychonoff) which is not an AR(Tychonoff).

Different classes Q

13. Let, for a while, Q and Q_1 be two classes out of $\alpha-\iota$, such that $Q \subset Q_1$. Let X be an ANR(Q). Under what conditions is it true that X is also an ANR(Q_1)? And, if X is an AR(Q), when is it an AR(Q_1)? We shall solve this problem for some classes Q and Q_1. In particular the problem will be solved when Q is any of the metric classes (i.e. $\eta-\iota$) and Q_1 is any of the classes that contains all metric spaces (i.e. $\alpha-\delta$). For the solution see theorems 17.1 and 17.2.

If X is an ANR(Q) then $X \in Q$ and hence $X \in Q_1$. Thus, if all Q_1-spaces are normal, X is an ANR(Q_1) if and only if X is an NES(Q_1) (see theorem 8.1).

Theorem 13.1. *Let all Q-spaces be fully normal, and let all Q_1-spaces be normal. Then an AR(Q) is an AR(Q_1) if and only if it is an ANR(Q_1).*

Proof. For, by theorem 12.4, an AR(Q) is contractible. Hence theorem 13.1 follows from theorem 12.6.

Therefore, if we know which ANR(Q)'s are ANR(Q_1)'s, we also know which AR(Q)'s are AR(Q_1)'s.

When Q_1 is the class of Tychonoff spaces the problems for ANR's and for AR's are different and will be treated separately.

Theorem 13.2. *Any ANR(comp. metr.) is an ANR(normal).*

Proof. For an ANR(comp. metr.) is a neighborhood retract of the Hilbert cube I_ω, and I_ω is an ES(normal). (Cf. 2.15 and example 2.19.)

Theorem 13.3. *Any ANR(compact) is an ANR(normal)* ([26] p. 95).

Proof. For an ANR(compact) is a neighborhood retract of a Tychonoff cube, and any cube is an ES(normal). (We mean by a Tychonoff cube the topological product of uncountably many closed intervals.)

Thus, for these two classes Q, any ANR(Q) is an ANR(Q_1) for any $Q_1 \supset Q$ such that all Q_1-spaces are normal. In fact, this is true even if Q_1 is the class of Tychonoff spaces (see theorem 16.2).

Theorem 13.4. *Any ANR(sep. metr.) is an ANR(metric).*

Proof. Fox ([13] p. 273) proved that an ANR(sep. metr.) X is an NES(sep. metr.). His proof can be trivially changed so that it shows that X is also an NES(metric).

Thus for metric spaces there is essentially one concept: ANR(metric). An ANR(sep. metr.) is a separable ANR(metric) and an ANR(comp. metr.) is a compact ANR(metric).

14. In all cases considered so far any ANR(Q) is an ANR(Q_1). However this is not in general true. The author proved in [14] p. 378 that a necessary and sufficient condition for an ANR(sep. metr.) to be an ANR(normal) is that it is an absolute G_δ. This result can be strengthened in various ways. Several of these are due to C. H. Dowker [11].

By an absolute G_δ we mean a metric space which, whenever imbedded in a metric space, is a G_δ, i.e. a countable intersection of open sets. All locally compact metric spaces are absolute G_δ's (cf. lemma 16.4). The class of all absolute G_δ's is known to be the same as the class of all topologically complete spaces, i.e. spaces which can be given a complete metric. (Cf. [20] Chapter 3.)

Theorem 14.1. *Any metric ANR(fully normal) is an absolute G_δ.*

Proof. This proof will be a modification of the proof of theorem 4.2 of [14] p. 378. Let X be a metric ANR(fully normal) and let X be a subset of any metric space Y. We have to prove that X is a G_δ in Y.

We construct a new space Z. The points of Z shall be in 1–1-correspondence with the points of Y. Let $h(z) \in Y$ be the point corresponding to $z \in Z$ under this 1–1-correspondence. Let $X' = h^{-1}(X)$. We define a topology on Z by taking as its open sets all sets of the form

$$h^{-1}(O) \cup A,$$

where O is any open subset of Y and A any subset of $Z - X'$. Then Z is Hausdorff. Let us show that it is fully normal.

Let $\alpha = \{U_\lambda\}$ be an open covering of Z. Each U_λ is of the form

$$U_\lambda = h^{-1}(O_\lambda) \cup A_\lambda,$$

where O_λ is an open subset of Y and $A_\lambda \subset Z - X'$. The set

$$U = \bigcup_\lambda O_\lambda$$

is open in Y and $X \subset U$. U is a metric space, hence fully normal. Thus we can take an open star-refinement $\{V_\mu\}$ of $\{O_\lambda\}$. Then $\{h^{-1}(V_\mu)\}$ is an open collection in Z, covering $h^{-1}(U) \supset X'$. Complete this collection to an open covering β of Z by adding the collection of points of $Z - h^{-1}(U)$, each such point being an open set. That β is a star-refinement of α is easily verified. Since α is arbitrary, this proves that Z is fully normal.

Now X' is homeomorphic to X, hence an ANR(fully normal). Since X' is closed in Z and Z is fully normal, X' is a neighborhood retract of Z.

That X is a G_δ in Y now follows as in [14] p. 379. A simplification of the arguments has been given in [11].

Theorem 14.2 (C. H. Dowker). *Any ANR(metric) which is an absolute G_δ is an ANR(coll. normal).*

For the proof see [11]. We mention at the same time the following fact.

Theorem 14.3. *Any Banach space is an AR(coll. normal).*[1]

Proof. This follows here from theorem 14.2. For it is known that a Banach space is an AR(metric) (cf. [12] p. 357), and a Banach space has a complete metric and is therefore an absolute G_δ. However, when proving theorem 14.2, C. H. Dowker needs theorem 14.3 for the case of a generalized (i.e. not necessarily separable) Hilbert space, and he gives a direct proof for this case. This proof can be applied to an arbitrary Banach space.

Theorem 14.4. *A collection of non-void mutually disjoint open sets in an NES(normal) is at most countable.*

Proof. Let $\{O_\lambda\}$ be a collection of non-void disjoint open sets in the NES (normal) X. Suppose $\Lambda = \{\lambda\}$ is uncountable.

Bing ([4] p. 184) has shown that for every uncountable set $\Lambda = \{\lambda\}$ there is a normal space Y with a locally finite collection $\{y_\lambda\}$ of disjoint points which do not have disjoint neighborhoods. The space Y is therefore not collectionwise normal.

Let now B be the subset of this space Y consisting of all the points y_λ. Then, since $\{y_\lambda\}$ is locally finite, B is closed in Y. Define a function $f: B \to X$ by selecting for each y_λ some point $f(y_\lambda) \in O_\lambda$. Since $\{y_\lambda\}$ is locally finite, f is continuous. Hence there is a neighborhood extension of f, say $F: O \to X$. But the sets $F^{-1}(O_\lambda)$ are disjoint neighborhoods of the points y_λ. This is a contradiction, which proves theorem 14.4.

Theorem 14.5. *Any fully normal NES(normal) is a Lindelöf space.*

Proof. Let X be a fully normal NES(normal) and $\alpha = \{U_\lambda\}$ an open covering of X. Let $\beta = \{V_\mu\}$ be an open star-refinement of α. We may assume that no V_μ is void. Using Zorn's lemma (cf. [23] p. 5) we take a collection $\gamma = \{V_{\mu_\varkappa}\}$ of disjoint elements of β such that for each $V_\mu \in \beta$ there is an element of γ meeting V_μ. Then

$$\delta = \{\mathrm{St}(V_{\mu_\varkappa}, \beta)\}$$

is an open covering of X. But from theorem 14.4 we have that γ is countable. Now δ, which is also countable, is a refinement of α. This proves that X is a Lindelöf space (cf. lemma 6.2).

[1] That any Banach space is an AR(fully normal) was known to Arens ([2] p. 18).

Theorem 14.6 (C. H. DOWKER). *Any metric NES(normal) is separable.*

Proof. For by theorem 14.5 it is a metric Lindelöf space. Hence by lemma 6.5 it is separable.

Theorem 14.7. *Any ANR(sep. metr.) which is an absolute G_δ is an ANR (normal).*

For the proof see [14] p. 380.

15. Now let Q_1 be the class of Tychonoff spaces. Then we cannot apply theorem 12.6, but have to consider ANR(Tychonoff) and AR(Tychonoff) separately.

We need some preliminaries on Tychonoff spaces. As is well-known (cf. [23] p. 29) the Tychonoff spaces are those spaces which can be imbedded in a suitable Tychonoff cube. By a Tychonoff cube we mean the topological product of uncountably many closed intervals. Let $\Lambda=\{\lambda\}$ be the uncountable index set, and choose for each λ a closed interval $I_\lambda=\{t_\lambda \mid 0\leqq t_\lambda \leqq 1\}$. Then a point of the Tychonoff cube I', which is the topological product of the intervals I_λ, can be written in the form $\{t_\lambda\}$, where for each λ, t_λ is a number of the interval I_λ. Let o denote the point of I' having $t_\lambda=0$ for all λ's, and denote for each λ by I'_λ the set of all points in I' for which t_λ is arbitrary but $t_{\lambda_1}=0$ for $\lambda_1 \neq \lambda$. I'_λ is a closed line segment in I'.

Lemma 15.1. *Let O'_n be a sequence of neighborhoods of o. Then*

$$\bigcap_{n=1}^{\infty} O'_n$$

contains all segments I'_λ except for at most a countable number of indices λ. Hence, since Λ is uncountable, there is a λ such that

$$I'_\lambda \subset O'_n \text{ for every } n.$$

Proof. This follows from the fact that each neighborhood of o contains all segments I'_λ except for at most a finite number of indices λ.

Lemma 15.2. *The space $I'-\{o\}$ is not normal.*

Proof. Let us change the notations slightly. Denote the Tychonoff cube by $I\times I'$ where $I=\{t \mid 0\leqq t\leqq 1\}$ and I' is a Tychonoff cube. Let $o\in I'$ be as above. We want to show that the space

$$I\times I'-\{0\}\times\{o\}$$

is not normal.

Consider the two closed subsets

$$A=(I-\{0\})\times\{o\} \quad\text{and}\quad B=\{0\}\times(I'-\{o\}).$$

They are disjoint. Let us show that for any neighborhood U of A we have that $\bar{U}$ meets B.

In fact let $t_n\to 0$ be a sequence of positive numbers in I $\left(\text{e.g. } t_n=\frac{1}{n}\right)\cdot$ Then, since U is a neighborhood of $\{t_n\}\times\{o\}$, U contains a set of the form $\{t_n\}\times O'_n$, where O'_n is a neighborhood of o in I'. Using lemma 15.1, we get an I'_λ such that $I'_\lambda\subset O'_n$ for each n. Then

$$\{t_n\} \times I'_\lambda \subset U \text{ for each } n.$$

Hence

$$\{0\} \times (I'_\lambda - \{o\}) \subset \overline{U},$$

so that $\overline{U}$ meets B. This proves lemma 15.2.

Remark 15.3. The arguments of this proof are essentially the same as those used by TYCHONOFF in [32] p. 553. Instead of the product $I \times I'$ he considers the product of two spaces in which the points are some ordinals. (Cf. [18] p. 154.)

Now let us take up the question: which metric spaces are ANR(Tychonoff)? This is answered by theorems 15.4 and 16.6. The corresponding problem for AR(Tychonoff) is solved by theorems 15.5 and 16.3.

Theorem 15.4. *Any metric ANR(Tychonoff) is separable and locally compact.*

Proof. Let X be a metric ANR(Tychonoff). Then, by theorem 11.1, X is an NES(normal) and therefore, by theorem 14.6, separable.

Imbed X in a Hilbert cube I_ω. Let I' be a Tychonoff cube. In the product

$$I_\omega \times I',$$

which is again a Tychonoff cube, we consider the set

$$T = X \times \{o\} \cup I_\omega \times (I' - \{o\}).$$

T is a Tychonoff space and $X \times \{o\}$ is a closed subset of T. Since $X \times \{o\}$ is homeomorphic to X and X is an ANR(Tychonoff), there is an open set O in T containing $X \times \{o\}$ and a retraction $r: O \to X \times \{o\}$.

Suppose that X is not locally compact. We assert that then there is a point $u \in I_\omega$ and a neighborhood O' of o in I' satisfying

$$u \in \overline{X} \text{ (closure in } I_\omega), \tag{1}$$

$$u \notin X, \tag{2}$$

$$\{u\} \times (O' - \{o\}) \subset O. \tag{3}$$

We shall prove below that (1), (2), and (3) lead to a contradiction.

Since X is not locally compact there is a point $x_0 \in X$ such that no neighborhood of x_0 in X is compact. We have $\{x_0\} \times \{o\} \subset O$. Since O is open in T, there is therefore a closed neighborhood U of x_0 in I_ω and a neighborhood O' of o in I' such that

$$\{x_0\} \times \{o\} \subset (U \times O') \cap T \subset O.$$

We intend to choose $u \in U$. Then (3) is immediately satisfied.

The set $U \cap X$ is a neighborhood of x_0 in X, hence it is not compact. Therefore $U \cap X$ cannot be closed in U, since U is compact. Thus we can take a $u \in U - X$ for which $u \in \overline{U \cap X}$. This point therefore satisfies (1) and (2).

Now to get a contradiction out of (1), (2), and (3) take for each $n = 1, 2, \ldots$

$$U_n = S\left(u, \frac{1}{n}\right) \cap X,$$

where $S\left(u, \frac{1}{n}\right)$ stands for the $\frac{1}{n}$-sphere of u in I_ω. From (2) we have

$$\bigcap_{n=1}^{\infty} U_n = \emptyset \tag{4}$$

and from (1) we see that no U_n is void. Hence let us take $x_n \in U_n$. Then $x_n \to u$. The set U_n is open in X, so that $r^{-1}(U_n \times \{o\})$ is open in T. Hence, since

$$\{x_n\} \times \{o\} \subset r^{-1}(U_n \times \{o\}),$$

we have a neighborhood O'_n of o in I' such that

$$\{x_n\} \times O'_n \subset r^{-1}(U_n \times \{o\}),$$

or

$$r(\{x_n\} \times O'_n) \subset U_n \times \{o\}. \tag{5}$$

Now apply lemma 15.1 to the sequence $O', O'_1, O'_2, \ldots$ We get an I'_λ contained in all sets $O', O'_1, O'_2, \ldots$ By (3) and (5) we obtain

$$\{u\} \times (I'_\lambda - \{o\}) \subset O, \tag{6}$$

$$r(\{x_n\} \times I'_\lambda) \subset U_n \times \{o\}. \tag{7}$$

For any $m > n$ we have by (7), since $U_m \subset U_n$,

$$r(\{x_m\} \times I'_\lambda) \subset U_n \times \{o\}.$$

Thus, since r is continuous and by (6) defined on $\{u\} \times (I'_\lambda - \{o\})$, and since $x_m \to u$,

$$r(\{u\} \times (I'_\lambda - \{o\})) \subset \overline{U}_n \times \{o\} \text{ for every } n \tag{8}$$

(closure in X). But $\overline{U}_n \subset U_{n-1}$, so that (8) contradicts (4). Thus theorem 15.4 is proved.

Theorem 15.5. *Any metric AR(Tychonoff) is compact.*

Proof. Let X be a metric AR(Tychonoff). As in the previous proof we see that X is separable. Imbed X in I_ω and consider the spaces $I_\omega \times I'$ and T as above. Since X is an AR(Tychonoff) we have a retraction $r: T \to X \times \{o\}$.

If X is not compact, X is not closed in I_ω. Take any point u satisfying (1) and (2). Then (3) is true for instance for $O' = I'$ (since $O = T$). The same contradiction as above now proves theorem 15.5.

16. Theorem 16.1. *Any AR(compact) is an AR(Tychonoff).*

Proof. Suppose X is any AR(compact). Let X be a closed subset of a Tychonoff space Z. Imbed Z in a Tychonoff cube I'. Then X, being compact, is closed in I'. But X is an AR(compact) and I' is compact. Therefore X is a retract of I'. Hence X is also a retract of Z.

Theorem 16.2. *Any ANR(compact) is an ANR(Tychonoff).*

Proof. This is proved in an analogous way.

Theorem 16.3 (Hu). *Any AR(comp. metr.) is an AR(Tychonoff)* ([17] p. 1053).

Proof. This follows from theorem 16.1. For by theorems 13.1 and 13.2 an AR(comp. metr.) is an AR(compact).

Lemma 16.4. *Let a locally compact space X be imbedded in a Hausdorff space X'. Let $\bar{X}$ denote the closure of X in X'. Then X is open in $\bar{X}$.* (Cf. [7] p. 69.)

Proof. Let $x \in X$ and let U be a compact neighborhood of x in X. That U is a neighborhood of x can be expressed by

$$x \notin \overline{X - U} \tag{1}$$

(closure in X'). Now, since U is compact, $\bar{U} = U$. Hence

$$\bar{X} = \overline{(X - U) \cup U} = \overline{X - U} \cup U,$$

so that

$$\bar{X} - X \subset \overline{X - U}. \tag{2}$$

From (1) and (2):

$$x \notin \overline{\bar{X} - X}.$$

Since x is arbitrary this implies that $\bar{X} - X$ is closed. Hence X is open in $\bar{X}$.

Remark 16.5. Among Tychonoff spaces this property characterizes the locally compact spaces. For then X' can be chosen compact, hence X is an open subset of the compact space $\bar{X}$.

Theorem 16.6. *Any separable, locally compact ANR(metric) is an ANR (Tychonoff).*

Proof. Let X be a separable ANR(metric). If X is compact it is an ANR (Tychonoff) by theorems 13.2 and 16.2. Let X be locally compact but not compact. Suppose X is a closed subset of any Tychonoff space Z.

We need the following spaces and sets:

a) A Tychonoff cube I', three subsets Z, X, $\bar{X}$.

b) A Hilbert cube I_ω, two subsets Y, $\tilde{Y}$.

They are defined as follows. Imbed Z in a Tychonoff cube I'. Let $\bar{X}$ denote the closure of X in I'. Then $\bar{X}$ is compact. Let further Y be a space homeomorphic to X. Use the well-known fact that a locally compact space Y can be imbedded in a compact space $\tilde{Y}$, such that $\tilde{Y} - Y$ consists of a single point, say $\tilde{y}$ (cf. [23] p. 23). Since Y is separable metric, $\tilde{Y}$ is separable metric and can therefore be imbedded in a Hilbert cube I_ω.

Let $h: X \to Y$ be a homeomorphism. Our theorem is proved, when we have shown that there is an extension of h to some neighborhood of X in Z.

First, extend h relative to $\tilde{Y}$ to a mapping $j: \bar{X} \to \tilde{Y}$ defined by

$$j(x) = h(x) \quad \text{for } x \in X,$$
$$j(x) = \tilde{y} \quad \text{for } x \in \bar{X} - X.$$

To prove that j is continuous let U be open in $\tilde{Y}$. Then if $\tilde{y} \in U$, $\tilde{Y} - U$ is a compact subset of Y. Hence

$$j^{-1}(\tilde{Y} - U) = h^{-1}(\tilde{Y} - U)$$

is compact. Thus $j^{-1}(U)$ is the complement of a compact subset of $\bar{X}$, hence open in $\bar{X}$. If $\tilde{y} \notin U$, $j^{-1}(U) = h^{-1}(U)$ is open in X. But X is open in $\bar{X}$ by lemma 16.4. Hence $j^{-1}(U)$ is open in $\bar{X}$. Thus j is continuous.

Secondly, extend j relative to I_ω to a mapping $k: I' \to I_\omega$. This is always possible since $\bar{X}$ is closed in I' and I_ω is an ES(normal) (cf. example 2.19).

Now Y, being homeomorphic to X, is an ANR(metric). The set $\tilde{Y} = Y \cup \{\tilde{y}\}$ is closed in I_ω so that Y is closed in $I_\omega - \{\tilde{y}\}$. Therefore we have a retraction $r: O \to Y$ of a set O, which is open in $I_\omega - \{\tilde{y}\}$, hence also in I_ω. The set $k^{-1}(O)$ is therefore open in I'. It contains X. Hence

$$rk \mid k^{-1}(O) \cap Z : k^{-1}(O) \cap Z \to Y$$

is a neighborhood extension in Z of $rk \mid X = h$. This proves theorem 16.6.

Remark 16.7. The fact that X is closed in Z was never used in this proof. That this assumption is superfluous can be seen directly from lemma 16.4. For since X is open in $\bar{X}$ (closure in Z), X is closed in some open set in Z, i.e. in a neighborhood of X in Z.

17. Now let us sum up the results about ANR(metric) and AR(metric) obtained in §§ 13–16.

Theorem 17.1. *Let X be an ANR(metric).*

a) *If X is an ANR(fully normal) (in particular if X is an ANR(coll. normal)), then X is an absolute G_δ.* (See 14.1.)

b) *If X is an absolute G_δ, X is an ANR(coll. normal) (hence also an ANR(fully normal)).* (See 14.2.)

c) *X is an ANR(normal) if and only if X is a separable absolute G_δ.* (See 14.1, 14.6, 14.7.)

d) *X is an ANR(Tychonoff) if and only if X is separable and locally compact.* (See 15.4, 16.6.)

Theorem 17.2. *Let X be an AR(metric).*

a) *If X is an AR(fully normal) (in particular if X is an AR(coll. normal)), then X is an absolute G_δ.* (See 14.1.)

b) *If X is an absolute G_δ, X is an AR(coll. normal) (hence also an AR(fully normal)).* (See 13.1, 14.2.)

c) *X is an AR(normal) if and only if X is a separable absolute G_δ.* (See 13.1, 14.1, 14.6, 14.7.)

d) *X is an AR(Tychonoff) if and only if X is compact.* (See 15.5, 16.3.)

Example 17.3. A closed interval is a compact AR(metric) and so it is an AR(Tychonoff). Also a Tychonoff cube is an AR(Tychonoff) (by theorem 16.1). (Cf. example 2.19.)

Example 17.4. A real line is a separable locally compact AR(metric). Hence it is also an absolute G_δ and therefore an AR(normal). It is an ANR(Tychonoff) and contractible. But it is not an AR(Tychonoff) since it is not compact.

Example 17.5. The topological product of a countable number of real lines is an AR(normal) (cf. 2.17). Hence it is an absolute G_δ. It is not locally compact, however, and therefore it is not an ANR(Tychonoff).

Example 17.6. Any Banach space is an AR(coll. normal). If it is not separable it is not an AR(normal).

Example 17.7. The author gave in [14] p. 381 an example of a space which is an AR(sep. metr.) but not an absolute G_δ. This is therefore not an AR(fully normal). Whether it is an AR(Lindelöf) is still an unsolved problem.

Example 17.8. It is easy to prove that if a product $X = X_1 \times X_2$ of Tychonoff spaces X_1 and X_2 is an ANR(Tychonoff) then so are also X_1 and X_2. Thus we see from example 17.5 that the product of uncountably many real lines is not an ANR(Tychonoff). This space is a Tychonoff space which is not normal ([29] p. 981). It is an ES(normal). (Cf. example 2.20.)

Example 17.9. Let X be the space $I' - \{o\}$ of lemma 15.2. It is not normal. It is an open subset of I' which is an ES(normal). Hence X is an NES(normal) (cf. 2.16). Since X is contractible it is an ES(normal) by theorem 12.3.

We assert that X is an ANR(Tychonoff). In fact this can be proved by the method used in the proof of 16.6. For X is locally compact, and adding the single point o to X we get the compact space I' which is an ES(normal). We leave the details to the reader. (Cf. example 23.3.)

Local NES (Q)

18. We now take up the following problem: Is the property of a space X to be an NES(Q) a local property? The answer is yes if all Q-spaces are fully normal. If Q is the class of collectionwise normal spaces or the class of normal spaces it is true when X satisfies some conditions. This problem has been studied for compact metric spaces by YAJIMA [35] and for separable metric spaces by the author ([15] p. 392).

First we need some facts on coverings and their nerves. Let $\alpha = \{U_\lambda\}$ be a covering of a space X. By the nerve of α we mean the abstract simplicial complex whose vertices are the sets U_λ and in which

$$\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$$

is a simplex if and only if

$$\bigcap_{i=1}^{n} U_{\lambda_i} \neq \emptyset.$$

This simplex will often be denoted by

$$(\lambda_1, \ldots, \lambda_n).$$

If σ and σ_1 are simplices of nerve α we mean by

$$\sigma \prec \sigma_1 \text{ or } \sigma_1 \succ \sigma$$

that σ is a proper face of σ_1. Let St σ denote the star of σ, i.e. the set of simplices σ_1 such that $\sigma_1 \succeq \sigma$.

Some properties of the covering α correspond to certain properties of nerve α. Thus α is star-finite if and only if nerve α is locally finite, i.e. the star of each vertex is a finite complex.

Now suppose that in nerve α the star of each vertex is finite dimensional. Then for each $U_\lambda \in \alpha$ there is an integer n_λ such that each point of U_λ belongs to at most n_λ elements of α. We call such a covering elementwise uniformly point-finite.

C. H. Dowker ([9] p. 209) has proved the following lemma.

Lemma 18.1. *Let α be a locally finite open covering of a normal space X. Then α has a locally finite open refinement which is elementwise uniformly point-finite.*

19. Definition 19.1. A space X is called a *local NES(Q)* if each point of X has a neighborhood which is an NES(Q).

This terminology is justified by 2.16. A local NES(Q) has an open covering by NES(Q)'s.

We give in §§ 21 and 22 the proofs of the following three theorems.

Theorem 19.2. *Let all Q-spaces be fully normal. Then any local NES(Q) is an NES(Q).*

Theorem 19.3. *Let all Q-spaces be collectionwise normal. Then any fully normal local NES(Q) is an NES(Q).*

Theorem 19.4. *Let all Q-spaces be normal. Then any Lindelöf space which is a local NES(Q) is an NES(Q).*

The proofs will be essentially the same for all three theorems. Since they are technically a little complicated, let us first give the main ideas.

Consider the simple case when X is the union of two open NES(Q)'s, say $X = O_1 \cup O_2$. Let (Y, B) be a Q-pair and $f: B \to X$ a mapping. Then we shall prove that f has a neighborhood extension (cf. [15] p. 392). This proof will be divided into two parts.

The first part consists of some preliminaries, which in this simple case are rather trivial. B is covered by the two open sets

$$f^{-1}(O_1) \text{ and } f^{-1}(O_2),$$

and Y is covered by the two open sets

$$f^{-1}(O_1) \cup (Y - B) \text{ and } f^{-1}(O_2) \cup (Y - B).$$

Since Y is normal there is a closed refinement of this covering, say $Y = Y_1 \cup Y_2$. Put

$$B_1 = Y_1 \cap B \text{ and } B_2 = Y_2 \cap B.$$

To avoid a trivial case, let $B_1 \cap B_2 \neq \emptyset$. We have

$$f(B_1) \subset O_1 \text{ and } f(B_2) \subset O_2.$$

In the second part we use the fact that O_1 and O_2 are NES(Q)'s. We extend $f | B_1$ relative to O_1 to a mapping $F_1: U_1 \to O_1$, where U_1 is a neighborhood of

B_1 in Y_1. Similarly we take a neighborhood extension $F_2: U_2 \to O_2$ of $f \mid B_2$ relative to O_2.

Now if

$$(1) \qquad F_1(u) = F_2(u) \text{ for } u \in U_1 \cap U_2,$$

we could define a mapping $F: U_1 \cup U_2 \to X$ uniquely by

$$F(u) = F_1(u) \text{ for } u \in U_1,$$
$$F(u) = F_2(u) \text{ for } u \in U_2.$$

Then F would be the sought-for neighborhood extension of f.

However, in order to get the equality (1) we have to start by taking a neighborhood extension of $f \mid B_1 \cap B_2$ in $Y_1 \cap Y_2$ relative to $O_1 \cap O_2$. The whole process can be described as follows.

The nerve of the covering $\{Y_i\}$ consists of three elements, the 1-simplex (1, 2) and the 0-simplices (1) and (2). Starting with the set $Y_1 \cap Y_2$, which corresponds to the 1-simplex (1, 2) we take a neighborhood extension of $f \mid B_1 \cap B_2$ in $Y_1 \cap Y_2$ relative to $O_1 \cap O_2$. Thereafter define neighborhood extensions in the two sets

$$Y_1 - (Y_1 \cap Y_2) \text{ and } Y_2 - (Y_1 \cap Y_2),$$

which correspond to the 0-simplices (1) and (2). This will be done in such a way that the final function defined by the three extensions, is continuous.

In the general case we shall have instead of the two sets Y_1 and Y_2 a closed covering $\{Y_\lambda\}$ of a closed neighborhood $\tilde{Y}$ of B in Y. This covering $\{Y_\lambda\}$ will be locally finite and elementwise uniformly point-finite. The first part of the proofs of theorems 19.2, 19.3, and 19.4 will be to construct this covering. This will be done in § 21.

The second part will be an induction. For each $\sigma \in$ nerve $\{Y_\lambda\}$ we will make a neighborhood extension. These extensions will be taken in such an order that the extension corresponding to a simplex σ is taken before the extension corresponding to any face of σ. Since $\{Y_\lambda\}$ is elementwise uniformly point-finite, $\operatorname{St}\sigma$ is finite dimensional for each $\sigma \in$ nerve $\{Y_\lambda\}$. Because of this the induction will work.

This second part will be found in § 22. In § 20 we prove some lemmas, used in §§ 21 and 22.

20. Lemma 20.1. *Let (Y, B) be a normal pair, $\{U_\lambda\}$ a locally finite open covering of B and $\{V_\lambda\}$ a locally finite open covering of Y such that*

$$V_\lambda \cap B \subset U_\lambda.$$

Then there is a closed neighborhood $\tilde{Y}$ of B in Y and a closed covering $\{Y_\lambda\}$ of $\tilde{Y}$ such that

a) *$\{Y_\lambda\}$ is locally finite,*

b) *$Y_\lambda \cap B \subset U_\lambda$,*

c) *$Y_{\lambda_1} \cap \ldots \cap Y_{\lambda_n} \neq \emptyset$ implies $U_{\lambda_1} \cap \ldots \cap U_{\lambda_n} \neq \emptyset$ for any finite collection of indices.*

Proof. We shall take Y_λ as a subset of V_λ. Then a) and b) are satisfied. Y is normal and $\{V_\lambda\}$ is locally finite. Apply lemma 3.4. We get an open covering $\{W_\lambda\}$ of Y such that

$$\overline{W}_\lambda \subset V_\lambda.$$

Take any point $y \in Y$. Since $\{\overline{W}_\lambda\}$ is locally finite, y is contained in only a finite number of sets $\overline{W}_\lambda$, say $\overline{W}_{\lambda_1}, \ldots, \overline{W}_{\lambda_n}$. Let us call y admissible if for the corresponding sets $U_{\lambda_1}, \ldots, U_{\lambda_n}$ we have

$$U_{\lambda_1} \cap \ldots \cap U_{\lambda_n} \neq \emptyset.$$

Evidently all points of B are admissible.

Let us prove that the set of all admissible points is open in Y. In fact, by lemma 3.5, an admissible point y has a neighborhood meeting only the sets $\overline{W}_{\lambda_1}, \ldots, \overline{W}_{\lambda_n}$ that contains y. But then all points of this neighborhood are admissible.

Since Y is normal we can take a closed neighborhood $\tilde{Y}$ of B in Y such that all points of $\tilde{Y}$ are admissible. Put

$$Y_\lambda = \overline{W}_\lambda \cap \tilde{Y}.$$

Then $\{Y_\lambda\}$ is a closed covering of $\tilde{Y}$ and $Y_\lambda \subset V_\lambda$. We still have to prove c).

Suppose

$$Y_{\lambda_1} \cap \ldots \cap Y_{\lambda_n} \neq \emptyset.$$

Take a point

$$y \in Y_{\lambda_1} \cap \ldots \cap Y_{\lambda_n}.$$

Then $y \in \tilde{Y}$, so that y is admissible. But this implies

$$U_{\lambda_1} \cap \ldots \cap U_{\lambda_n} \neq \emptyset.$$

Lemma 20.2. *Let B be a subset of the space Y and $\{Y_\lambda\}$ a locally finite closed covering of Y. Suppose that for each λ there is a neighborhood C_λ of $Y_\lambda \cap B$ in Y_λ. Then*

$$C = \bigcup_\lambda C_\lambda$$

is a neighborhood of B in Y.

Proof. Let $b \in B$. We want to show that b is an interior point of C. Since $\{Y_\lambda\}$ is locally finite b belongs to a finite number of sets Y_λ, say $Y_{\lambda_1}, \ldots, Y_{\lambda_n}$. Then by lemma 3.5 there is an open neighborhood O' of b in Y such that

$$Y_\lambda \cap O' \neq \emptyset \text{ only for } \lambda = \lambda_1, \ldots, \lambda_n.$$

Since C_{λ_i} $(i = 1, \ldots, n)$ is a neighborhood of b in Y_{λ_i} there is an open set O_{λ_i} in Y for which

$$b \in O_{\lambda_i} \cap Y_{\lambda_i} \subset C_{\lambda_i}.$$

Then

$$O = O' \cap O_{\lambda_1} \cap \ldots \cap O_{\lambda_n}$$

is open in Y and contains b. Finally we shall show

$$O \subset C. \tag{1}$$

But if $y \in O$, $y \in Y_\lambda$ for some λ. From $y \in O'$ we obtain that λ is one of $\lambda_1, \ldots, \lambda_n$, say λ_1. Then $y \in O_{\lambda_1}$ implies

$$y \in O_{\lambda_1} \cap Y_{\lambda_1} \subset C_{\lambda_1} \subset C.$$

This proves (1).

Lemma 20.3. *Let (Y, B) be a normal pair and Y' a closed subset of Y. Put $B' = Y' \cap B$. Suppose there is given a closed neighborhood C' of B' in Y'. Then there exists a closed neighborhood C of B in Y such that*

$$C' = Y' \cap C. \tag{2}$$

Hence we can write

$$C = C' \cup C^*,$$

where

$$C^* \subset Y - Y'.$$

Proof. Let U' be an open neighborhood of B' in Y' contained in C'. Then $Y' - U'$ and B are two disjoint closed sets in the normal space Y. Therefore we can take a closed neighborhood V of B in Y such that

$$V \cap (Y' - U') = \emptyset.$$

Hence

$$V \cap Y' \subset U' \subset C'. \tag{3}$$

Put

$$C = C' \cup V.$$

Since V is a neighborhood of B in Y, C is a neighborhood of B in Y, and since C' and V are closed in Y, C is closed in Y. That (2) holds, follows from (3).

21. Proof of theorem 19.2 (first part). Suppose that all Q-spaces are fully normal. Let X be a local NES(Q), and let (Y, B) be any Q-pair and $f: B \to X$ any mapping. We want to find a neighborhood extension of f.

Let $\alpha = \{O'_\varkappa\}$ be a covering of X by open NES(Q)'s. Then $\{f^{-1}(O'_\varkappa)\}$ is an open covering of B. Since B is fully normal there is a locally finite open refinement $\{U_\lambda\}$ of $\{f^{-1}(O'_\varkappa)\}$. Because of lemma 18.1 we may assume that $\{U_\lambda\}$ is elementwise uniformly point-finite. For each λ choose an element of α, say O_λ, such that $f(U_\lambda) \subset O_\lambda$. By lemma 7.4 there is a locally finite open covering $\{V_\lambda\}$ of Y such that $V_\lambda \cap B = U_\lambda$.

Apply lemma 20.1. We get a closed neighborhood $\tilde{Y}$ of B in Y and a locally finite closed covering $\{Y_\lambda\}$ of $\tilde{Y}$. Since $Y_\lambda \cap B \subset U_\lambda$ we have

$$f(Y_\lambda \cap B) \subset O_\lambda.$$

Finally $\{Y_\lambda\}$ is elementwise uniformly point-finite. For this follows from 20.1 c) and from the fact that $\{U_\lambda\}$ is elementwise uniformly point-finite.

Proof of theorem 19.3 (first part). Suppose that all Q-spaces are collectionwise normal. Let X be a fully normal local NES(Q), and let (Y, B) be any Q-pair and $f: B \to X$ any mapping. We want to find a neighborhood extension of f.

Let $\{O_\lambda\}$ be a covering of X by open NES(Q)'s. Since X is fully normal we may assume that $\{O_\lambda\}$ is locally finite (cf. 2.16), and because of lemma 18.1, we may assume that $\{O_\lambda\}$ is elementwise uniformly point-finite. Put $f^{-1}(O_\lambda) = U_\lambda$. Then $\{U_\lambda\}$ is a locally finite and elementwise uniformly point-finite open covering of B. By lemma 7.3 there is a locally finite open covering $\{V_\lambda\}$ of Y such that $V_\lambda \cap B \subset U_\lambda$.

Apply lemma 20.1. We get a closed neighborhood $\tilde{Y}$ of B in Y and a locally finite and elementwise uniformly point-finite closed covering $\{Y_\lambda\}$ of $\tilde{Y}$. We have

$$f(Y_\lambda \cap B) \subset O_\lambda.$$

Proof of theorem 19.4 (first part). Suppose that all Q-spaces are normal. Let X be a Lindelöf space which is a local NES(Q) and let (Y, B) be any Q-pair and $f: B \to X$ any mapping. We want to find a neighborhood extension of f.

Let $\{O_\lambda\}$ be a covering of X by open NES(Q)'s. Since X is a Lindelöf space we may assume that $\{O_\lambda\}$ is countable and star-finite (see lemma 6.3). Put $f^{-1}(O_\lambda) = U_\lambda$. Then $\{U_\lambda\}$ is a countable star-finite open covering of B. By lemma 7.2 we have a locally finite open covering $\{V_\lambda\}$ of Y such that $V_\lambda \cap B = U_\lambda$. Since $\{U_\lambda\}$ is star-finite it is also elementwise uniformly point-finite.

Apply lemma 20.1. We get a closed neighborhood $\tilde{Y}$ of B in Y and a locally finite and elementwise uniformly point-finite closed covering $\{Y_\lambda\}$ of $\tilde{Y}$. We have

$$f(Y_\lambda \cap B) \subset O_\lambda.$$

22. Proof of theorems 19.2, 19.3, and 19.4 (second parts). We have the following set-up:

A class Q such that all Q-spaces are normal;

A space X;

Some open subsets O_λ of X, each O_λ being an NES(Q);

A Q-pair $(\tilde{Y}, B)$;

A locally finite and elementwise uniformly point-finite closed covering $\{Y_\lambda\}$ of $\tilde{Y}$;

A mapping $f: B \to X$ such that $f(Y_\lambda \cap B) \subset O_\lambda$.

We want to find a neighborhood extension of $f: B \to X$ in $\tilde{Y}$ (for $\tilde{Y}$ is a neighborhood of B in Y, cf. lemma 2.1).

Let Σ be the nerve of the covering $\{Y_\lambda\}$. Σ is a simplicial complex. Since $\{Y_\lambda\}$ is elementwise uniformly point-finite the star of each vertex of Σ is finite dimensional. Hence for each simplex $\sigma \in \Sigma$ there is an upper bound for the dimension of those simplices σ_1 which have σ as a face. Thus if we define $I(\sigma)$ by

$$I(\sigma) = \max_{\sigma_1 \in \mathrm{St}\,\sigma} (\dim \sigma_1 - \dim \sigma),$$

$I(\sigma)$ is a finite non-negative integer. Notice that

(1) $$\sigma_1 \succ \sigma \text{ implies } I(\sigma_1) < I(\sigma).$$

We need some sets in $\tilde{Y}$. Put

$$D = \tilde{Y},$$

$$D_\sigma = \bigcap_{i=1}^{n} Y_{\lambda_i} - \bigcup_{\lambda \neq \lambda_i} Y_\lambda \text{ for } \sigma = (\lambda_1, \ldots, \lambda_n),$$

$$B_\sigma = D_\sigma \cap B.$$

Then $\{D_\sigma\}$ is a covering of D by sets which are mutually disjoint, and $\{B_\sigma\}$ is the corresponding covering of B. D is a Q-space and B is closed in D.

Furthermore put

$$D_{\mathrm{St}\,\sigma} = \bigcup_{\sigma_1 \in \mathrm{St}\,\sigma} D_{\sigma_1} \text{ and } B_{\mathrm{St}\,\sigma} = \bigcup_{\sigma_1 \in \mathrm{St}\,\sigma} B_{\sigma_1}.$$

Then we have

(2) $$D_{\mathrm{St}\,\sigma} = \bigcap_{i=1}^{n} Y_{\lambda_i} \text{ for } \sigma = (\lambda_1, \ldots, \lambda_n).$$

Hence $\{D_{\mathrm{St}\sigma}\}$ is a closed covering of D. Let us show that $\{D_{\mathrm{St}\sigma}\}$ is locally finite. In fact $\{Y_\lambda\}$ is locally finite so that if $y \in D$ there is a neighborhood W of y in D meeting only a finite number of sets Y_λ, say $Y_{\lambda'_1}, \ldots, Y_{\lambda'_m}$. Then, because of (2), W meets $D_{\mathrm{St}\sigma}$ only if each λ_i is one of $\lambda'_1, \ldots, \lambda'_m$. But this is possible only for a finite number of σ's.

Put, for $\sigma = (\lambda_1, \ldots, \lambda_n)$,

$$O_\sigma = \bigcap_{i=1}^{n} O_{\lambda_i}.$$

Then O_σ is an NES(Q) (cf. 2.16). We have

(3) $$\sigma_1 \succ \sigma \text{ implies } O_{\sigma_1} \subset O_\sigma$$

and

(4) $$f(B_\sigma) \subset O_\sigma.$$

We want to define an extension $F: C \to X$ of f to a neighborhood C of B in D. Suppose for a moment that this has been done. Then if we set $C_\sigma = D_\sigma \cap C$ we have

(5) $$C = \bigcup_{\sigma \in \Sigma} C_\sigma,$$

and if we put

(6) $$F_\sigma = F \mid C_\sigma,$$

$F_\sigma: C_\sigma \to X$ is an extension of $f \mid B_\sigma$.

However we shall do the converse. We shall for each $\sigma \in \Sigma$ define a set C_σ and a mapping $F_\sigma: C_\sigma \to X$ and so define C and F by (5) and (6). We shall do this succesively by an induction on increasing $I(\sigma)$. C_σ and $F_\sigma: C_\sigma \to X$ have to satisfy

$$B_\sigma \subset C_\sigma \subset D_\sigma, \tag{7}$$

$$F_\sigma \mid B_\sigma = f \mid B_\sigma, \tag{8}$$

$$F_\sigma (C_\sigma) \subset O_\sigma. \tag{9}$$

Put

$$C_{\mathrm{St}\,\sigma} = \bigcup_{\sigma_1 \in \mathrm{St}\,\sigma} C_{\sigma_1},$$

and define a function $F_{\mathrm{St}\,\sigma}: C_{\mathrm{St}\,\sigma} \to X$ by

$$F_{\mathrm{St}\,\sigma} \mid C_{\sigma_1} = F_{\sigma_1} \text{ for } \sigma_1 \in \mathrm{St}\ \sigma.$$

Since $C_{\sigma_1} \subset D_{\sigma_1}$ and the sets D_{σ_1} are mutually disjoint, $F_{\mathrm{St}\,\sigma}$ is uniquely determined. We now also require

$$C_{\mathrm{St}\,\sigma} \text{ is a neighborhood of } B_{\mathrm{St}\,\sigma} \text{ in } D_{\mathrm{St}\,\sigma}, \tag{10}$$

$$C_{\mathrm{St}\,\sigma} \text{ is closed in } D, \tag{11}$$

$$F_{\mathrm{St}\,\sigma} \text{ is continuous.} \tag{12}$$

Let σ be any simplex of Σ with $I(\sigma) = n$. Then $n \geqq 0$. If $n > 0$ we assume, when defining C_σ and F_σ, that C_{σ_1} and F_{σ_1} are already defined for all σ_1 with $I(\sigma_1) < n$, in particular for all $\sigma_1 \succ \sigma$ (see (1)).

Put

$$D'_\sigma = D_{\mathrm{St}\,\sigma} - D_\sigma = \bigcup_{\sigma_1 \succ \sigma} D_{\sigma_1},$$

$$C'_\sigma = \bigcup_{\sigma_1 \succ \sigma} C_{\sigma_1},$$

$$B'_\sigma = B_{\mathrm{St}\,\sigma} - B_\sigma = \bigcup_{\sigma_1 \succ \sigma} B_{\sigma_1}.$$

Then

$$B'_\sigma \subset C'_\sigma \subset D'_\sigma.$$

Define a function $F'_\sigma: C'_\sigma \cup B_\sigma \to X$ by

$$F'_\sigma \mid C_{\sigma_1} = F_{\sigma_1} \text{ for } \sigma_1 \succ \sigma,$$

$$F'_\sigma \mid B_\sigma = f \mid B_\sigma.$$

Then we have

$$F'_\sigma \mid C_{\mathrm{St}\,\sigma_1} = F_{\mathrm{St}\,\sigma_1} \text{ for } \sigma_1 \succ \sigma,$$

$$F'_\sigma \mid B_{\mathrm{St}\,\sigma} = f \mid B_{\mathrm{St}\,\sigma}.$$

But $F_{\mathrm{St}\,\sigma_1}$ is continuous (see (12)) and $f \mid B_{\mathrm{St}\,\sigma}$ is continuous. Hence, since the sets $C_{\mathrm{St}\,\sigma_1}$ and the set $B_{\mathrm{St}\,\sigma}$ together make up a locally finite closed covering of $C'_\sigma \cup B_\sigma$, F'_σ is continuous by lemma 3.3. By (3), (4), and (9) we get

$$F'_\sigma (C'_\sigma \cup B_\sigma) \subset O_\sigma. \tag{13}$$

We can write

$$D'_\sigma = \bigcup_{\sigma_1 \succ \sigma} D_{\mathrm{St}\sigma_1}.$$

Since $\{D_{\mathrm{St}\sigma_1}\}$ is locally finite and since each $D_{\mathrm{St}\sigma_1}$ is closed we see by lemma 3.2 that

(14) $$D'_\sigma \text{ is closed in } D.$$

Similarly, since (11) is true for each $\sigma_1 \succ \sigma$,

(15) $$C'_\sigma \text{ is closed in } D.$$

Furthermore lemma 20.2 proves that

(16) $$C'_\sigma \text{ is a neighborhood of } B'_\sigma \text{ in } D'_\sigma.$$

For B'_σ is a subset of D'_σ, $\{D_{\mathrm{St}\sigma_1};\ \sigma_1 \succ \sigma\}$ is a locally finite closed covering of D'_σ, and $C_{\mathrm{St}\sigma_1}$ is a neighborhood of $B_{\mathrm{St}\sigma_1} = D_{\mathrm{St}\sigma_1} \cap B'_\sigma$ in $D_{\mathrm{St}\sigma_1}$ (see (10)).

The function F'_σ is the part of $F_{\mathrm{St}\sigma}$ that is already defined. We want to extend it in order to get $F_{\mathrm{St}\sigma}$.

Apply lemma 20.3 on the normal pair $(D_{\mathrm{St}\sigma}, B_{\mathrm{St}\sigma})$, the closed subset D'_σ (see (14)), and the closed neighborhood C'_σ of B'_σ (see (15) and (16)). Then we get a closed neighborhood of $B_{\mathrm{St}\sigma}$ in $D_{\mathrm{St}\sigma}$ of the form

$$C'_\sigma \cup C^*_\sigma,$$

where

$$C^*_\sigma \subset D_{\mathrm{St}\sigma} - D'_\sigma = D_\sigma.$$

Now, O_σ is an NES(Q). The set

$$C'_\sigma \cup B_\sigma = C'_\sigma \cup B_{\mathrm{St}\sigma}$$

is closed in $C'_\sigma \cup C^*_\sigma$ (see (15)), and $C'_\sigma \cup C^*_\sigma$ is a Q-space, since it is closed in $D_{\mathrm{St}\sigma}$ and therefore in D. Because of (13) we can find a neighborhood extension of F'_σ in $C'_\sigma \cup C^*_\sigma$ relative to O_σ of the form

$$g_\sigma : C'_\sigma \cup C_\sigma \to O_\sigma,$$

where $C_\sigma \subset C^*_\sigma$ and $C'_\sigma \cup C_\sigma$ is closed in $C'_\sigma \cup C^*_\sigma$, hence also in D. This defines C_σ. Finally let $F_\sigma : C_\sigma \to X$ be defined by

$$F_\sigma(y) = g_\sigma(y) \text{ for } y \in C_\sigma,$$

i.e. F_σ and $g_\sigma | C_\sigma$ are the same mappings except that F_σ is into X and $g_\sigma | C_\sigma$ is into O_σ. Let us verify that (7)–(12) are satisfied.

We already know that $C_\sigma \subset C^*_\sigma \subset D_\sigma$. Since

$$C'_\sigma \cup C_\sigma \supset C'_\sigma \cup B_\sigma \supset B_\sigma$$

and

$$C'_\sigma \cap B_\sigma = \emptyset,$$

we have $B_\sigma \subset C_\sigma$. This shows (7). From

$$F_\sigma \mid B_\sigma = F'_\sigma \mid B_\sigma = f \mid B_\sigma$$

we get (8). (9) is immediate. Since

$$C_{\mathrm{St}\,\sigma} = C'_\sigma \cup C_\sigma$$

we see that (11) is true. (10) follows by lemma 2.1 from the fact that $C_{\mathrm{St}\,\sigma}$ is a neighborhood of $B_{\mathrm{St}\,\sigma}$ in $C'_\sigma \cup C^*_\sigma$, and $C'_\sigma \cup C^*_\sigma$ is a neighborhood of $B_{\mathrm{St}\,\sigma}$ in $D_{\mathrm{St}\,\sigma}$. Finally

$$F_{\mathrm{St}\,\sigma}: C_{\mathrm{St}\,\sigma} \to X \text{ and } g_\sigma: C_{\mathrm{St}\,\sigma} \to O_\sigma$$

take the same values for each $y \in C_{\mathrm{St}\,\sigma}$. Since g_σ is continuous so is $F_{\mathrm{St}\,\sigma}$. This proves (12).

Hence we have shown that we can define C_σ and $F_\sigma: C_\sigma \to X$ satisfying (7)–(12). By the induction on increasing $I(\sigma)$ we can do this for all simplices σ.

Now let C and $F: C \to X$ be defined by (5) and (6). We have

$$C = \bigcup_{\sigma \in \Sigma} C_{\mathrm{St}\,\sigma}.$$

Hence, since $\{D_{\mathrm{St}\,\sigma}\}$ is a locally finite closed covering of D, (10) and lemma 20.2 show that C is a neighborhood of B in D. From (6) we get

$$F \mid C_{\mathrm{St}\,\sigma} = F_{\mathrm{St}\,\sigma},$$

which is continuous by (12). Since $C_{\mathrm{St}\,\sigma} \subset D_{\mathrm{St}\,\sigma}$, $\{C_{\mathrm{St}\,\sigma}\}$ is locally finite, and (11) and lemma 3.3 show that F is continuous.

Now $F \mid B = f$. For we have

$$F \mid B_\sigma = F_\sigma \mid B_\sigma = f \mid B_\sigma$$

and $\{B_\sigma\}$ covers B. Hence F is an extension of f to a neighborhood C of B in D.

Thus theorems 19.2, 19.3, and 19.4 are completely proved.

23. Theorem 23.1. *Let all Q-spaces be normal. Then any finite union of open NES(Q)'s is an NES(Q).*

Proof. This theorem is proved as theorem 19.4. We do not need now to have the assumption that the space be a Lindelöf space. For this was used only to get the countable, star-finite covering $\{O_\lambda\}$. However we already have a finite covering.

Example 23.2. Let X be a space with the discrete topology. Then X is a local NES(Q) for any Q. Therefore if X has a countable number of points X is an NES (normal) by theorem 19.4. However if X has uncountably many points it is not an NES (normal), for then by theorem 14.5 it would be a Lindelöf space, which it is not. But it is an NES (coll. normal) by theorem 19.3.

Example 23.3. As in the proof of lemma 15.2, we consider in the space

$$Z = I \times I' - \{0\} \times \{o\}$$

the two closed subsets

$$A = (I - \{0\}) \times \{o\} \text{ and } B = \{0\} \times (I' - \{o\}).$$

Put

$$X = A \cup B.$$

Then X is a locally compact space and A and B are two open subsets of X. A is known to be an ES (normal) and an ANR (Tychonoff) (cf. theorems 17.1 and 17.2). The same facts are known about B (cf. example 17.9). Then, by theorem 23.1, X is an NES (normal). However, it is not an ANR (Tychonoff). For it is closed in the Tychonoff space Z and it is not a neighborhood retract of Z, since A and B do not have disjoint neighborhoods in Z (cf. lemma 15.2).

Example 23.4. BORSUK ([6] p. 226) proved that if $X = X_1 \cup X_2$, where each of X_1, X_2, and $X_1 \cap X_2$ is an ANR (comp. metr.), then X is an ANR (comp. metr.). This is not true, however, even for the class of all compact spaces. For take in the topological product $I \times I'$ the sets

$$X_1 = I \times \{o\},$$

$$X_2 = \{0\} \times I',$$

$$X = X_1 \cup X_2.$$

Then X_1 is a closed interval, X_2 a Tychonoff cube, and $X_1 \cap X_2$ a single point. Hence each of them is an AR (compact). However, X is not a neighborhood retract of $I \times I'$, since $X - \{0\} \times \{o\}$ is not a neighborhood retract of $I \times I' - \{0\} \times \{o\}$.

Infinite polyhedra

24. It is known that a finite simplicial polyhedron with the usual Euclidean topology is an ANR (comp. metr.) (cf. [6] p. 227). Hence, by theorems 13.2 and 16.2, it is also an ANR (Tychonoff). Let us now turn to infinite simplicial polyhedra.

All our polyhedra will be simplicial polyhedra and we shall therefore usually drop the word simplicial. A polyhedron is infinite if it has an infinite number of simplices or, what is the same, an infinite number of vertices.

By a subpolyhedron of a polyhedron X we mean any union of closed simplices of the simplicial decomposition of X.

We shall give an infinite polyhedron two, in general different, topologies. They both satisfy the following two conditions:

a) Any subpolyhedron is a closed subset.

b) Any finite subpolyhedron has, considered as a subspace, the Euclidean topology.

First, let the polyhedron X be locally finite (i.e. the star of each vertex is a finite polyhedron). Then a) and b) determine a unique topology for X. It can be proved that this topology makes X into a metrizable locally compact space. Each point of X has a neighborhood which is a finite subpolyhedron and therefore an NES (coll. normal). Hence, by theorem 19.3, X is an NES (coll. normal). If X has a countable number of vertices (cf. lemma 27.1) it is an ANR (Tychonoff) by theorem 16.6.

However, if the polyhedron is not locally finite, we can define two different topologies satisfying a) and b). We call them the weak topology and the metric topology.

The weak topology is defined as follows. Let X be the polyhedron. A set $A \subset X$ is closed if and only if for each simplex $\sigma \subset X$ the set $A \cap \sigma$ is closed in σ in the Euclidean topology for σ. As is easily verified, conditions a) and b) are satisfied. A set $O \subset X$ is open if and only if for each σ the set $O \cap \sigma$ is open in σ. This topology makes X into a CW-complex in the sence of J. H. C. Whitehead ([33] p. 223). It is known to be a normal space ([33] p. 225). (It can also be proved to be fully normal, but we do not need this fact.)

In order to define the metric topology we need the following notations. Let the vertices of X be $\{p_\lambda\}$. A point $x \in X$ is determined by its barycentric coordinates $\{x_\lambda\}$. They satisfy

$$0 \leqq x_\lambda \leqq 1 \text{ for each } \lambda,$$

$$x_\lambda \neq 0 \text{ only for a finite number of } \lambda\text{'s},$$

$$\sum_\lambda x_\lambda = 1.$$

Now for two points $x, x' \in X$ put

$$d(x, x') = \sum_\lambda |x_\lambda - x'_\lambda|.$$

Then $d(x, x')$ is a metric. The topology defined by this metric is the same as the one defined by the metric

$$d_1(x, x') = \sqrt{\sum_\lambda (x_\lambda - x'_\lambda)^2},$$

called by Lefschetz ([24] p. 9) the natural metric. In this topology a sequence of points $x^n = \{x_\lambda^n\}$ converges to $x = \{x_\lambda\}$ if and only if, for each λ, $x_\lambda^n \to x_\lambda$. A function $f: Y \to X$ is continuous when each coordinate $(f(y))_\lambda$ of $f(y)$ is continuous in y. The topology satisfies conditions a) and b).

When the polyhedron is not locally finite these two topologies do not coincide. For we see that the weak topology is not metrizable since it does not satisfy the first countability axiom (cf. [23] p. 6). Note that in both topologies the open star of the vertex p_λ, i.e. the set

$$\operatorname{St} p_\lambda = \{x \mid x_\lambda > 0\},$$

is open. For its complement in X is a subpolyhedron and hence closed in X by a). (Conversely: that all sets $\operatorname{St} p_\lambda$ are open implies a).)

The purpose of the next two paragraphs will be to show that a polyhedron with either of the two topologies is an NES (metric). In the case of the metric topology we can use this and our previous results in order to determine when the polyhedron is an ANR(Q) for the classes $\alpha-\delta$. This will be done in § 27 (see theorem 27.4). The corresponding problems in the case of the weak topology are unsolved.

It is a standard trick within the theory of finite polyhedra to imbed the polyhedron in a simplex having the same vertices as the polyhedron. In order to use this trick in our case we need the following definition.

Definition 24.1. A polyhedron is called *full* if each finite subcollection of its vertices spans a simplex.

Any polyhedron X can be imbedded in a full polyhedron Z with the same vertices. We give Z the same kind of topology as X. Since X is a subpolyhedron of Z, X is a closed subset of Z. We shall see below that, in either topology, X is a neighborhood retract of Z.

25. Theorem 25.1. *Any simplicial polyhedron with the weak topology is an NES (metric).*

Theorem 25.2. *Any full simplicial polyhedron with the weak topology is an ES (metric).*

Proof of theorem 25.1. Any polyhedron X can be imbedded in a full polyhedron. Hence, because of 2.15, the theorem follows from theorem 25.2 and the following lemma.

For the proof of theorem 25.2 see after lemma 25.4 below.

Lemma 25.3. *Any subpolyhedron X of a simplicial polyhedron Z with the weak topology is a neighborhood retract of Z.*

Proof. The main trick will be to consider the barycentric subdivision Z' of Z (cf. [24] p. 8). We give to Z' the weak topology. The spaces Z and Z' are defined on the same set. Their topologies coincide. For this is true on each simplex σ of Z and follows in general from the definition of the weak topology.

By subdividing Z we get from X its subdivision X', which is a subpolyhedron of Z'. We claim that X' and Z' satisfy:

a) Let $p_1', \ldots, p_n'$ be vertices of a simplex $\sigma' \subset Z'$, and let them all belong to X'. Then $\sigma' \subset X'$.

For since the points $\{p_i'\}$ are vertices of a simplex of Z' they all lie in a simplex σ of Z having one of $\{p_i'\}$ as its barycenter, say p_1'. But $p_1' \in X'$, i.e. $p_1' \in X$. Hence $\sigma \subset X$ and $\{p_i'\}$ are certain vertices of the subdivision of σ. Therefore $\sigma' \subset X'$.

Let $\{p_\mu'\}$ be all vertices of Z', indexed by a set $\mathrm{M}=\{\mu\}$. A point $z \in Z'$ is determined by its barycentric coordinates $z=\{z_\mu\}$, where

$$\sum_{\mu \in \mathrm{M}} z_\mu = 1.$$

Let $\mathrm{M}_0 \subset \mathrm{M}$ be the set of all indices for which $p_\mu' \in X'$. Consider the real-valued function defined by

$$a(z) = \sum_{\mu \in \mathrm{M}_0} z_\mu .$$

We have:

(1) $$a(z) \text{ is continuous,}$$

(2) $$U = \{z \mid a(z) > 0\} \text{ is open in } Z',$$

(3) $$U \supset X'.$$

Here (3) is trivial since $a(z)=1$ for $z \in X'$, and (2) is a consequence of (1). A function on a polyhedron with the weak topology is continuous if it is continuous on each simplex. But on a simplex the function a is a finite sum of continuous functions. Hence a is continuous.

Now define a retraction $r: U \to X'$ by taking as the image point $r(z)$ of a point $z \in U$ the point whose barycentric coordinates are

$$(r(z))_\mu = \frac{z_\mu}{a(z)} \quad \text{for } \mu \in \mathrm{M}_0,$$

$$(r(z))_\mu = 0 \text{ for } \mu \in \mathrm{M} - \mathrm{M}_0.$$

These formulas determine a point $r(z)$ of Z' lying in X'. For let $\sigma' \subset Z'$ be the simplex of lowest dimension such that $z \in \sigma'$. Then $z_\mu > 0$ if and only if p'_μ is a vertex of σ'. Hence, since

$$\sum_\mu (r(z))_\mu = 1,$$

there is a point $r(z) = \{(r(z))_\mu\}$ of σ'. Since $(r(z))_\mu = 0$ for $\mu \notin \mathrm{M}_0$, $r(z)$ is a point of a simplex with all vertices in X'. Hence, by a), $r(z) \in X'$.

We want to prove that r is continuous. Then we have to show that $r \mid \sigma' \cap U$ is continuous for each simplex σ'. But this is true since each coordinate $(r(z))_\mu$ is continuous.

Finally, for $z \in X'$ we have $a(z)=1$ and hence $r(z)=z$.

Therefore $r: U \to X'$ is a retraction of a neighborhood U of X' in Z'. This proves lemma 25.3.

Lemma 25.4. *Let X be a polyhedron with the weak topology and Y a metric space. Suppose $f: Y \to X$ is a mapping. Then if $\{\mathrm{St}\, p_\lambda\}$ is the open covering of X by the star of its vertices, $\alpha = \{f^{-1}(\mathrm{St}\, p_\lambda)\}$ is a locally finite open covering of Y.*

Proof. Clearly α is an open covering of Y so that we have to prove that it is locally finite.

Suppose α is not locally finite. Then there is a point $y_0 \in Y$ such that every neighborhood of y_0 meets an infinite number of elements of α. Since $f(y_0)$ is a point of some simplex of X, the point y_0 itself only belongs to a finite number of sets $f^{-1}(\mathrm{St}\, p_\lambda)$, say for $\lambda = \lambda'_1, \ldots, \lambda'_m$.

Now we construct a sequence of points $y_n \in Y$ and a sequence of indices λ_n such that (for $n = 1, 2, \ldots$)

(4) $$y_n \to y_0,$$

(5) $$y_n \in f^{-1}(\mathrm{St}\, p_{\lambda_n}),$$

(6) $$\lambda_n \neq \lambda_i' \text{ for each } i = 1, \ldots, m,$$

(7) $$\lambda_n \neq \lambda_{n_1} \text{ for } n \neq n_1.$$

This is possible by induction. For when choosing λ_n we have to avoid the finite number of indices

$$\lambda_1', \ldots, \lambda_m', \lambda_1, \ldots, \lambda_{n-1},$$

and there are, for each neighborhood U of y_0, infinitely many sets $f^{-1}(\mathrm{St}\, p_\lambda)$ meeting U. Hence we can take y_n in the intersection of a set $f^{-1}(\mathrm{St}\, p_{\lambda_n})$ and a suitable neighborhood U_n of y_0.

Note that (6) and the definition of the indices λ_i' imply

$$y_0 \notin f^{-1}(\mathrm{St}\, p_{\lambda_n}).$$

Hence, by (5),

(8) $$f(y_0) \neq f(y_n).$$

Now, let O be the complement in X of the set which consists of all points $f(y_n)$, $n = 1, 2, \ldots$ By (8), O contains $f(y_0)$. By (5) and (7) each simplex σ of X contains only a finite number of points $f(y_n)$. Therefore $O \cap \sigma$ is open in σ. Hence O is open.

Finally, (4) and the continuity of f implies

$$f(y_n) \to f(y_0).$$

But this contradicts the fact that O is a neighborhood of $f(y_0)$ containing no point $f(y_n)$. This proves lemma 25.4.

Proof of theorem 25.2. We use theorem 12.3 and prove that a full polyhedron X with the weak topology is a contractible NES (metric).

Let $\{x_\lambda\}$ be the barycentric coordinates for a point $x \in X$. Choose some $\lambda_0 \in \Lambda = \{\lambda\}$. Define $h: X \times I \to X$ by

$$(h(x, t))_\lambda = (1-t)\, x_\lambda \text{ for } \lambda \neq \lambda_0,$$

$$(h(x, t))_{\lambda_0} = (1-t)\, x_{\lambda_0} + t.$$

For each (x, t) these formulas define a point $h(x, t) \in X$. In fact, the polyhedron X is full, only a finite number of coordinates x_λ are $\neq 0$, and

$$\sum_{\lambda \in \Lambda} (h(x, t))_\lambda = (1-t) \sum_{\lambda \in \Lambda} x_\lambda + t = 1.$$

The continuity of h follows from the fact that $h \mid \sigma \times I$ is continuous for each σ (cf. [33] p. 228). The homotopy h is a contraction of X into the vertex p_{λ_0}.

In order to prove that X is an NES (metric) let (Y, B) be any metric pair and $f: B \to X$ any mapping. Put

$$U_\lambda = f^{-1}(\mathrm{St}\, p_\lambda).$$

Then $\{U_\lambda\}$ is an open covering of B. It is locally finite by lemma 25.4. Applying lemma 7.4 we get a locally finite open covering $\{V_\lambda\}$ of Y such that

$$U_\lambda = V_\lambda \cap B.$$

Each coordinate $(f(y))_\lambda$ of $f(y)$ is a continuous function $f_\lambda(y)$ of y defined on B. Hence $f_\lambda: B \to I$ is a mapping into the closed interval I. We have

$$U_\lambda = \{y \mid y \in B,\ f_\lambda(y) > 0\}.$$

Now extend f_λ to a mapping $g_\lambda: Y \to I$ by putting

$$g_\lambda(y) = 0 \text{ for } y \notin V_\lambda$$

and applying Tietze's extension theorem. Take the function

$$a(y) = \sum_{\lambda \in \Lambda} g_\lambda(y),$$

which is finite and continuous since $\{V_\lambda\}$ is locally finite. If $y \in B$ we have

$$a(y) = \sum_{\lambda \in \Lambda} f_\lambda(y) = 1.$$

Hence the set

$$O = \{y \mid a(y) > 0\}$$

is a neighborhood of B.

Define $F: O \to X$ by

$$(F(y))_\lambda = \frac{g_\lambda(y)}{a(y)}.$$

For each $y \in O$ this formula defines a point $F(y) \in X$. F is continuous, for so is $(F(y))_\lambda$ and $\{V_\lambda\}$ is locally finite. Since $F \mid B = f$, F is a neighborhood extension of f.

26. Theorem 26.1. *Any simplicial polyhedron with the metric topology is an NES (metric).*

Theorem 26.2. *Any full simplicial polyhedron with the metric topology is an ES (metric).*

Theorem 26.1 is the consequence of theorem 26.2 and the following lemma.

Lemma 26.3. *Any subpolyhedron X of a simplicial polyhedron Z with the metric topology is a neighborhood retract of Z.*

Proof. We use the same method as in the proof of lemma 25.3 and only need to point out the places where the argument depends upon the topology of Z.

First we need the fact that the metric topology of Z coincides with the metric topology of Z'. However this has been proved by LEFSCHETZ ([24] p. 21). Next we shall prove the continuity of

$$a(z) = \sum_{\mu \in M_0} z_\mu.$$

But this follows from

$$|a(z)-a(z')| \leqq \sum_{\mu \in M_0} |z_\mu - z'_\mu| \leqq d(z, z').$$

Finally each $(r(z))_\mu$ is continuous. Hence $r(z)$ is continuous. Therefore $r: U \to X'$ is a neighborhood retraction. This proves lemma 26.3.

Proof of theorem 26.2. We want to use the fact (proved by DUGUNDJI [12] p. 358) that any convex set of a Banach space is an ES(metric).

Let X be a full polyhedron with the metric topology. Let $\{p_\lambda\}$ be its vertices, Λ the index set. We imbed X in the Banach space S consisting of all $s = \{s_\lambda\}$ where s_λ are real numbers and $\sum_{\lambda \in \Lambda} |s_\lambda|$ is convergent. The norm of an element of S is defined by

$$\|s\| = \sum_{\lambda \in \Lambda} |s_\lambda|.$$

The imbedding of X in S is the obvious one: if x_λ are the barycentric coordinates of x, $\{x_\lambda\}$ denotes a point of S. If x is identified with this point, X is imbedded in S. This imbedding is metric since

$$d(x, x') = \sum_{\lambda \in \Lambda} |x_\lambda - x'_\lambda| = \|x - x'\|,$$

where x and x' are two points of X and d is the metric on X.

Since the polyhedron X is full X is a convex set in S. Thus Dugundji's theorem completes the proof.

27. Now we shall combine theorem 26.1 with theorem 17.1. Therefore we want to know when a polyhedron with the metric topology is separable, locally compact, or an absolute G_δ.

Lemma 27.1. *A simplicial polyhedron with the metric topology is separable if and only if it has a countable number of vertices.*

Proof. If it has uncountably many vertices it is certainly not separable. If it has a countable number of vertices it is the union of a countable number of simplices, hence separable.

Lemma 27.2. *A simplicial polyhedron with the metric topology is locally compact if and only if it is locally finite.*

Proof. If it is locally finite it is certainly locally compact. If it is not locally finite there is some vertex belonging to an infinite number of 1-simplices. Hence this vertex has no compact neighborhood.

Lemma 27.3. *A simplicial polyhedron with the metric topology is an absolute G_δ if and only if it contains no infinite full subpolyhedron.*

The condition is for instance satisfied if the star of each vertex is finite dimensional. It is certainly not satisfied if the polyhedron itself is an infinite full polyhedron.

Proof. Sufficiency. Suppose the polyhedron X with the metric topology contains no infinite full subpolyhedron. I claim that the space X with the metric d is complete.

Let X be imbedded in the Banach space S as in the proof of theorem 26.2. Let $x^n = \{x_\lambda^n\}$ be a Cauchy sequence in X. Then, since S is a complete metric space, x^n converges to a point $s = \{s_\lambda\}$ of S. We shall show that s belongs to X.

From $x^n \to s$ we have, for each λ,

$$x_\lambda^n \to s_\lambda, \tag{1}$$

so that, since $0 \leqq x_\lambda^n \leqq 1$,

$$0 \leqq s_\lambda \leqq 1. \tag{2}$$

We also conclude from $x^n \to s$ that $\| x^n \| \to \| s \|$ i.e., since $\| x^n \| = 1$,

$$\| s \| = \sum_\lambda s_\lambda = 1. \tag{3}$$

In $\Lambda = \{\lambda\}$ let Λ' be the set of all indices for which $s_\lambda > 0$. Let Λ_0 be any finite subset of Λ'. Then, by (1), for some sufficiently large n

$$x_\lambda^n > 0 \text{ for } \lambda \in \Lambda_0.$$

Hence, since $x^n \in X$, the simplex σ_0 spanned in S by the vertices p_λ, $\lambda \in \Lambda_0$ is a face of a simplex in X, so that $\sigma_0 \subset X$.

From this we obtain that the vertices p_λ, $\lambda \in \Lambda'$ span a full subpolyhedron X' of X. But then Λ' must be finite and X' a simplex. By (2), (3), and the definition of Λ' we have $s \in X'$. Hence $s \in X$.

Therefore X is a complete metric space. As was previously remarked, this means that X is an absolute G_δ.

Necessity. Let X be a polyhedron with the metric topology. Suppose that X is an absolute G_δ containing an infinite full subpolyhedron A. We shall show that this is impossible.

Without loss of generality we may assume that A has a countable number of vertices. A is a subpolyhedron of X and therefore closed in X. Hence, since X is an absolute G_δ, A is also an absolute G_δ. Therefore A can be given a complete metric.

But in A the open stars of the vertices are a countable collection of open dense sets with a void intersection. By BAIRE's theorem (cf. [18] p. 160) this is impossible in a complete metric space. This completes the proof of lemma 27.3.

Now by lemmas 27.1, 27.2, and 27.3 and theorems 17.1 and 26.1:

Theorem 27.4. *Let X be a simplicial polyhedron with the metric topology. Then*

a) *X is an ANR (coll. normal) if and only if X contains no infinite full subpolyhedron.*

b) *X is an ANR (normal) if and only if X has a countable number of vertices and contains no infinite full subpolyhedron.*

c) *X is an ANR (Tychonoff) if and only if X is locally finite and has a countable number of vertices.*

Example 27.5. Take in a full polyhedron the subpolyhedron X consisting of all 1-simplices and all 0-simplices. Give to X the metric topology. Then X is an ANR (coll. normal). It is an ANR (normal) if and only if it has a countable number of vertices and it is an ANR (Tychonoff) if and only if it has a finite number of vertices.

Homotopy theorems

28. In this final paragraph we shall study homotopy properties of ANR(Q)'s. They are all generalizations of theorems already known in the case when Q is the class of separable metric spaces. Since many of the proofs are similar to those given in the separable metric case we shall omit the details.

We need to use lemma 12.1. Since this lemma is proved only for the classes δ–ι, we have to restrict ourselves to these classes.

Theorem 28.1. *Let Q be any of the classes δ–ι. Then any $ANR(Q)$ is locally contractible.*

This is proved by a method similar to the one proving theorem 12.4. (Cf. [15] p. 397, the first half of the proof of theorem 4.2.)

Theorem 28.2. *Let Q be any of the classes δ–ι. Then the homotopy extension theorem holds for mappings of Q-spaces into an $NES(Q)$.*

Explicitely this means that if X is an NES(Q), (Y, B) any Q-pair, $F_0: Y \to X$ any mapping and $f_t: B \to X$ any homotopy such that $f_0 = F_0 | B$, then there exists a homotopy $F_t: Y \to X$ such that $F_t | B = f_t$.

For the proof see [9] p. 205 or [18] p. 86.

Theorem 28.3. *Let Q be any of the classes δ–ι. Then a Q-space X is an $ANR(Q)$ if and only if for each point $x \in X$ there exists a neighborhood V of x such that for any Q-pair (Y, B) any mapping $f: B \to V$ has an extension relative to X.*

For the proof see [15] p. 398. For the sufficiency we need theorem 19.2.

Theorem 28.4. *Let Q be any of the classes δ–ι. If the homotopy extension theorem holds for mappings of Q-spaces into a locally contractible space X then X is an $NES(Q)$.*

For the proof see [15] p. 398.

Definition 28.5. Let $\alpha = \{U_\lambda\}$ be an open covering of X. We call a homotopy $f_t: Y \to X$ an *α-homotopy* if for each $y \in Y$ there is a U_λ such that $f_t(y) \in U_\lambda$ for $0 \leq t \leq 1$. A space Z is said to *dominate* a space X if there exists two mappings $\varphi: X \to Z$ and $\psi: Z \to X$ such that $\psi\varphi: X \to X$ is homotopic to the identity mapping $i: X \to X$. If this homotopy is an α-homotopy Z is said to *α-dominate* X.

Dugundji has proved a theorem ([12] p. 365) which can be reformulated thus:

Theorem 28.6. *Let α be any open covering of an ANR (metric) X. Then X is α-dominated by a polyhedron with the weak topology.*

He asks the question whether it is true that among metric spaces this property characterizes ANR (metric)'s. That this is the case is proved by theorem 28.8.

Theorem 28.7. *If for each open covering α of a metric space X there exists an NES (metric) α-dominating X, then X is an ANR (metric).*

The proof is essentially the same as in [15] (proof of theorems 7.1 and 7.2).

Theorem 28.8. *If for each open covering α of a metric space X there exists a polyhedron with the weak topology α-dominating X, then X is an ANR (metric).*

This is a consequence of theorems 25.1 and 28.7.

Remark 28.9. Instead of an α-dominating polyhedron for each covering α, it is sufficient in theorem 28.8 to assume that there is a suitable sequence of polyhedra dominating X, as in [15] p. 405.

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