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A note on a problem of Boas

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Boas has in a paper [1] generalized certain theorems of Plancherel and Pólya [2] concerning simultaneous convergence of

$$\int_{-\infty}^{+\infty} |f(x)|^p \, dx \quad \text{and} \quad \sum_{-\infty}^{+\infty} |f(\lambda_n)|^p$$

for entire functions $f(z)$ of exponential type. He also puts the question how to treat corresponding problems for functions regular in a half-plane, especially what may be precisely stated as follows.

Problem. *Let $f(z)$ be regular for $x \geq 0$, $z = x + iy$, and such that, if $z \to \infty$ in this half-plane,*

$$\limsup \frac{\log |f(z)|}{|z|} = c, \qquad 0 < c < \infty. \tag{1}$$

Let $\varphi(t)$ be, for $t \geq 0$, a non-decreasing convex function of $\log t$ with $\varphi(0) = 0$.

Consider further a sequence of positive numbers $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ such that

$$\inf_n (\lambda_{n+1} - \lambda_n) \geq 2\delta > 0. \tag{2}$$

Does then

$$\int_0^{\infty} \varphi\{|f(x)|\} \, dx < \infty \tag{3}$$

imply

$$\sum_0^{\infty} \varphi\{e^{-c\delta} |f(\lambda_n)|\} < \infty? \tag{4}$$

At first I thought the answer ought to be negative. Therefore the following affirmative proof was not finally elaborated until I heard Professor Lennart Carleson express a contrary opinion: that my original condition (5) was superfluous.

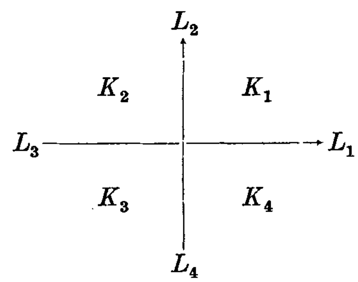
By means of an example Boas shows that the factor $e^{-c\delta}$ in (4) cannot be dropped. It signifies a number arbitrarily close to 1 since—as soon as (2) is fulfilled—δ may be chosen arbitrarily close to 0.

The proof is given in two steps.

Theorem I. *Condition (3) implies (4) if we impose the additional condition*

$$\int_0^{\frac{1}{2}} \frac{\varphi(t)}{t \log \frac{1}{t}} dt < \infty. \tag{5}$$

Proof. We denote the coordinate half-axes and the quadrants, counted in the positive direction and starting with the positive x-axis, by $L_1, \ldots, L_4$, and $K_1, \ldots, K_4$ respectively.



We do not repeat here a proof of the fact (proved e.g. by Boas) that (1) and (3) imply that it is possible, for each $c' > c$, to find a constant B such that

$$|f(z)| \leq B e^{c' |y|}. \tag{6}$$

Choosing $c'' > c'$ we consider the function

$$\psi(z) = \varphi \{ e^{-c'' |y|} |f(x+iy)| \}. \tag{7}$$

The function $\psi(z)$ is subharmonic both within K_1 and within K_4. In K_1, for instance, we have $\psi(z) = \varphi\{|e^{ic''z} f(z)|\}$ which is subharmonic in consequence of the properties of φ. From (6) we infer that $\psi(z)$ is bounded.

The essential point of the proof is to make sure of the convergence for $y \neq 0$ of

$$\Psi(y) = \int_0^\infty \psi(x+iy)\, dx; \tag{8}$$

according to (3) we know that $\Psi(0)$ is finite.

In order to estimate (8) we now form in K_1—and analogously in K_4—the harmonic majorant of the bounded subharmonic function $\psi(z)$ whose boundary values agree with those of ψ. We so obtain for, e.g., $z \in K_1$:

$$\psi(z) \leq h_1(z) + h_2(z), \tag{9}$$

where $h_1(z)$ is harmonic and $= \psi(x)$ on L_1, $= 0$ on L_2, and $h_2(z)$ harmonic and $= 0$ on L_1, $= \psi(iy)$ on L_2. By reflection in L_2 we extend $h_1(z)$ to be harmonic in $K_1 + K_2$ with boundary values $-\psi(-x)$ on L_3. Analogously $h_2(z)$ will be defined in $K_1 + K_4$ with the values $-\psi(-iy)$ on L_4.

According to the representation formula for functions harmonic in a half-plane we obtain

$$h_1(x+iy)=\frac{1}{\pi}\int_0^\infty y\,\psi(\xi)\left\{\frac{1}{(\xi-x)^2+y^2}-\frac{1}{(\xi+x)^2+y^2}\right\}d\xi \tag{10}$$

and

$$h_2(x+iy)=\frac{1}{\pi}\int_0^\infty x\,\psi(i\eta)\left\{\frac{1}{(\eta-y)^2+x^2}-\frac{1}{(\eta+y)^2+x^2}\right\}d\eta. \tag{11}$$

From this we obtain

$$\int_0^\infty h_1(x+iy)\,dx=\frac{2}{\pi}\int_0^\infty \psi(\xi)\operatorname{arctg}\frac{\xi}{y}\,d\xi\le\int_0^\infty \psi(\xi)\,d\xi=\Psi(0) \tag{12}$$

and

$$\int_0^\infty h_2(x+iy)\,dx=\frac{1}{\pi}\int_0^\infty \psi(i\eta)\log\left|\frac{\eta+y}{\eta-y}\right|d\eta\le A+C\int_{2\delta}^\infty\frac{\psi(i\eta)}{\eta}\,d\eta, \tag{13}$$

where A and C may be taken as fixed constants if we restrict the variation of y to some interval $0\le y\le\delta$.

From (7), (6) and the fact that φ is non-decreasing we obtain $\psi(i\eta)\le\varphi(t)$, where $t=Bc^{-(c''-c')\eta}$. After introducing t as variable instead of η we then obtain (5) as a condition for convergence in (13) and also—after addition of (12) and (13)—as a condition for uniform boundedness of $\Psi(y)$ for $0\le y\le\delta$. Because of the analogous situation in K_4 we may here just as well write $|y|\le\delta$.

To finish the proof we again follow the paper quoted. From the foregoing we infer that

$$\int_{-\delta}^{+\delta}\int_0^\infty \psi(x+iy)\,dx\,dy<\infty. \tag{14}$$

For $|y|\le\delta$ it holds true that

$$\psi(x+iy)\equiv\varphi\{e^{-c''|y|}|f(x+iy)|\}\ge\varphi\{e^{-c''\delta}|f(x+iy)|\}.$$

This latter function is subharmonic for $|y|\le\delta$, whereas ψ is not. In the strip $|y|\le c\delta/c''$ it holds that $\psi\ge\varphi\{e^{-c\delta}|f|\}$. Each number $\varphi\{e^{-c\delta}|f(\lambda_n)|\}$ is accordingly less than or equal to the average of ψ over a circular disc around λ_n, i.e. (14) implies (4) and Theorem I is proved.

It now remains to examine the condition (5). If this be *not* satisfied it must hold true that

$$\int_0^{1/2}\frac{\varphi(t)\,dt}{t\log 1/t}=+\infty. \tag{15}$$

Theorem II. *The class of functions $f(z)$, satisfying* (1), (3) *and* (15) *is empty.*

Proof. Let us study $\log |f(z)|$ in K_1. The weak growth according to (1) implies that $\log |f(z)|$ within K_1 is dominated by the harmonic function which is obtained in form of an integral of the boundary values along $L_1 + L_2$. Let $z_0 = x_0 + i y_0$ be an arbitrary point within K_1. We then obtain

$$\log |f(z_0)| \leq A_0 + \frac{y_0}{\pi} \int_0^\infty \log |f(x)| \frac{4 x x_0 \, dx}{[(x-x_0)^2 + y_0^2]\,[(x+x_0)^2 + y_0^2]}. \tag{16}$$

Here A_0 denotes a convergent integral along L_2 corresponding to (11), and the second integral corresponds to (10). Since $|f(x)|$ is bounded this integral either converges or diverges towards $-\infty$ at the same time as

$$\int_1^\infty \frac{\log |f(x)|}{x^3} dx \quad \text{or} \quad - \int_1^\infty \frac{\log \frac{1}{|f(x)|}}{x^3} dx.$$

For simplicity's sake we set $u(x) = \log \{1/|f(x)|\}$ and $g\{u(x)\} = \varphi\{|f(x)|\}$. Thus $g(u)$ is a steadily decreasing function of u. Formula (3) can now be written

$$\int_0^\infty g\{u(x)\}\, dx < \infty. \tag{17}$$

The substitution $u = -\log t$ transforms (15) into

$$\int_{\log 2}^\infty \frac{g(u)}{u} du = +\infty. \tag{18}$$

We shall show that these relations imply that the integral in (16) diverges, i.e. that

$$\int_1^\infty \frac{u(x)}{x^3} dx = +\infty. \tag{19}$$

Let $v(x)$ be obtained from $u(x)$ by a measure-conserving rearrangement, such that $v(x)$ is steadily increasing (towards $+\infty$ according to (17)). Then

$$\int_0^\infty g\{v(x)\}\, dx = \int_0^\infty g\{u(x)\}\, dx \text{ and } \int_0^\infty \frac{v(x)}{x^3} dx \leq \int_0^\infty \frac{u(x)}{x^3} dx. \tag{20}$$

Summing up, the present problem may be stated as follows:
From

$$\int_0^\infty g\{v(x)\}\, dx < \infty \tag{21}$$

and
$$\int_1^{\infty} \frac{g(v)}{v} dv = +\infty \tag{18}$$

we want to deduce
$$\int_1^{\infty} \frac{v(x)}{x^3} dx = +\infty. \tag{22}$$

In (18) we may consider $v = v(x)$, where x varies from some number $a > 0$ to $+\infty$. By partial integration we may restate (21) and (18) as

$$\overline{\lim_{A \to +\infty}} \left\{ [g \cdot x]_a^A + \int_a^A x\, d(-g) \right\} < \infty \tag{23}$$

and

$$\lim_{A \to +\infty} \left\{ [g \log v]_a^A + \int_a^A \log v\, d(-g) \right\} = +\infty. \tag{24}$$

Here g is non-negative and $-g$ steadily increasing; accordingly there must exist arbitrarily large numbers x_n such that

$$\log v(x_n) > x_n \quad \text{or} \quad v(x_n) > e^{x_n}. \tag{25}$$

Since then

$$\int_{x_n}^{\infty} \frac{v(x)}{x^3} dx > \int_{x_n}^{\infty} \frac{e^{x_n}}{x^3} dx = \frac{e^{x_n}}{2 x_n^2} \to +\infty \quad \text{as } x_n \to +\infty,$$

the validity of (22) is obvious. For each z_0 in K_1 we thus obtain $f(z_0) = 0$, i.e. $f(z) \equiv 0$ and, therefore, without interest in connection with functions of exponential type.

Thus the condition (5) is redundant, and it is demonstrated that (3) implies (4).

REFERENCES

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