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On distribution functions with a limiting stable distribution function

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1. Introduction[1]

The general stable d.f.'s[2] have been introduced by P. LÉVY[3] who defined them implicitly by help of their characteristic functions and explicitly as limiting distribution functions. To every α, $0<\alpha\leq 2$ there belong stable distribution functions $G_\alpha(x)$ and these have the following property.[4] If $\ast$ denotes the convolution and σ_1, σ_2 and σ are positive numbers with

$$\sigma_1^\alpha+\sigma_2^\alpha=\sigma^\alpha \tag{1}$$

we have

$$G_\alpha\left(\frac{x}{\sigma_1}\right)\ast G_\alpha\left(\frac{x}{\sigma_2}\right)=G_\alpha\left(\frac{x}{\sigma}\right). \tag{2}$$

In the following we shall only use the property (2) and the fact that $G_\alpha(x)$ has derivatives of bounded variation of all orders.[5]

Let now $F(x)$ denote a d.f. and let $F^{*n}(x)$ denote the n-fold convolution of $F(x)$ with itself. W. DOEBLIN has given necessary and sufficient conditions which $F(x)$ must satisfy, if $F^{*n}(b_n x)$ shall converge to a stable d.f. $G_\alpha(x)$, $0<\alpha<2$.[6] If $\alpha=2$ then $G_\alpha(x)$ is the normal d.f. and the conditions for convergence are then well known.

Our method can be used to get the conditions for convergence and we shall return to this problem later. Here we shall give estimations of the remainder term

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[1] Mr. KAI LAI CHUNG drew my attention to the general stable d.f.'s in a discussion which I had with him on the application of my methods.

[2] d.f. — read distribution function(s).

[3] P. LÉVY (1), pp. 94–97, 198–204.

[4] We call α the exponent of the stable d.f.

[5] We omit the singular case, when $G_\alpha(x)$ is discontinuous.

[6] W. DOEBLIN (1), pp. 71–96.

$$F^{*n}(x) - G_\alpha^{*n}(x)$$

when $F^{*n}(n^{\frac{1}{\alpha}} x)$ converges to $G_\alpha^{*n}(n^{\frac{1}{\alpha}} x) = G_\alpha(x)$.[1] Our main result is the following theorem.[2]

Theorem. *Let $G(x)$ denote a stable d.f. of exponent α, $0 < \alpha \leq 2$, and let $g(x)$ denote a non decreasing function such that there exist positive numbers λ_1 and λ_2 with $\lambda_2 \leq [\lambda_1] + 1$ for which*

$$\frac{1}{a}\left(\frac{x}{y}\right)^{\lambda_1} \leq \left(\frac{x}{y}\right)^{\alpha} \frac{g(x)}{g(y)} \leq a \left(\frac{x}{y}\right)^{\lambda_2}$$

with a constant a when $x > y > 0$.[3] *Further suppose that $F(x)$ is a d.f. which satisfies the following conditions*

1⁰ $$\int_{-\infty}^{\infty} |x|^{\alpha} g(|x|)\, |d[F(x) - G(x)]| < \infty,$$

2⁰ $$\int_{-\infty}^{\infty} x^{\nu}\, d[F(x) - G(x)] = 0$$

for $\nu = 0, 1, \ldots, \nu_0$. Putting

$$\delta(n) = \begin{cases} g^{-1}(n^{\frac{1}{\alpha}}) & \text{if } \nu_0 \geq [\lambda_1] \\ n^{-\frac{\nu_0 + 1 - \alpha}{\alpha}} & \text{if } \nu_0 < [\lambda_1] \end{cases}$$

we then have for fixed ν and large n

(i) $$\binom{n}{\nu} G^{*n-\nu}(x) * [F(x) - G(x)]^{*\nu} = \begin{cases} O[\delta^{\nu}(n)] & \text{always,} \\ o[\delta^{\nu}(n)], & \text{if } \nu_0 \geq [\lambda_1] \\ & \phantom{\text{if }} \lambda_2 < [\lambda_1] + 1. \end{cases}$$

Assuming that

3⁰ $$\sum_{n=1}^{\infty} n^{-1} g^{-(s+1)}(n^{\frac{1}{\alpha}}) < \infty,$$

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[1] In the general case $F^{*n}(b_n x)$ converges to $G_\alpha(x)$ when b_n is some suitable possibly more complicated increasing function of n. Compare DOEBLIN loc. cit.

[2] In the following we write $G(x)$ instead of $G_\alpha(x)$.

[3] We can for instance consider $g(x) = x^{\varrho}$ with some exponent $\varrho > 0$ or $g(x) = \log x$. If $\lim_{x \to \infty} \frac{g(cx)}{g(x)} = \gamma(c) \neq 0$ for some c in the interval $0 < c < 1$ we may choose $\lambda_1 - \alpha$ and $\lambda_2 - \alpha$ arbitrarily close to $\frac{\log \gamma(c)}{\log c}$ and then $g(x)$ has the mentioned property.

we have the asymptotic expansion

(ii) $$F^{*n}(x)=\sum_{\nu=0}^{s}\binom{n}{\nu}G^{*n-\nu}(x)*[F(x)-G(x)]^{*\nu}+r_n^{(s+1)}$$

with

(iii) $$r_n^{(s+1)}=\begin{cases}0\,[\delta^{s+1}(n)]+0\,(n^{-\frac{1}{\alpha}}) & \textit{always},\\ o\,[\delta^{s+1}(n)]+0\,(n^{-\frac{1}{\alpha}}), & \textit{if}\quad \nu_0\geq[\lambda_1],\\ & \lambda_2<[\lambda_1]+1.\end{cases}$$

If furthermore the condition[1]

4° $$\lim_{p\to\infty} V\{F^{*p}(x)*[F(x)-G(x)]\}=0$$

is satisfied then $0\,(n^{-\frac{1}{\alpha}})$ *may be omitted in* (iii).

Remark I. The estimations of $r_n^{(s+1)}$ in (iii) can be given such that they are independent on other quantities than s, α, λ_1, λ_2, ν_0 and n. If 4° holds and $0\,(n^{-\frac{1}{\alpha}})$ is omitted the estimation is more dependent on $F(x)$.

Remark II. Independently on the condition 3° the expansion (ii) is asymptotic with

$$r_n^{(s+1)}=\begin{cases}0\,[\delta^{s+1}(n)]+0\,[\delta^{\frac{1}{\lambda_2}}(n)]+0\,(n^{-\frac{1}{\alpha}}) & \text{always}\\ o\,[\delta^{s+1}(n)]+o\,[\delta^{\frac{1}{\lambda_2}}(n)]+0\,(n^{-\frac{1}{\alpha}}), & \text{if}\quad \nu_0\geq[\lambda_1],\\ & \lambda_2<[\lambda_1]+1.\end{cases}$$

Remark III. If we change the assumptions 1° and 2° to

1° a $$\int_{-\infty}^{\infty}|x|^{\alpha-1}g(|x|)\,|F(x)-G(x)|\,dx<\infty$$

and

2° a $$\int_{-\infty}^{\infty}x^{\nu}[F(x)-G(x)]\,dx=0$$

for $\nu=0, 1, \ldots, \nu_0-1$, the theorem still holds.

Remark IV. The theorem can easily be generalized to the case of multidimensional d.f.[2]

[1] $V\{\ \}$ denotes the total variation.
[2] Compare H. Bergström (2).

If we can chose $g(x)=x^r$ with some $r>0$, it follows from (ii) and (iii) that $F^{*n}(x)$ cannot have jumps of larger order of magnitude than $n^{-\frac{1}{\alpha}}$. In the case $\alpha=2$ we know that $F^{*n}(x)$ has jumps of the order $n^{-\frac{1}{\alpha}}$, if for instance $F(x)$ is the Bernoullian distribution function.

We shall prove the theorem by the same method that we have used for the proof of the corresponding special theorem for the normal d.f.[1] In fact we have only to change that proof at some points in order to get the proof of the general theorem. However, we shall give the complete proof here.

We start by proving a lemma.

Lemma. *Let $g(x)$, $F(x)$, $G(x)$, λ_1, λ_2 and α be defined as in the theorem, and suppose that the condition 1° is satisfied. Then the moments*

$$\int_{-\infty}^{\infty} x^\nu\, d\,[F(x)-G(x)]=\beta^{(\nu)}$$

exist for $\nu=0, \ldots, [\lambda_1]$, and if $\psi(x)$ is a function with bounded continuous derivatives of all orders $\leq [\lambda_1]+1$ for all x, we have for $p>0$

$$\text{(i)}\qquad \psi\left(\frac{x}{p}\right) * [F(x)-G(x)]=\sum_{\nu=0}^{[\lambda_1]} \frac{(-1)^\nu \beta^{(\nu)}}{\nu!\, p^\nu}\, \psi^{(\nu)}\left(\frac{x}{p}\right)+$$

$$+\,0\,[p^{-\alpha} g^{-1}(p)]\left\{\operatorname*{Max}_x \left|\psi^{([\lambda_1])}\left(\frac{x}{p}\right)\right| + \operatorname*{Max}_x \left|\psi^{([\lambda_1]+1)}\left(\frac{x}{p}\right)\right|\right\}.$$

Here 0 *may be changed against* o *if $\lambda_2<[\lambda_1]+1$.*

Proof: We put $h(x)=x^\alpha g(x)$, $[\lambda_1]=\lambda$. Expanding $\psi\left(\frac{x-t}{p}\right)$ by Taylor's formula we get

$$\text{(3)}\qquad \psi\left(\frac{x-t}{p}\right)=\sum_{\nu=0}^{\lambda} \frac{(-1)^\nu t^\nu}{\nu!\, p^\nu}\, \psi^{(\nu)}\left(\frac{x}{p}\right)+g_\lambda(x,t),$$

where the remainder term may be written in either of the forms

$$\text{(4 a)}\qquad g_\lambda(x,t)=\frac{(-1)^\lambda}{\lambda!}\left(\frac{t}{p}\right)^\lambda \left[\psi^{(\lambda)}\left(\frac{x-\Theta t}{p}\right)-\psi^{(\lambda)}\left(\frac{x}{p}\right)\right],\quad 0<\Theta<1$$

or

$$\text{(4 b)}\qquad g_\lambda(x,t)=\frac{(-1)^{\lambda+1}}{(\lambda+1)!}\left(\frac{t}{p}\right)^{\lambda+1} \psi^{(\lambda+1)}\left(\frac{x-\Theta_1 t}{p}\right),\quad 0<\Theta_1<1.$$

Owing to 1° the moments $\beta^{(\nu)}$ exists for $\nu=0, \ldots, \lambda$. Applying (4 a) and (4 b) we then get

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[1] H. BERGSTRÖM (2), p. 5.

$$\psi\left(\frac{x}{p}\right) * [F(x)-G(x)] = \int_{-\infty}^{\infty} \psi\left(\frac{x-t}{p}\right) d[F(t)-G(t)] =$$

$$= \sum_{\nu=0}^{\lambda} \frac{(-1)^{\nu} \beta^{(\nu)}}{\nu!\, p^{\nu}} \psi^{(\nu)}\left(\frac{x}{p}\right) + \int_{-\infty}^{\infty} g_{\lambda}(x, t)\, d[F(t)-G(t)].$$

Using (4 a) for $|t|>b$ and (4 b) for $|t|\leq b$ and observing that, owing to the made assumption about $g(x)$,

$$|t^{\lambda}| \leq a \frac{b^{\lambda}}{h(b)} h(|t|) \quad \text{for} \quad |t| \geq b,$$

$$|t^{\lambda+1}| \leq a \frac{b^{\lambda+1}}{h(b)} h(|t|) \quad \text{for} \quad |t| \leq b,$$

we get

$$\left| \int_{-\infty}^{\infty} g_{\lambda}(x, t)\, d[F(t)-G(t)] \right| \leq$$

$$\leq \frac{2\, b^{\lambda} a}{\lambda!\, h(b)\, p^{\lambda}} \operatorname{Max}_{x} \left| \psi^{(\lambda)}\left(\frac{x}{p}\right) \right| \int_{|t|>b} h(|t|)\, |d[F(t)-G(t)]| +$$

$$+ \frac{b^{\lambda+1} a}{(\lambda+1)!\, h(b)\, p^{\lambda+1}} \operatorname{Max}_{x} \left| \psi^{(\lambda+1)}\left(\frac{x}{p}\right) \right| \int_{|t|\leq b} h(|t|)\, |d[F(t)-G(t)]|.$$

Putting

$$r(b) = \int_{|t|\geq b} h(|t|)\, |d[F(t)-G(t)]|,$$

we now have to consider the quantities

$$\frac{b^{\lambda} r(b)}{h(b)\, p^{\lambda}} \quad \text{and} \quad \frac{b^{\lambda+1}}{h(b)\, p^{\lambda+1}}. \tag{5}$$

Here $r(b)$ is bounded for $b=0$ and nonincreasing for $b>0$ with $\lim_{b\to\infty} r(b)=0$. If we choose $b=p$ the quantities (5) are both $0[h^{-1}(p)]$. Therefore (i) holds. But for $b<p$ we have more over

$$\begin{cases} \dfrac{b^{\lambda} r(b)}{h(b)\, p^{\lambda}} = \left(\dfrac{b}{p}\right)^{\lambda} \dfrac{h(p)}{h(b)} \dfrac{r(b)}{h(p)} \leq a \dfrac{r(b)}{h(p)} \left(\dfrac{p}{b}\right)^{\lambda_2-\lambda} \\[2ex] \dfrac{b^{\lambda+1}}{h(b)\, p^{\lambda+1}} = \left(\dfrac{b}{p}\right)^{\lambda+1} \dfrac{h(p)}{h(b)} \dfrac{1}{h(p)} \leq \dfrac{a}{h(p)} \left(\dfrac{b}{p}\right)^{1+\lambda-\lambda_2}. \end{cases} \tag{6}$$

If $\lambda_2 < 1 + \lambda$ it is obviously possible to let $b = b(p) < p$ tend to infinity in such a way that the right sides of (6) are $o\left[\frac{1}{h(p)}\right]$. Thus the lemma is proved.

We are now going to prove the theorem. Then it is sufficient to consider the case $\nu_0 \geq [\lambda_1]$, for if $\nu_0 < [\lambda_1]$ we may choose $g(x) = x^{\nu_0+1-\alpha}$.

We get the relation (i) of the theorem in the case $\nu = 1$, $\nu_0 \geq \lambda = [\lambda_1]$ if we apply the lemma with

$$\psi\left(\frac{x}{p}\right) = G^{*\, n-1}(x) = G\left(\frac{x}{p}\right), \quad p = (n-1)^{\frac{1}{\alpha}}.$$

Generally (i) of the theorem may be proved by help of induction. Putting

$$\psi\left(\frac{x}{p}\right) = G\left(\frac{x}{p}\right) \ast [F(x) - G(x)]^{*\,\nu-1}, \quad p = (n-\nu+1)^{\frac{1}{\alpha}}$$

we get (i) of the lemma. Here is

$$\psi^{(\mu)}\left(\frac{x}{p}\right) = G^{(\mu)}\left(\frac{x}{p}\right) \ast [F(x) - G(x)]^{*\,\nu-1}$$

and we can apply the lemma again putting now

$$\psi\left(\frac{x}{p}\right) = G^{(\mu)}\left(\frac{x}{p}\right) \ast [F(x) - G(x)]^{*\,\nu-2}$$

for $\mu = \lambda$ and $\mu = \lambda + 1$. In this way we prove (i) of the theorem by help of induction.

In order to prove (iii) we shall also use induction. We assume that the inequality

$$|r_\varrho^{(s+1)}| < C\,\delta_1^{s+1}(\varrho) \tag{7}$$

holds for $\varrho < n$ with

$$\delta_1(\varrho) = \begin{cases} \operatorname{Max}\,[g^{-(s+1)}(\varrho^{\frac{1}{\alpha}}),\ \varrho^{-\frac{1}{\alpha}}], & \text{if} \quad \nu_0 \geq \lambda \\ \operatorname{Max}\,[\varrho^{-(s+1)\frac{(\nu_0+1-\alpha)}{\alpha}},\ \varrho^{-\frac{1}{\alpha}}], & \text{if} \quad \nu_0 < \lambda \end{cases}$$

and a constant C and then we prove that

$$r_n^{(s+1)} < C(n)\,\delta_1^{s+1}(n) \tag{8}$$

where $C(n) < C$ if n is larger than some constant n_0 and also

$$C(n) = o(n), \quad \text{if} \quad \nu_0 \geq [\lambda_1],\ \lambda_2 < [\lambda_1] + 1$$

and

$$n^{-\frac{1}{\alpha}} = o\,[\delta_1^{s+1}(n)].$$

Obviously (7) holds with a sufficiently large constant C for $n \le n_0$. Thus the theorem follows by induction.

In order to prove (8), we consider

$$r_n^{(s+1)}(x) * \phi\left(\frac{x}{p_1}\right),$$

where $\phi(x)$ is the normal d.f. with the mean value 0 and the dispersion 1, and give an estimation in the form

$$\left| r_n^{(s+1)}(x) * \phi\left(\frac{x}{p_1}\right) \right| < M(n, p_1), \tag{9}$$

where $M(n, p_1)$ depends on n, p_1 and C. Then applying a lemma for Weierstrass singular integral[1], and observing that

$$\frac{d}{dx} G^{*n}(x) = n^{-\frac{1}{\alpha}} G'(n^{-\frac{1}{\alpha}} x),$$

we obtain

$$| r_n^{s+1} | \le \text{Max}\, [k\, M(n, p_1),\ \beta\, p_1\, n^{-\frac{1}{\alpha}}] \tag{10}$$

with constants k and β. By suitable choice of p_1 it can then be proved that (8) holds.

For abbreviation we put

$$\binom{n}{\nu} G^{*\,n-\nu}(x) * [F(x) - G(x)]^{*\nu} = \Delta_n^{(\nu)}.$$

Owing to an identical expansion we have[2]

$$r_n^{(s+1)} = \sum_{\mu=s+1}^{n} \binom{\mu-1}{s} F^{*\,n-\mu} * (F-G)^{*\,s+1} * G^{*\,\mu-s-1},$$

i.e.

$$r_n^{(s+1)} = \sum_{\mu=s+1}^{n} F^{*\,n-\mu} * \Delta_{\mu-1}^{(s)} * (F-G) \tag{11}$$

and

$$F^{*n} = \sum_{\nu=0}^{s} \Delta_n^{(\nu)} + r_n^{(s+1)}. \tag{12}$$

Let $m_1 = \left[\frac{n}{2}\right]$, $n > n_0$ where n_0 is a suitable fixed integer. Observing that then owing to the property of $g(x)$

$$\frac{\delta(m_1)}{\delta(n)} > c_0$$

[1] H. Bergström (1), p. 143.

[2] H. Bergström (2), p. 2.

with a constant c_0,[1] we get in the same way as we have found (i) of the theorem

$$(13)\qquad \left|\sum_{\mu=m_1+1}^{n} F^{*n-\mu} * \Delta_{\mu-1}^{(s)} * (F-G)\right| \le \operatorname{Max}_x \sum_{\mu=m_1+1}^{n} \left|\Delta_{\mu-1}^{(s)} * (F-G)\right| < a_0(n)\,\delta^{s+1}(n)$$

where $a_0(n)$ is smaller than some constant c_1 for all n and tends to zero for large n if $\nu_0 \ge [\lambda_1]$, $\lambda_2 < [\lambda_1]+1$. In order to estimate the members in the right side of (11) for $\mu \le m_1$, we express F^{*n} by (12) changing n to $n-\mu$. Then we get

$$(14)\qquad \sum_{\mu=s+1}^{m_1} F^{*n-\mu} * \Delta_{\mu-1}^{(s)} * (F-G) = \sum_{\mu=s+1}^{m_1} \sum_{\nu=0}^{s} \Delta_{n-\mu}^{(\nu)} * \Delta_{\mu-1}^{(s)} * (F-G) + \sum_{\mu=s+1}^{m_1} r_{n-\mu}^{(s+1)} * \Delta_{\mu-1}^{(s)} * (F-G).$$

Analogous to (13) we find for $n > n_0$

$$(15)\qquad \left|\sum_{\mu=s+1}^{m_1} \sum_{\nu=0}^{s} \Delta_{n-\mu}^{(\nu)} * \Delta_{\mu-1}^{(s)} * (F-G)\right| < a_1(n)\,\delta^{s+1}(n)$$

where $a_1(n)$ is defined in the same way as $a_0(n)$.

Further we have with an integer $m_2 \le m_1$.

$$(16)\qquad \left|\sum_{\mu=s+1}^{m_1} r_{n-\mu}^{(s+1)} * \Delta_{\mu-1}^{(s)} * (F-G) * \phi\left(\frac{x}{p_1}\right)\right| \le$$

$$\le \sum_{\mu=m_2+1}^{m_1} \binom{\mu-1}{s} \operatorname{Max}_x \left| r_{n-\mu}^{(s+1)} * G^{*\mu-s-1} * (F-G)^{*s+1}\right| + \sum_{\mu=s+1}^{m_2} \binom{\mu-1}{s} \operatorname{Max}_x \left| r_{n-\mu}^{(s+1)} * \phi\left(\frac{x}{p_1}\right) * (F-G)^{*s+1}\right|.$$

In the members of the first sum of the right side of (16) we apply the lemma with

$$\psi\left(\frac{x}{p}\right) = r_{n-\mu}^{(s+1)} * G\left(\frac{x}{p}\right), \qquad p = (\mu-s-1)^{\frac{1}{\alpha}}$$

and observe that for any positive integer ν

$$\left|\psi^{(\nu)}\left(\frac{x}{p}\right)\right| = \left| r_{n-\mu}^{(s+1)} * G^{(\nu)}\left(\frac{x}{p}\right)\right| \le \operatorname{Max}_x \left| r_{n-\mu}^{(s+1)}\right| \int_{-\infty}^{\infty} \left|G^{(\nu+1)}(x)\right| dx.^{2}$$

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[1] The "constants" c in the following depend only on the quantities a, s, λ_1, λ_2, α and ν_0 defined in the theorem.

[2] All derivatives of $G(x)$ are of bounded variation, what for instance easily may be obtained from BERGSTHÖM (3).

Assuming that (7) holds for $\varrho < n$ and observing that

$$\delta_1 (n-\mu) < c_2 \, \delta_1 (n)$$

with a constant c_2 for $\mu \leq m_1$ we get

$$| r_{n-\mu}^{(s+1)} | < C \, c_3 \, \delta_1^{(s+1)} (n)$$

with a constant c_3 and thus according to the lemma

$$\sum_{\mu=m_2+1}^{m_1} \binom{\mu-1}{s} \operatorname*{Max}_x | r_{n-\mu}^{(s+1)} * G^{*\mu-s-1} * (F-G)^{*s+1} | < \\ < C \, \delta_1^{s+1} (n) \, a_2 (m_2) \sum_{\mu=m_2+1}^{m_1} \frac{1}{\mu-1} \delta^{s+1} (\mu-1), \tag{17}$$

where $a_2 (m_2)$ is defined in the same way as $a_0 (n)$. In the members of the second sum of the right side of (16) we apply the lemma with

$$\psi \left(\frac{x}{p} \right) = r_{n-\mu}^{(s+1)} * \phi \left(\frac{x}{p_1} \right), \quad p = p_1 .$$

Then we get in the same way as we have obtained (17)

$$\sum_{\mu=s+1}^{m_2} \binom{\mu-1}{s} \operatorname*{Max}_x \left| r_{n-\mu}^{(s+1)} * \phi \left(\frac{x}{p_1} \right) * (F-G)^{*s+1} \right| < \\ < C \, \delta_1^{s+1} (n) \, a_3 (p_1) \binom{m_2}{s+1} p_1^{-\alpha (s+1)} \delta^{s+1} (p_1^{\alpha}) \tag{18}$$

where $a_3 (p_1)$ is defined in the same way as $a_0 (n)$. Combining (11), (13), (15), (17) and (18), we find that we can choose $M (n, p_1)$ in (9) equal to

$$M (n, p_1) = \delta^{s+1} (n) \, [a_0 (n) + a_1 (n)] + \\ + \delta_1^{s+1} (n) \, C \left[a_2 (m_2) \sum_{\mu=m_2+1}^{m_1} \frac{1}{\mu-1} \delta^{s+1} (\mu-1) + \right. \\ \left. + a_3 (p_1) \binom{m_2}{s+1} p_1^{-\alpha (s+1)} \delta^{s+1} (p_1^{\alpha}) \right] . \tag{19}$$

Owing to (10) we have then to show that

$$\operatorname*{Max}_x \, [k \, M (n, p_1), \; \beta \, p_1 \, n^{-\frac{1}{\alpha}}] < C \, \delta_1^{s+1} (n).$$

Now the sum in the right side of (19) is the partial sum of a convergent series. (If $\delta (n) = n^{-\frac{\nu_0+1-\alpha}{\alpha}}$ we may assume $\nu_0 + 1 - \alpha > 0$ for otherwise there is nothing to prove.)

Therefore we may choose m_2 so large that

$$k\, a_2(m_2) \sum_{\mu=m_2+1}^{m_1} \frac{1}{\mu-1}\, \delta^{s+1}(\mu-1) < \frac{1}{3}$$

where k is given in (10). Having thus determined m_2, we may choose p_1 so large that

$$k\, a_3(p_1) \binom{m_2}{s+1} p_1^{-\alpha(s+1)}\, \delta^{s+1}(p_1^{\alpha}) < \frac{1}{3}.$$

If further C is so large that

$$k\,[a_0(n) + a_1(n)] < \frac{C}{3}$$

we have

$$k\, M(n, p_1) < C\, \delta^{s+1}(n).$$

If C also is so large that

$$\beta\, p_1 < C$$

where β is given in (10) it follows from (10) that (8) holds with $C(n) = C$, if (7) is satisfied for $\varrho < n$ and n is larger than some value n_0. But (8) holds with a sufficiently large constant C for $n \le n_0$. Thus (2) holds with $C(n) = C$ for all n.

In order to prove the stronger inequality when $\nu_0 \ge [\lambda_1]$, $\lambda_2 < [\lambda_1] + 1$, we have only consider the case

$$n^{-\frac{1}{\alpha}} = 0\,[\delta^{s+1}(n)], \quad \nu_0 \ge [\lambda_1].$$

Then we know by the just proved theorem that (8) holds with a constant C and we get (19) with this known constant. It is then possible to let p_1 tend to infinity in such a way that

$$\beta\, p_1\, n^{-\frac{1}{\alpha}} = 0\,[\delta^{s+1}(n)]$$

and to let m_2 tend to infinity in such a way that

$$\binom{m_2}{s+1} p_1^{-\alpha(s+1)}\, \delta^{(s+1)}(p_1^{\alpha}) = o\,(1),$$

and then is also

$$\sum_{\mu=m_2+1}^{m_1} \frac{1}{\mu-1}\, \delta^{\mu-1}(\mu-1) = o\,(1)$$

when n tends to infinity. Therefore (iii) holds.

Let now the condition 4° be satisfied. Then it follows from a general theorem[1] that $0\,(n^{-\frac{1}{\alpha}})$ may be omitted.

[1] H. BERGSTRÖM (2), p. 4.

At last we have to prove the statement in the remarks of our theorem. The contents of the remarks I and III are immediately drawn from the proof given above. In order to prove the remark II, we only need to consider the case $\delta(n) = g^{-1}(n^{\frac{1}{\alpha}})$, $\lambda_1 = \alpha$. If we put

$$\delta_2^{s+1}(n) = \text{Max}\,[\delta^{s+1}(n),\ \delta^{\frac{1}{\lambda_2}}(n),\ n^{-\frac{1}{\alpha}}]$$

and assume that (7) holds with δ_2 instead of δ_1 we obviously obtain (19) with $\delta_2(n)$ instead of $\delta_1(n)$. Choosing $m_1 = m_2$, putting $p_1 = n^{\frac{1}{\alpha}} q$ and observing that for $0 < q < 1$

$$\frac{\delta(p_1^\alpha)}{\delta(n)} = \frac{g(n^{\frac{1}{\alpha}})}{g(n^{\frac{1}{\alpha}} q)} < a\, q^{-(\lambda_2 - \alpha)} \tag{20}$$

we may write (19) in the form

$$M(n, p_1) < \delta^{s+1}(n)\,[a_0(n) + a_1(n)] + C\, a_4(n^{\frac{1}{\alpha}} q)\, q^{-\lambda_2 (s+1)}\, \delta_2^{s+1}(n)\, \delta^{s+1}(n). \tag{21}$$

Further (10) may be written

$$|r_n^{(s+1)}| \le \text{Max}\,[k\, M(n, p_1),\ \beta q]. \tag{22}$$

Now we choose

$$q = C^{\frac{1}{\lambda_2 (s+1)+1}}\, \delta_2^{\frac{1}{\lambda_2}}(n).$$

If then C is so large that

$$k\,[a_0(n) + a_1(n)] < \frac{C}{2},$$

$$k\, a_4(n^{\frac{1}{\alpha}} q)\, C^{-\frac{\lambda_2 (s+1)}{\lambda_2 (s+1)+1}} < \frac{1}{2}$$

and

$$\beta\, C^{-\frac{\lambda_2 (s+1)}{\lambda_2 (s+1)+1}} < 1$$

for $n > n_0$ and n_0 is so large that $q < 1$ for $n > n_0$, it follows from (22) that

$$r_n^{(s+1)} < C\, \delta_2^{s+1}(n). \tag{23}$$

Since this inequality obviously holds for $n \le n_0$ if C is sufficiently large it then holds for all n with suitable C. (If $\delta_2(n)$ doesn't tend to zero when n tends to infinity it isn't perhaps possible to get $q < 1$ but then (23) is trivial.)

In order to prove the stronger inequality when $\nu_0 \ge [\lambda_1]$, $\lambda_2 < [\lambda_1] + 1$, we now start from the true inequality (23) and then get (21). We have only to consider the case

$$n^{-\frac{1}{\alpha}} = o\,[\delta^{s+1}(n)] + o\,[\delta_2^{\frac{1}{\lambda_2}}(n)]$$

and then it is possible to choose

$$q = o\left[\delta_2^{\frac{1}{\lambda_2}}(n)\right]$$

in such a way that $n^{\frac{1}{\alpha}} q \to \infty$ and

$$a_4\left(n^{\frac{1}{\alpha}} q\right) q^{-\lambda_2 (s+1)} \delta_2^{s+1}(n) = o(1).$$

Then the right side of (21) is $o\left[\delta^{s+1}(n)\right] + o\left[\delta^{\frac{1}{\lambda_2}}(n)\right]$.

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