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Fourier transforms of the class $\mathfrak{L}_p$

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It is well known that the theorem of RIESZ-FISCHER and the theorem of PLANCHEREL, dealing with Fourier transforms of the classes $\mathfrak{L}_2$ on the circle and line, respectively, have analogues for other classes $\mathfrak{L}_p$ $(1 < p < 2)$. Thus the theorem of YOUNG-HAUSDORFF states that if f is any function on $[0, 2\pi]$ such that $\int_0^{2\pi} |f(x)|^p\,dx < \infty$, then the numbers $c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{inx} f(x)\,dx$ have the property that

$$\sum_{n=-\infty}^{\infty} |c_n|^{p'} < \infty, \tag{1}$$

where $p' = \frac{p}{p-1}$, and

$$\left[\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right]^{\frac{1}{p}} \geq \left[\sum_{n=-\infty}^{\infty} |c_n|^{p'}\right]^{\frac{1}{p'}} \tag{2}$$

(See for example [5], pp. 189–202.) An analogous theorem, proved by TITCHMARSH (see [3], pp. 96–107), shows that every function f in $\mathfrak{L}_p(-\infty, \infty)$ admits a Fourier transform of class $\mathfrak{L}_{p'}$ with norm in $\mathfrak{L}_{p'}(-\infty, \infty)$ majorized by a constant times the $\mathfrak{L}_p$ norm of f. For both of these cases, examples can be given to show that not all sequences of class $l_{p'}$ or functions of class $\mathfrak{L}_{p'}$ can be obtained as Fourier transforms of the class $\mathfrak{L}_p$. (See [5], p. 190, and [3], pp. 111–112.) It is the purpose of the present note to show that this phenomenon must appear for all infinite locally compact Abelian groups.

Throughout the present note, let G stand for a locally compact Abelian group. Integration with regard to a suitably normalized Haar measure on G is indicated by expressions such as

$$\int_G f(x)\,dx. \tag{3}$$

For all numbers $r \geq 1$, the symbol $\mathfrak{L}_r$ denotes the space of all complex-valued Haar measurable functions f such that

$$\int_G |f(x)|^r\,dx < \infty, \tag{4}$$

under the usual definitions of addition and multiplication by complex numbers. $\mathfrak{L}_r$ is normed by

$$(5) \qquad \|f\|_r = \Big[\int_G |f(x)|^r \, dx\Big]^{\frac{1}{r}}.$$

Let $\mathfrak{C}_{\infty\infty}(G)$ denote the space of all continuous complex-valued functions on G each of which vanishes outside of some compact set. Let G^* be the group of all continuous characters of G, topologized in the usual fashion ([4], pp. 99–100). The expression (x, y) is used to denote the value of the character $y \varepsilon G^*$ at the point $x \varepsilon G$, or, dually, the value of the character $x \varepsilon G$ at the point $y \varepsilon G^*$.

For a function $f \varepsilon \mathfrak{C}_{\infty\infty}(G)$, the Fourier transform Tf is defined by the usual expression

$$(6) \qquad Tf(y) = \int_G (x, y) f(x) \, dx.$$

Throughout the present note, for every number $p > 1$, let $p' = \frac{p}{p-1}$. A. WEIL has shown ([4], pp. 116–117), by using the convexity theorem of M. RIESZ, that the mapping T of (6) has the property that $Tf \varepsilon L_{p'}(G^*)$ and that $\|Tf\|_{p'} \leq \leq \|f\|_p$, for $1 < p \leq 2$. Thus T can be extended by continuity to a linear transformation T_p with domain $\mathfrak{L}_p(G)$ and range contained in $\mathfrak{L}_{p'}(G^*)$ such that:

$$(7) \qquad T_p(\mathfrak{L}_p(G)) \text{ is dense in } \mathfrak{L}_{p'}(G^*);$$

$$(8) \qquad T_p \text{ is linear;}$$

$$(9) \qquad \|T_p f\|_{p'} \leq \|f\|_p.$$

(Assertion (7) requires separate proof.) It follows immediately that T_p is a one-to-one mapping. Our aim is to prove the following fact.

Theorem. If G is a locally compact Abelian infinite group and if $1 < p < 2$, then the image $T_p(\mathfrak{L}_p(G))$ is a dense set of the first category in $\mathfrak{L}_{p'}(G^*)$, and the functions in $\mathfrak{L}_{p'}(G^*)$ which are not Fourier transforms comprise a dense set of the second category.

This theorem was suggested by a question raised by I. SEGAL [2]. The proof is based on the following two lemmata.

Lemma A. Let G be any infinite locally compact group and let p be a number greater than 1. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $\mathfrak{L}_p(G)$ such that f_n converges weakly to zero in $\mathfrak{L}_p(G)$ and

$$(10) \qquad \|f_{n_1} + f_{n_2} + \cdots + f_{n_m}\|_p = m^{\frac{1}{p}}$$

for all subsets $\{f_{n_1}, f_{n_2}, \ldots, f_{n_m}\}$ of $\{f_n\}_{n=1}^{\infty}$ $(m = 1, 2, 3, \ldots)$.

Suppose first that G is discrete. Then let $x_1, x_2, x_3, \ldots$ be any countably infinite sequence of distinct points in G, and let $f_n(x) = 1$ or 0 as $x = x_n$ or $x \neq x_n$. For an arbitrary bounded linear functional M on $\mathfrak{L}_p(G)$, there exists a function $h \varepsilon \mathfrak{L}_{p'}(G)$ such that $M(f) = \int_G h(x) f(x)\,dx = \sum_{x \varepsilon G} h(x) f(x)$ for all $f \varepsilon \mathfrak{L}_p(G)$. Since $\sum_{x \varepsilon G} |h(x)|^{p'} < \infty$, we have $\lim_{n\to\infty} M(f_n) = \lim_{n\to\infty} h(x_n) = 0$. The equality (10) clearly holds for this sequence $\{f_n\}_{n=1}^{\infty}$.

If G is not discrete, then the Haar measure μ of every open set U containing the identity is positive but can be made arbitrarily small for appropriately chosen U. It is then apparent that there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint measurable sets in G such that $\mu(A_n) > 0$ $(n = 1, 2, 3, \ldots)$ and $\lim_{n\to\infty} \mu(A_n) = 0$. Write $\mu(A_n)$ as α_n; and define $f_n(x)$ as being either $\alpha_n^{-\frac{1}{p}}$ or 0 as $x \varepsilon A_n$ or x non εA_n. It is plain that (10) holds for this sequence $\{f_n\}_{n=1}^{\infty}$. To show that f_n converges weakly to zero, consider first any bounded measurable function φ on G. We then have

$$\left|\int_G f_n(x)\varphi(x)\,dx\right| \le \sup_{x \varepsilon G} |\varphi(x)| \cdot \alpha_n^{1-\frac{1}{p}},$$

and thus $\lim_{n\to\infty} \int_G f_n(x)\varphi(x)\,dx = 0$. For an arbitrary function $h \varepsilon \mathfrak{L}_{p'}(G)$ and $\varepsilon > 0$, there exists a bounded measurable function φ such that $\|\varphi - h\|_{p'} < \varepsilon$. Applying HÖLDER's inequality, we find

$$\left|\int_G [\varphi(x) - h(x)] f_n(x)\,dx\right| \le \|f_n\|_p \cdot \|\varphi - h\|_{p'} = \|\varphi - h\|_{p'} < \varepsilon.$$

From this, it follows that $\overline{\lim_{n\to\infty}} \left|\int_G h(x) f_n(x)\,dx\right| \le \varepsilon$, and hence f_n converges weakly to zero.

Lemma B. Let G be any locally compact group, let q be a number ≥ 2, and let $\{g_n\}_{n=1}^{\infty}$ be any sequence of functions in $\mathfrak{L}_q(G)$ which converges weakly to zero. Then there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ and a positive constant A such that

$$\|g_{n_1} + g_{n_2} + \cdots + g_{n_m}\|_q \le A\, m^{\frac{1}{2}}$$

for $m = 1, 2, 3, \ldots$.

This lemma has been proved for real spaces $\mathfrak{L}_q$ by BANACH and MAZUR. (See [1], pp. 197–199.) Their proof is stated for real $\mathfrak{L}_q$ on [0, 1] but can be carried over *verbatim* for real $\mathfrak{L}_q$ on an absolutely arbitrary measure space. To apply the proof of BANACH-MAZUR to the present case, which treats a complex space $\mathfrak{L}_q$, we need only note that a sequence $\{g_n\}_{n=1}^{\infty}$ converges weakly to zero if and only if the real and imaginary parts $\{\mathfrak{R} g_n\}_{n=1}^{\infty}$ and $\{\mathfrak{I} g_n\}_{n=1}^{\infty}$ converge weakly to zero with respect to bounded linear functionals on $\mathfrak{L}_q$ which are real for real functions.

We remark also that Lemmata A and B hold for general measure spaces.

We can now prove our Theorem. Suppose that $1 < p < 2$, that G is an infinite locally compact Abelian group, and assume that every function in $\mathfrak{L}_{p'}(G^*)$ is the Fourier transform of a function in $\mathfrak{L}_p(G)$. The transformation T_p thus maps $\mathfrak{L}_p(G)$ continuously onto $\mathfrak{L}_{p'}(G^*)$. A theorem of BANACH ([1], p. 41, Théorème 5) shows that the inverse transformation T_p^{-1} is also continuous. Thus there exists a constant $C > 0$ such that

$$(11) \qquad \|Tf\|_{p'} \leq \|f\|_p \leq C \|Tf\|_{p'}$$

for all $f \varepsilon \mathfrak{L}_p(G)$. Now consider the sequence $\{f_n\}_{n=1}^{\infty}$ described in Lemma A, for the space $\mathfrak{L}_p(G)$. It is plain that the sequence $\{Tf_n\}_{n=1}^{\infty}$ converges weakly to zero in $\mathfrak{L}_{p'}(G^*)$. By Lemma B, there exist a subsequence $\{Tf_{n_k}\}_{k=1}^{\infty}$ and a positive constant A such that

$$(12) \qquad \|\sum_{k=1}^{m} Tf_{n_k}\|_{p'} \leq A m^{\frac{1}{2}}.$$

Combining (10), (11), and (12), we see that

$$(13) \qquad m^{\frac{1}{p}} = \|\sum_{k=1}^{m} f_{n_k}\|_p \leq \|\sum_{k=1}^{m} Tf_{n_k}\|_{p'} \leq A C m^{\frac{1}{2}}.$$

As (13) holds for $m = 1, 2, 3, \ldots$, we see at once that $\frac{1}{p} \leq \frac{1}{2}$, which contradicts our hypothesis. Hence T cannot map $\mathfrak{L}_p(G)$ onto $\mathfrak{L}_{p'}(G^*)$. A theorem of BANACH ([1], p. 38, Théorème 3) shows that $T_p(\mathfrak{L}_p(G))$ must be of the first category; since $\mathfrak{L}_{p'}(G^*)$ is complete, the set of functions in $\mathfrak{L}_{p'}(G^*)$ which are not Fourier transforms must be of the second category and accordingly dense.

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