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Inclusion relations among methods of summability compounded from given matrix methods

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1. Introduction. It is our object to clarify and generalize some theorems on inclusion among matrix methods of summability given by RUDBERG [1944], and to give applications involving the Cesàro, Abel, Euler, Borel, binary, and other methods. While RUDBERG gives no references, we observe that some fundamental ideas underlying the paper of RUDBERG and this one were used by HARDY and CHAPMAN [1911], JACOBSTHAL [1920] and KNOPP [1920]. Other references appear later.

For each $r = 1, 2, 3, \ldots$, let $A(r)$ be a triangular matrix of elements $a_{nk}(r)$ such that

$$a_{nk}(r) \geqq 0,\ a_{nn} > 0, \qquad 0 \leqq k \leqq n;\ n = 0, 1, \ldots \tag{1.01}$$

$$\lim_{n\to\infty} a_{nk}(r) = 0 \qquad k = 0, 1, 2, \ldots \tag{1.02}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}(r) = 1. \tag{1.03}$$

Then, for each r, $A(r)$ determines a regular Silverman-Toeplitz transformation

$$\sigma_n(r) = \sum_{k=0}^{n} a_{nk}(r) s_k \tag{1.1}$$

by means of which a given sequence s_n is evaluable to s if $\sigma_n(r) \to s$ as $n \to \infty$. Our terminology agrees with that of HARDY [1949].

Let the elements of a given sequence $s_0, s_1, s_2, \ldots$ be denoted by $s_0(0)$, $s_1(0)$, $s_2(0), \ldots$. Let $s_0(1), s_1(1), s_2(1), \ldots$ denote the $A(1)$ transform of $s_0(0)$, $s_1(0)$, $s_2(0), \ldots$; let $s_0(2), s_1(2), s_2(2), \ldots$ denote the A_2 transform of $s_0(1), s_1(1)$, $s_2(1), \ldots$; and so on. Then, for each $r = 1, 2, 3, \ldots$

$$s_n(r) = \sum_{k=0}^{\infty} a_{nk}(r) s_k(r-1) \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

The elements of these sequences form the double sequence

$$(1.3)\qquad \begin{array}{l} s_0(0),\ s_1(0),\ s_2(0),\ s_3(0),\ \ldots \\ s_0(1),\ s_1(1),\ s_2(1),\ s_3(1),\ \ldots \\ s_0(2),\ s_1(2),\ s_2(2),\ s_3(2),\ \ldots \\ s_0(3),\ s_1(3),\ s_2(3),\ s_3(3),\ \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array}$$

For each $r = 1, 2, 3, \ldots$, the elements of the r-th row below the first in (1.3) constitute the transform of the given sequence $s_0, s_1, \ldots$ by the product matrix $B(r)$ defined by

$$(1.4)\qquad B(r) = A(r)A(r-1)\ldots A(2)A(1).$$

Thus, for each $r = 1, 2, 3, \ldots$

$$(1.5)\qquad s_n(r) = \sum_{k=0}^{n} b_{nk}(r)s_k \qquad n = 0, 1, 2, \ldots$$

where the numbers $b_{nk}(r)$ are the elements of the matrix $B(r)$.

Suppose now that s_n is a particular sequence evaluable to s by one of the matrices $B(r)$, say $B(r_0)$. Then the r_0-th row below the first in (1.3) is convergent to s, and it follows from regularity of the matrices A_r that each lower row is likewise convergent to s. This means, roughly speaking, that $s_n(r)$ is near s whenever $r \geqq r_0$ and n is large in comparison to r. Hence, as is well known and easily shown, there is a sequence R_n such that $R_n \to \infty$ and the relation

$$(1.6)\qquad \lim_{n\to\infty} s_n(r_n) = s$$

holds for each sequence r_n such that $r_0 \leqq r_n \leqq R_n$ for each n. This implies that if s_n is a sequence evaluable to s by one of the matrices $B(r)$, then there is a sequence $r_0, r_1, \ldots$ such that $r_n \to \infty$ and the sequence s_n is evaluable to s by the matrix which transforms s_n into

$$(1.7)\qquad s_n(r_n) = \sum_{k=0}^{n} b_{nk}(r_n)s_k, \qquad n = 0, 1, 2, \ldots.$$

While $B(r_0, r_1, \ldots)$ and $b_{nk}(r_0, r_1, \ldots)$ are natural notations for this matrix and its elements, we see that $b_{nk}(r_0, r_1, \ldots) = b_{nk}(r_n)$. Hence we shall use the simpler notations $B(r_n)$ and $b_{nk}(r_n)$ for the matrix and its elements. We observe that if $r_n = r$ for each n, then $B(r_n) = B(r)$. While it could be presumed that different sequences r_n would be required for different sequences s_n, we shall see in Section 3 that this is not so. In other words, a single sequence r_n can be chosen in such a way that the matrix $B(r_n)$ defines a method of summability which includes each one of the methods $B_1, B_2, \ldots$. Moreover, relations among some such methods $B(r_n)$ will be obtained. We shall sometimes refer to a matrix $B(r_n)$, which results from combining matrices $A(r)$ and selecting from matrices $B(r)$, as a *compounded* matrix.

2. Two examples. Before passing to constructive theorems and significant examples, we look briefly at two trivial examples. Suppose first that $A(r)$ is, for each $r = 1, 2, \ldots$, the identity matrix which transforms each sequence into itself. Then each row of the double sequence (1.3) is identical with the first row. It is therefore clear that the transformation $B(r_n)$ defined by (1.7) is regular if and only if $r_n \to \infty$, and, moreover, that two transformations $B(r_n)$ and $B(r'_n)$ for which $r_n \to \infty$ and $r'_n \to \infty$ are equivalent to convergence and hence to each other.

Our second trivial example involves the double sequence

$$\begin{array}{l} 2,\ s_1(0),\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \ldots \\ 2,\ s_1(1),\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots \\ 2,\ s_1(2),\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \ldots \\ 2,\ s_1(3),\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots \\ 2,\ s_1(4),\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{2.1}$$

in which $s_1(0), s_1(1), \ldots$ is a given sequence of real numbers such that $-2 = s_1(0) < s_1(1) < s_1(2) < \ldots$ and $s_1(r) < -1$ for each $r = 0, 1, 2, \ldots$. For each $r = 2, 3, 4, \ldots$ the row containing $s_1(r)$ is identical with the row containing $s_1(r-2)$ except that $s_1(r) \neq s_1(r-2)$. For each $r = 1, 2, 3, \ldots$, let $A(r)$ be a matrix which carries the row of (2.1) containing $s_1(r-1)$ into the row containing $s_1(r)$ and which is not only regular but satisfies the stronger conditions

$$a_{nk}(r) > 0, \qquad 0 \leqq k \leqq n;\ n = 0, 1, 2, \ldots \tag{2.2}$$

and

$$\sum_{k=1}^{n} a_{nk}(r) = 1 \qquad n = 0, 1, 2, \ldots. \tag{2.3}$$

It is easy to see that such matrices $A(r)$ exist, but are not uniquely determined by our specification of the sequences obtained by starting with the one particular sequence given in the first row of (2.1). The service of this example, to which one may profitably refer occasionally, is to establish the truth of assertions such as the following. The hypotheses on $A(r)$ given in Section 1, even when supplemented by the stronger hypotheses in (2.2) and (2.3), imply neither equivalence nor consistency of the two transformations $B(r_n)$ and $B(r'_n)$ of the form (1.7) for which r_n and r'_n are the sequences $1, 2, 1, 2, 1, 2, \ldots$ and $2, 3, 2, 3, 2, 3, \ldots$.

3. Inclusion theorems. The theorems of this section are generalizations of Theorem 1 of RUDBERG [1944]. The matrices $T_1, T_2, \ldots$ introduced at the top of page 2 of Rudberg's paper correspond to the matrices $A(1), A(2), \ldots$ of this paper. Rudberg assumed, in the notation of this paper, that $a_{nk} > 0$ when $0 \leqq k \leqq n$; and he needed the condition $\sum_{k=0}^{n} a_{nk}(r) = 1$, $n = 0, 1, 2, \ldots$, to obtain some of his equalities. It should be pointed out that the matrices $A(r)$

and $B(r)$, which are different except in very special cases, are denoted by the same symbol T_r in Rudberg's paper; accordingly one cannot interpret the arguments and results of that paper without giving very close attention to the details.

Theorem 3.1. *There is a monotone increasing sequence* $R_1, R_2, \ldots$ *such that* $R_n \geqq 1$, $R_n \to \infty$, *and the matrix transformation* $B(r_n)$ *defined by* (1.7) *is regular whenever* $1 \leqq r_n \leqq R_n$ *for each* n.

Proof. Since the matrix $B(r)$, defined by (1.4), is the product of regular matrices with nonnegative elements, it has these same properties. Let

$$F(R, n) = \sum_{k=0}^{R} \sum_{r=1}^{R} b_{nk}(r) \qquad R = 1, 2, \ldots. \tag{3.11}$$

Then

$$\lim_{n\to\infty} F(R, n) = 0 \qquad R = 1, 2, \ldots. \tag{3.12}$$

Hence there is a sequence R'_n such that $R'_n \to \infty$ and $F(R'_n, n) \to 0$ as $n \to \infty$. If $1 \leqq r_n \leqq R'_n$, then

$$\lim_{n\to\infty} b_{nk}(r_n) = 0 \qquad k = 0, 1, 2, \ldots. \tag{3.13}$$

Let

$$G(R, n) = \sum_{r=1}^{R} \left| \sum_{k=0}^{n} b_{nk}(r) - 1 \right|. \tag{3.14}$$

Then

$$\lim_{n\to\infty} G(R, n) = 0 \qquad R = 1, 2, \ldots. \tag{3.15}$$

Hence there is a sequence R''_n such that $R''_n \to \infty$ and $G(R''_n, n) \to 0$. If $1 \leqq$ $\leqq r_n \leqq R''_n$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_{nk}(r_n) = 1. \tag{3.16}$$

If, for each n, R_n is the minimum of the positive integers $R'_n, R''_n, R'_{n+1}, R''_{n+1}, R'_{n+2}, \ldots$ then $1 \leqq R_1 \leqq R_2 \leqq \ldots$, $R_n \to \infty$, and both (3.13) and (3.16) hold when $1 \leqq r_n \leqq R_n$. Since $b_{nk}(r_n) \geqq 0$, it follows that $b_{nk}(r_n)$ is regular when $1 \leqq r_n \leqq R_n$ and Theorem 3.1 is proved.

Theorem 3.2. *If* r'_n *and* r_n *are sequences such that* r_n *is monotone increasing, if* $r'_n \geqq r_n$ *for all sufficiently great* n, *and if the transformations* $B(r'_n)$ *and* $B(r_n)$ *are regular, then* $B(r'_n)$ *includes* $B(r_n)$.

Proof. Since the alteration of a finite set of elements of the sequence r'_n has no effect on $B(r'_n)$ evaluability of sequences, we can and shall assume that $r'_n \geqq r_n$ for each $n = 1, 2, 3, \ldots$. Let n be a fixed positive integer. We make temporary use of the abbreviation $\sigma = L[q_1, q_2, \ldots, q_p]$ to signify that σ is a linear combination, with nonnegative multipliers, of the numbers in brackets. Since $r'_n \geqq r_n$, our hypotheses on the matrices $A(r)$ imply that

$$(3.21) \qquad \begin{aligned} s_n(r'_n) &= L[s_0(r'_n - 1), s_1(r'_n - 1), \ldots, s_n(r'_n - 1)] = \cdots \\ &= L[s_0(r_n), s_1(r_n), \ldots, s_n(r_n)] \\ &= L[s_0(r_n), \ldots, s_{n-1}(r_n)] + c_{nn} s_n(r_n) \end{aligned}$$

where $c_{nn} > 0$. Since $r_k \geqq r_{k-1}$, this reduction can be continued until we obtain nonnegative coefficients c_{nk} such that

$$(3.22) \qquad s_n(r'_n) = \sum_{k=0}^{n} c_{nk} s_k(r_k) \qquad n = 1, 2, 3, \ldots.$$

Since $c_{nk} \geqq 0$, we can show that this transformation is regular, and hence that $B(r'_n)$ includes $B(r_n)$, by showing that

$$(3.23) \qquad \lim_{n \to \infty} c_{nk} = 0 \qquad k = 0, 1, 2, \ldots.$$

and

$$(3.24) \qquad \lim_{n \to \infty} \sum_{k=0}^{n} c_{nk} = 1.$$

To prove (3.23), we use (3.22) and (1.7) to obtain

$$(3.25) \qquad \begin{aligned} s_n(r'_n) &= \sum_{j=1}^{n} c_{nj} s_j(r_j) \\ &= \sum_{j=0}^{n} c_{nj} \sum_{k=0}^{j} b_{jk}(r_j) s_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} c_{nj} b_{jk}(r_j) \right] s_k. \end{aligned}$$

Since the matrix $B(r'_n)$ which transforms s_k into $s_n(r'_n)$ is regular by hypothesis, it must be true that

$$(3.251) \qquad \lim_{n \to \infty} \sum_{j=k}^{n} c_{nj} b_{jk}(r_j) = 0, \qquad k = 0, 1, 2, \ldots.$$

Since all terms appearing in these sums are nonnegative, it follows that

$$(3.252) \qquad \lim_{n \to \infty} c_{nk} b_{kk}(r_k) = 0, \qquad k = 0, 1, 2, \ldots.$$

Since the hypothesis that $a_{kk}(r) > 0$ for each k and r implies that $b_{kk}(r) > 0$ for each k and r, the conclusion (3.23) follows from (3.252). To prove (3.24), we observe from (1.7) that $s_n(r_n)$ and $s_n(r'_n)$ are the $B(r_n)$ and $B(r'_n)$ transforms of the sequence s_n for which $s_n = 1$ when $n = 0, 1, 2, \ldots$. Since $B(r_n)$ and $B(r'_n)$ are regular by hypothesis, it follows that $s_n(r_n) = 1 + \varepsilon_n$ and $s_n(r'_n) = 1 + \varepsilon'_n$ where $\varepsilon_n \to 0$ and $\varepsilon'_n \to 0$ as $n \to \infty$. Use of (3.22) then gives

$$(3.26) \qquad 1 + \varepsilon'_n = \sum_{k=0}^{n} c_{nk}(1 + \varepsilon_k).$$

Since $c_{nk} \geqq 0$ and (3.23) holds, this implies (3.24) and Theorem 3.2 is proved.

The fact that the conclusion of Theorem 3.2 will fail to be obtainable if we delete one or the other of the hypotheses that $B(r'_n)$ and $B(r_n)$ are regular can be seen from the example of the unique sequence of matrices $A(r)$ for which the double sequence (1.3) takes the form

$$
\begin{array}{l}
s_0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \ldots \\
s_0,\ 2s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \ldots \\
s_0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \ldots \\
s_0,\ s_1,\ s_2,\ 2s_3,\ s_4,\ s_5,\ \ldots \\
\ldots\ldots\ldots\ldots\ldots
\end{array} \tag{3.27}
$$

Here $s_n(r) = s_n$ except that $s_r(r) = 2s_r$ when r is odd. Another example is obtained by making $s_r(r) = 2s_r$ for each $r > 0$. If, however, the matrices $A(r)$ satisfy the additional condition

$$\sum_{k=0}^{n} a_{nk}(r) = 1 \qquad n = 0, 1, 2, \ldots \tag{3.28}$$

then the matrices $B(r)$ satisfy the condition

$$\sum_{k=0}^{n} b_{nk}(r) = 1 \qquad n = 0, 1, 2, \ldots; \tag{3.281}$$

in this case the numbers ε'_n and ε_k in (3.26) are zero and we obtain the conclusion of Theorem (3.2) without the hypothesis that $B(r_n)$ is regular.

Theorem 3.3. *There is a monotone increasing sequence $R_1, R_2, \ldots$ such that $R_n \geqq 1$, $R_n \to \infty$, and the matrix transformation $B(r_n)$ includes each one of $B(1)$, $B(2)$, $B(3)$, ... whenever $1 \leqq r_n \leqq R_n$ and $r_n \to \infty$.*

Proof. With the sequence $R_1, R_2, \ldots$ determined as in Theorem 3.1, we obtain the desired conclusion with the aid of Theorem 3.2.

Theorem 3.4. *The regular transformations of the form $B(r_n)$ for which $r_1 \leqq r_2 \leqq r_3 \leqq \ldots$ constitute a consistent family.*

Proof. Let r'_n and r''_n denote monotone increasing sequences of integers such that $B(r'_n)$ and $B(r''_n)$ are regular. For each n, let r_n be the maximum of r'_n and r''_n. Then, for each n, one or the other of the two formulas

$$a_{nk}(r_n) = a_{nk}(r'_n), \quad a_{nk}(r_n) = a_{nk}(r'_n) \tag{3.41}$$

holds when $0 \leqq k \leqq n$. Thus regularity of $B(r'_n)$ and $B(r''_n)$ implies that of $B(r_n)$. Hence Theorem 3.2 implies that $B(r_n)$ includes both $B(r'_n)$ and $B(r''_n)$. Therefore $B(r'_n)$ and $B(r''_n)$ must be consistent and Theorem 3.4 is proved.

4. The condition $a_{nk}(r) \geqq 0$. We assumed in (1.01) that $a_{nk}(r) \geqq 0$ for each n, k, and r. If this hypothesis were deleted and replaced by the hypothesis

$$\sum_{k=0}^{n} |a_{nk}(r)| > M,$$

then the matrices $A(r)$ would still be regular, but Theorem 3.1 and our deductions from it would fail. The following example provides proof. Let T be the regular transformation which transforms s_n into $s_0, 2s_0 - s_1, 2s_1 - s_2, 2s_2 - s_3, \ldots$. Let $A(r) = T$ for each $r = 1, 2, \ldots$ so that $B(r) = T^r$. It is readily verified that, for this example, $B(r_n)$ is regular if and only if the sequence r_n is bounded.

5. Applicability of Theorem 3.4. Theorem 3.4 establishes existence of matrix methods $B(r_n)$ of summability such that $B(r_n)$ includes $B(r)$ for each $r = 1, 2, 3, \ldots$. This does not necessarily imply that $B(r_n)$ includes $A(r)$ for each $r = 1, 2, \ldots$. Suppose, for example, that A_1 and A_2 are two inconsistent methods. Then no method can include both A_1 and A_2. The point is, of course, that the relations $B \supset A_1$ and $B \supset A_2 A_1$ do not imply that $B \supset A_2$.

It therefore becomes of particular interest to know what given sequences of matrix methods are representable in the form $B(1), B(2), \ldots$ defined in Section 1. The answer is obvious. A given sequence $\tilde{B}(1), \tilde{B}(2), \ldots$ has the form if and only if (i) for each $r = 1, 2, 3, \ldots$ the matrix $\tilde{B}(r)$ is a regular triangular matrix of nonnegative elements which has an inverse and (ii) the matrices $\tilde{A}(1), \tilde{A}(2) \ldots$ defined by $\tilde{A}(1) = \tilde{B}(1)$ and

$$\tilde{A}(r) = \tilde{B}(r)\tilde{B}^{-1}(r-1) \qquad r = 2, 3, \ldots \tag{5.1}$$

are such that, for each $r = 1, 2, \ldots, \tilde{A}(r)$ is a regular triangular matrix of nonnegative elements which has an inverse. When a given sequence $\tilde{B}(r)$ has the properties (i) and (ii), putting $A(r) = \tilde{A}(r)$ gives $B(r) = \tilde{B}(r)$.

There is an important case in which the hypothesis that

$$B(r_n) \supset A(r)A(r-1)\ldots A(1), \qquad r = 1, 2, \ldots \tag{5.2}$$

implies that $B(r_n) \supset A(r)$ for each $r = 1, 2, \ldots$. Suppose the matrices $A(r)$, $r = 1, 2, \ldots$ satisfy the conditions of Section 1 and, in addition, constitute a commutative family in the sense that $A(r)A(s) = A(s)A(r)$ for each $r, s = 1, 2, \ldots$. Then

$$A(r)A(r-1)\ldots A(1) = A(1)A(2)\ldots A(r) \supset A(r) \tag{5.3}$$

and accordingly (5.2) implies that $B(r_n) \supset A(r)$ for each $r = 1, 2, \ldots$.

The Cesàro and Euler matrices are members of a large family of commutative matrices first studied by HURWITZ and SILVERMAN [1917] and by HAUSDORFF [1921].

6. Cesàro methods. For each positive number α, let $C(\alpha)$ denote the Cesàro matrix of order α which transforms a given sequence s_n into

$$\sigma_n(\alpha) = \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} \binom{n+\alpha}{n}^{-1} s_k. \tag{6.01}$$

If $\alpha_1, \alpha_2, \ldots$ is a monotone increasing sequence of positive numbers, then for each fixed r, the matrices $C(\alpha_r)$ and $C(\alpha_r)C^{-1}(\alpha_{r-1})$ are regular triangular matrices of nonnegative elements which have inverses; see HAUSDORFF [1921]. Hence our theorems apply to the case in which the matrices $B(r)$ are the Cesàro matrices $C(\alpha_n)$. It is well known and easy to show that, since the elements of Cesàro matrices of positive order are positive and satisfy the strong condition in (3.281), the compounded Cesàro matrix $C(\alpha_n)$ is regular if and only if its elements $C_{nk}(\alpha_n)$ are such that $\lim_{n\to\infty} C_{nk}(\alpha_n) = 0$ for each k and hence if and only if $\lim \alpha(n)/n = 0$. The nonregular matrix $C(\alpha_n)$ for which $\alpha_n = n$, and a closely related regular matrix that includes $C(\alpha)$ for each fixed positive α, has been studied by OBRECHKOFF [1926]; see also KOGBETLIANTZ [1931, page 47]. RUDBERG [1944, Théorème III] compared methods of the form $C(\alpha_n)$ with the Abel power series method. We turn to this subject here because the stated results are supported by inadequate arguments and some are false.

A series Σu_n of complex terms is evaluable to s by the Abel power series method P if the series in

$$P(x) = \sum_{k=0}^{\infty} x^k u_k \tag{6.02}$$

converges when $0 < x < 1$ to a function $P(x)$ such that $P(x) \to s$ as $x \to 1$.

Theorem 6.1. *If Σu_n is a series such that $\Sigma x^k u_k$ converges when $0 < x < 1$, and if $x_1, x_2, \ldots$ is a sequence such that $0 < x_n < 1$ and $x_n \to 1$, then there is a regular compounded Cesàro matrix $C(\alpha_n)$ such that*

$$\lim_{n\to\infty} |\sigma_n(\alpha_n) - P(x_{p(n)})| = 0 \tag{6.11}$$

where $p(1), p(2), \ldots$ is a monotone increasing sequence of positive integers which contains each positive integer and for which $p(n) \to \infty$ as $n \to \infty$.

Let Σu_n be a given series satisfying the hypotheses of the theorem. For each $q = 1, 2, 3, \ldots$ let $k(q)$ be the least integer greater than q such that

$$\sum_{k=k(q)}^{\infty} |x_q^k u_k| < \frac{1}{q}. \tag{6.12}$$

The series-to-sequence form of the Cesàro transformation (6.01) is

$$\sigma_n(\alpha) = \sum_{k=0}^{n} \binom{n-k+\alpha}{n-k} \binom{n+\alpha}{n}^{-1} u_k \tag{6.13}$$

The coefficient of u_k in (6.13) is

$$c_{nk}(\alpha) = \frac{n(n-1)(n-2)\ldots(n-k+1)}{(n+\alpha)(n+\alpha-1)(n+\alpha-2)\ldots(n+\alpha-k+1)}. \tag{6.14}$$

Setting $x = (1+\alpha/n)^{-1}$, we put this in the form

$$c_{nk}(\alpha) = \frac{1-\frac{1}{n}}{1-\frac{x}{n}} \frac{1-\frac{2}{n}}{1-\frac{2x}{n}} \cdots \frac{1-\frac{k-1}{n}}{1-\frac{(k-1)x}{n}} x^k. \tag{6.15}$$

With positive integers $p(1)$, $p(2)$, ... to be determined below, we define α_n by the equivalent formulas

(6.16) $$\alpha_n = n\left(\frac{1}{x_{p(n)}} - 1\right), \quad x_{p(n)} = 1\Big/\left(1 + \frac{\alpha_n}{n}\right)$$

and put the $C(\alpha_n)$ transform of Σu_n in the form

(6.17) $$\sigma_n(\alpha_n) = \sum_{k=0}^{\infty} \gamma_{nk}\, x_{p(n)}^k\, u_k$$

where $\gamma_{n0} = 1$,

(6.18) $$\gamma_{nk} = \frac{1-\frac{1}{n}}{1-\frac{x_{p(n)}}{n}} \, \frac{1-\frac{2}{n}}{1-\frac{2x_{p(n)}}{n}} \cdots \frac{1-\frac{k-1}{n}}{1-\frac{(k-1)x_{p(n)}}{n}}$$

when $0 < k \leqq n$, and $\gamma_{nk} = 0$ when $k > n$. Since $0 \leqq \gamma_{nk} \leqq 1$, we find that

(6.2) $$|\sigma_n(\alpha_n) - P(x_{p(n)})| = \sum_{k=0}^{\infty} |\gamma_{nk} - 1|\, x_{p(n)}^k\, |u_k|$$
$$\leqq \sum_{k=0}^{k(p(n))} |\gamma_{nk} - 1|\, x_{p(n)}^k\, |u_k| + \sum_{k=k(p(n))}^{\infty} x_{p(n)}^k\, |u_k|.$$

Since (6.12) shows that the last term of (6.2) is less than $1/p(n)$, we can obtain the desired conclusion (6.11) by determining a sequence $p(n)$ of the required type such that

(6.21) $$\sum_{k=0}^{k(p(n))} |\gamma_{nk} - 1|\, x_{p(n)}^k\, |u_k| < \frac{1}{p(n)}$$

for each sufficiently great n. Using (6.18), we see that we can choose an integer $n_1 > 1$ such that (6.21) holds when $p(n) = 1$ and $n \geqq n_1$. Then choose $n_2 > n_1$ such that (6.21) holds when $p(n) = 2$ and $n \geqq n_2$. Continue the process to obtain an increasing sequence n_j of integers such that (6.21) holds when $p(n) = j$ and $n \geqq n_j$. On setting $p(n) = 1$ when $1 \leqq n < n_2$, $p(n) = 2$ when $n_2 \leqq n < n_3$, and so on, we obtain the sequence $p(n)$ and complete the proof of Theorem 6.1. The impossibility of proving (6.11) with $p(n) = n$ follows from consideration of the series Σu_n for which $u_n = 1$, $s_n = n + 1$ and $P(x) = 1/(1-x)$. In this case $\sigma_n(\alpha_n)$ always lies between 0 and $n + 1$ while $P(x_n)$ could be $(n+1)^2$.

Theorem 6.3. *If Σu_n is a series evaluable to s by the Abel power series method P, then there is a regular compounded Cesàro method $C(\alpha_n)$ by which the series is also evaluable to s.*

This follows immediately from Theorem 6.1, since (6.11) and the consequence $\lim P(x_{p(n)}) = s$ of Abel evaluability imply that $\lim \sigma_n(\alpha_n) = s$. The same argument shows that if the Abel transform $P(x)$ exists over $0 < x < 1$ but, as is true in the case of the example $P(x) = \sin\{1/(1-x)\}$, different sequences x_n give different limit points $\lim P(x_n)$ of $P(x)$, then each limit point must be the actual limit of $\sigma_n(\alpha_n)$ for some regular compounded Cesàro matrix $C(\alpha_n)$. Such a compounded Cesàro matrix generates a method of summability not included by the Abel method, and two such matrices can generate inconsistent methods which, by Theorem 3.4, cannot both have the form $C(\alpha_n)$ where α_n is monotone increasing.

7. Euler methods. For each α for which $0 < \alpha < 1$, let $E(\alpha)$ denote the Euler matrix of order α which transforms a given sequence s_n into

$$E_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} s_k. \tag{7.01}$$

If $\alpha_1, \alpha_2, \ldots$ is a monotone decreasing sequence of positive numbers for which $\alpha_1 \leqq 1$ then, for each r the matrices $E(\alpha_r)$ and $E(\alpha_r)E^{-1}(\alpha_{r-1})$ are regular triangular matrices which have inverses and nonnegative elements; for these and other facts relating to Euler transformations, see AGNEW [1944], references given there, and HARDY [1949]. Hence the theorems of Section 3 apply to cases in which the matrices $B(r)$ are the matrices $E(\alpha_n)$. A compounded Euler matrix $E(\alpha_n)$ with elements $e_{nk}(\alpha_n)$ is regular if and only if $\lim_{n\to\infty} e_{nk}(\alpha_n) = 0$ for each k and hence if and only if $\lim n\alpha_n = \infty$. RUDBERG [1944, Théorème IV] compared methods $E(1/n)$ with the Borel exponential method B^*.

A series Σu_n with partial sums s_n is evaluable to s^* by the Borel exponential method B^* if the series in

$$\sigma^*(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k \tag{7.02}$$

converges for each $x > 0$ and $\sigma^*(x) \to s^*$ as $x \to \infty$.

Theorem 7.1. *If $s_0, s_1, \ldots$ is a sequence such that $\Sigma(x_k/k!)s_k$ converges for each $x > 0$ and if $x_1, x_2, \ldots$ is a sequence such that $x_n > 0$ and $x_n \to \infty$, then there is a regular compounded Euler matrix $E(\alpha_n)$ such that*

$$\lim_{n\to\infty} |E_n(\alpha_n) - \sigma^*(x_{p(n)})| = 0 \tag{7.11}$$

where $p(1), p(2), \ldots$ is a monotone increasing sequence of positive integers which contains each positive integer and for which $p(n) \to \infty$ as $n \to \infty$.

Let $s_0, s_1, \ldots$ be a given sequence satisfying the hypotheses of the theorem. For each $q = 1, 2, 3, \ldots$ let $k(q)$ be the least integer greater than q such that

$$\sum_{k=k(q)}^{\infty} (x_q^k/k!)\,|s_k| < \frac{1}{q}. \tag{7.12}$$

Choose n_1 such that $x_1 < n_1$ and $x_2 < n_1$. Let $\alpha_n = p(n) = 1$ when $1 \leqq n \leqq n_1$. With positive integers $p(n_1+1), p(n_1+2) \ldots$ to be determined below to satisfy the conditions of the theorem and the additional condition

$$x_{p(n)}/n < 1, \qquad n \geqq n_1, \tag{7.13}$$

we define α_n when $n > n_1$ by the equivalent formulas

$$\alpha_n = x_{p(n)}/n, \quad x_{p(n)} = n\alpha_n \tag{7.14}$$

and put the $E(\alpha_n)$ transform of s_n in the form

$$E_n(\alpha_n) = \sum_{k=0}^{n} \binom{n}{k} \alpha_n^k (1-\alpha_n)^{n-k} s_k = \sum_{k=0}^{\infty} \psi_{nk} \frac{x_{p(n)}^k}{k!} s_k \tag{7.15}$$

where

$$\psi_{nk} = \frac{n!}{(n-k)!\, n^k} \left(1 - \frac{x_{p(n)}}{n}\right)^{n-k} \tag{7.16}$$

when $0 \leqq k \leqq n$ and $\psi_{nk} = 0$ when $k > n$. Since $0 \leqq \psi_{nk} \leqq 1$ and $0 \leqq \exp \cdot \cdot [-x_{p(n)}] \leqq 1$, we find that when $n > n_1$

$$|E_n(\alpha_n - \sigma^*(x_{p(n)})| \leqq \sum_{k=0}^{k(p(n))} |\psi_{nk} - e^{-x_{p(n)}}| \frac{x_{p(n)}^k}{k!} |s_k| + \sum_{k=k(p(n))}^{\infty} \frac{x_{p(n)}^k}{k!} |s_k|. \tag{7.17}$$

Since (7.12) shows that the last term of (7.17) is less than $1/(p-n)$, we can obtain the desired conclusion (7.11) by determining a sequence $p(n)$ of the required type such that

$$\sum_{k=0}^{k(p(n))} |\psi_{nk} - e^{-x_{p(n)}}| \frac{x_{p(n)}^k}{k!} |s_k| < \frac{1}{p(n)} \tag{7.18}$$

for each sufficiently great n. It follows from (7.16) that if $p(n)$ has a constant value q, then (7.18) will hold for all sufficiently great values of n. We have already defined n_1. When $j > 1$ and n_{j-1} has been defined, choose n_j such that $n_{j-1} < n_j$, $x_k < n_j$ when $k = 1, 2, \ldots, j$, and such that (7.13) holds when $n \geqq n_j$ and $p(n) = j - 1$. In terms of $n_1, n_2, n_3, \ldots$ we now complete the definition of the sequence $p(n)$ by setting $p(n) = 1$ when $n_1 \leqq n < n_2$ and, for each $j = 2, 3, \ldots, p(n) = j - 1$ when $n_j \leqq n < n_{j+1}$. Then $p(n) = 1$ when $1 \leqq n < n_3$ and (7.18) holds when $n \geqq n_2$. To show that (7.13) holds, we observe that if $n_1 \leqq n < n_2$ then $x_{p(n)} = x_1 < n_1 \leqq n$ and if $j \geqq 2$ and $n_j \leqq n < n_{j+1}$ then

$$x_{p(n)} = x_{j-1} \leqq n_{j-1} < n_j \leqq n. \tag{7.19}$$

It is now obvious that the sequences $p(n)$ and α_n have all of the required properties and Theorem 7.1 is proved.

Theorem 7.2. *If Σu_n is a series with partial sums s_n which is evaluable to s by the Borel exponential method B^*, then there is a regular compounded Euler method $E(\alpha_n)$ by which the series is also evaluable to s.*

This follows from Theorem 7.1 in the same way that Theorem 6.3 follows from Theorem 6.1. Obvious modifications of the remarks following Theorem 6.3 apply here.

8. The binary, Euler, and Borel methods. Corresponding to each α in the interval $0 < \alpha < 1$, let $T(\alpha)$ denote the matrix of the *binary* transformation of order α which transforms a given sequence $s_0, s_1, s_2, \ldots$ into the sequence

$$s_0,\ (1-\alpha)s_0 + \alpha s_1,\ (1-\alpha)s_1 + \alpha s_2,\ \ldots. \tag{8.1}$$

The transformations $T(\alpha)$ were used by HURWITZ [1926] to illustrate the theory of Tauberian theorems. The special transformation $T(1/2)$ and its powers are among those studied by SILVERMAN and SZASZ [1944] and by SZASZ [1944]. The matrices $T(\alpha)$ satisfy the conditions imposed upon the matrices $A(r)$ in Section 1. Hence we may set $A(r) = T(\alpha)$ for each $r = 1, 2, \ldots$ and obtain $B(r) = T^r(\alpha)$ for each $r = 1, 2, \ldots$.

For each $r = 0, 1, 2, \ldots$, let the transform of a given sequence s_n by the matrix $T^r(\alpha)$ be denoted by

$$s_0(r, \alpha),\ s_1(r, \alpha),\ s_2(r, \alpha),\ \ldots; \tag{8.2}$$

in particular, $s_n(0, \alpha) = s_n$. To facilitate the writing of formulas, let $s_n(r, \alpha)$ be defined for negative integer values of n by the formulas

$$s_n(r, \alpha) = s_0(r, \alpha) = s_0, \qquad n = -1, -2, -3, \ldots. \tag{8.3}$$

Then, for each $n = 0, \pm 1, \pm 2, \ldots$,

$$s_n(1, \alpha) = (1-\alpha)s_{n-1} + \alpha s_n \tag{8.41}$$

and

$$\begin{aligned} s_n(2, \alpha) &= (1-\alpha)s_{n-1}(1, \alpha) + \alpha s_n(1, \alpha) \\ &= (1-\alpha)^2 s_{n-2} + 2\alpha(1-\alpha)s_{n-1} + \alpha^2 s_n. \end{aligned} \tag{8.42}$$

Thus when $r = 1$ and when $r = 2$ the formula

$$s_n(r, \alpha) = \sum_{k=n-r}^{n} \binom{r}{n-k} \alpha^{r+k-n}(1-\alpha)^{n-k} s_k \tag{8.43}$$

holds when $n = 0, \pm 1, \pm 2, \ldots$; and it is easily shown by induction that (8.43) holds for each $r = 1, 2, 3, \ldots$. The double sequence whose rows consist of the transforms of s_n by the various powers of $T(\alpha)$ turn out to be very interesting. In particular, the sequence of elements on the main diagonal of the double sequence is generated by the regular transformation by which s_n is evaluable to s if $s_n(n, \alpha) \to s$ where

$$s_n(n, \alpha) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} s_k; \tag{8.5}$$

and (8.5) is precisely the Euler transformation $E(\alpha)$ of order α. It follows from Theorem 3.2 that $T(\alpha)$ and all of its powers are included by $E(\alpha)$. Using the very well known fact (see HARDY [1949, p. 183]) that $E(\alpha)$ is included by B^*, we conclude that $T(\alpha)$ and all of its powers are included by B^*.

Therefore the extensive family F of regular transformations, whose elements comprise the totality of binary transformations $T(\alpha)$ for which $0 < \alpha < 1$ and the totality of positive integer powers of these, constitutes a consistent family F of transformations included by the Borel exponential method B^*.

9. The binary, Nörlund, and generalized Abel methods. Let $T(\alpha)$ be the binary transformation defined in Section 8. Each sequence $p_0, p_1, p_2, \ldots$ for which $p > 0$, $p_n \geqq 0$, and $p_n/P_n \to 0$ where $P_n = p_0 + p_1 + \cdots + p_n$, generates a Nörlund transformation $N(p_n)$ by which a sequence s_n is evaluable to s if $\sigma_n \to s$ where

$$\sigma_n = (p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n)/P_n, \qquad n = 0, 1, \ldots. \tag{9.1}$$

For each $r = 1, 2, 3, \ldots$ the transform $s_n(r, \alpha)$ of a sequence s_n by the transformation $T^r(\alpha)$ is almost identical with the transform $\sigma_n(r, \alpha)$ by the Nörlund transformation $N(p_n(r, \alpha))$ generated by the sequence $p_0(r, \alpha)$, $p_1(r, \alpha), \ldots$ for which

$$p_n(r, \alpha) = \binom{r}{n} \alpha^{r-n} (1-\alpha)^n, \qquad n = 0, 1, \ldots, r, \tag{9.2}$$

and $p_n(r, \alpha) = 0$ when $n > r$; in fact (8.43) shows that

$$s_n(r, \alpha) = \sigma_n(r, \alpha) \qquad n \geqq r. \tag{9.3}$$

It follows from this that $T^{(r)}(\alpha)$ is equivalent to $N(p_n(r, \alpha))$, and using results of Section 8 we see that such a Nörlund method is included by the Euler method $E(r)$ and hence is also included by the Borel exponential method B^*.

Let a series Σu_n with partial sums s_n be called evaluable to σ by the generalized Abel method P^* if the series in

$$f(z) = (1-z) \sum_{k=0}^{\infty} z^k s_k \tag{9.4}$$

has a positive radius of convergence R and the function $f(z)$ defined by (9.4) when $|z| < R$ determines, by analytic extension along radial lines emanating from the origin, a function $f(z)$ such that $f(z) \to \sigma$ as $z \to 1$ over the real interval $0 < z < 1$. It was shown by SILVERMAN and TAMARKIN [1929] and by TAMARKIN [1932] that if $N(p_n)$ is regular and $p_n > 0$ then $N(p_n) \subset P^*$; and the same proof shows that if $N(p_n)$ is regular, $p_0 > 0$, and $p_n \geqq 0$, then $N(p_n) \subset P^*$. Therefore $T(\alpha)$ and all of its powers are included in P^*. This shows again that the family F, consisting of the binary transformations $T(\alpha)$ for which $0 < \alpha < 1$ and their powers, constitute a consistent family.

It is not true that the ordinary Abel method P includes all members of F. For example, it is easy to show that the sequence $s_n = (-2)^n$ is evaluable $T(2/3)$ to 0; but in this case the series $\Sigma z^n s_n$ has radius of convergence $1/2$ and therefore the sequence is nonevaluable by the Abel method P.

10. The symmetric binary method and its powers. Let T denote the symmetric binary method $T(1/2)$ which transforms a given sequence $s_0, s_1, s_2, \ldots$ into

(10.1) $$s_0, \frac{1}{2}(s_0 + s_1), \frac{1}{2}(s_1 + s_2), \frac{1}{2}(s_2 + s_3), \ldots.$$

SZASZ [1944] used this simple method T to illustrate many points in the theory of summability. In particular, he found it very easy to show that for each $r = 0, 1, 2, \ldots$ the alternating zeta series, the series in the right member of the familiar equation

(10.2) $$(1 - 2^{s-1})\zeta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

is evaluable T^r for each $s = \sigma + it$ in the half-plane $\sigma > -r$. This simple result and Sections 8 and 9 show that the alternating zeta series is evaluable over the entire plane by the Euler method $E(1/2)$ and hence also by the Borel exponential method B^* and the generalized Abel power series method P^*.

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