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On the existence of a largest subharmonic minorant of a given function

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1. Introduction

Suppose that E is an open connected set in a k-dimensional Euclidean space. We say that a real-valued function $u(x)$ in E is subharmonic if it satisfies the following conditions (Radó [3], §§ 1.1, 2.3):

(i) *$u(x)$ is bounded above on every compact subset of E.*

(ii) *$u(x)$ is upper semicontinuous on E, i.e. for every real number a the set of points x, where*

$$u(x) < a,$$

is an open set.

(iii) *For every k-dimensional compact sphere $S_R(\xi) \subset E$, with centre ξ and radius R, we have*

$$u(\xi) \leq \frac{1}{S_R} \int_{S_R(\xi)} u(x)\, dx, \tag{1}$$

where S_R denotes the volume of the sphere, and where the integration is carried out with respect to the (k-dimensional) Lebesgue measure.

Let us now suppose that $F(x)$ is a given non-negative, upper semicontinuous function on E. We allow the function to assume the value $+\infty$. We shall then consider the class $\{F\}$ of all subharmonic functions $u(x)$, such that

$$u(x) \leq F(x)$$

for every $x \in E$.

It is easy to realize that if two functions $u_1(x)$ and $u_2(x)$ belong to $\{F\}$, then the same is true for the function

$$\operatorname{Max}\{u_1(x), u_2(x)\}.$$

The corresponding property holds for any finite number of functions in $\{F\}$. Our aim in this paper is to show that under certain conditions also the function

$$M(x) = \sup_{u \in \{F\}} u(x)$$

belongs to $\{F\}$, i.e. that $F(x)$ has a largest subharmonic minorant.

Because of our assumption that $F \geq 0$, we see that for any $u \in \{F\}$

$$\operatorname{Max}\{u(x), 0\} \in \{F\}.$$

Hence, if we denote by $\{F^+\}$ the class of all non-negative functions in $\{F\}$, we obviously have

$$M(x) = \sup_{u \in \{F^+\}} u(x).$$

The above-mentioned problem is trivial if $k=1$, for then the class of subharmonic functions coincides with the class of convex functions. If $k \geq 2$, however, the question has been discussed in several papers. A main result is the following theorem by Sjöberg [4] and Brelot [1] (cf. also Green [2]).

Theorem 1. *$M(x)$ is subharmonic if and only if it is bounded on every compact subset of E.*

Hence we can confine ourselves to the problem of finding conditions on E and on $F(x)$, under which $M(x)$ is bounded. This problem was discussed for $k=2$ by Sjöberg [4]. Our results are, however, somewhat more general, and we shall use elementary methods, which can be carried over to the corresponding problem for more general classes of functions. Thus our results will remain true for the class of functions which we obtain if the right-hand member of the inequality (1) is exchanged for

$$\frac{B}{S_R} \int_{S_R(\xi)} u(x)\, dx,$$

where B is any fixed number ≥ 1. Moreover, the property "upper semicontinuous" in the stated definitions may be exchanged for "measurable" without affecting the results.

Theorem 1 is true even if we make these modifications. The necessity part is obvious, and we shall briefly sketch a proof of the sufficiency. We form the function

$$H(x) = \operatorname{Min}\{F(x), G(x)\},$$

where for any $\xi \in E$,

$$G(\xi) = \inf_{R \mid S_R(\xi) \subset E} \left\{ \sup_{u \in \{F^+\}} \frac{B}{S_R} \int_{S_R(\xi)} u(x)\, dx \right\}.$$

Obviously

$$u(x) \leq H(x) \leq F(x)$$

for any $x \in E$ and any $u \in \{F^+\}$. Furthermore it can be shown that $H(x)$ fulfils the three in the above-mentioned way modified conditions in the definition of subharmonicity. The only difficult part is the verification of (ii); however, for any fixed R, the function

$$\int_{S_R(\xi)} u(x)\, dx$$

is continuous, uniformly in u, at every point ξ with $S_R(\xi) \subset E$, and this has the effect that the functions

$$\sup_{u \in \{F^+\}} \frac{B}{S_R} \int\limits_{S_R(\xi)} u(x)\, dx$$

are continuous functions of ξ whenever they are defined, thus $G(\xi)$ is upper semicontinuous. Apparently the above-mentioned properties of $H(x)$ imply that it is the largest function in the class $\{F^+\}$. Hence the existence of a "subharmonic" $M(x)$ is secured.

It may be observed that with this change in the definitions, Theorem 1 is no longer trivial in the case $k=1$.

2. A general sufficient condition

In the case $k=2$ the following theorem was known by Beurling (cf. [4] p. 319). However, no proof has been published.

Theorem 2. *$M(x)$ is bounded on every compact subset of E if for some $\varepsilon > 0$*

$$\int\limits_E [\log^+ F(x)]^{k-1+\varepsilon}\, dx < \infty.$$

To prove the theorem we need the following lemma, where $u(x)$ is an arbitrary function in the class $\{F^+\}$, and where l_ν is the measure of the set E_ν where

$$e^\nu \le u(x) < e^{\nu+1}.$$

l_ν is finite for $\nu > 0$, if the condition in the theorem is fulfilled.

Lemma 1. *Let D be a positive constant and λ a positive integer, both so large that*

$$\frac{e}{D^k S_1} + \frac{1}{e^\lambda} \le 1,$$

where S_1 is the volume of the k-dimensional unit sphere. Then the following is true:

If for some integer ν and some point $x_\nu \in E$

$$u(x_\nu) \ge e^\nu \tag{2}$$

and

$$S_R(x_\nu) \subset E,$$

where

$$R > D(l_{\nu-\lambda} + l_{\nu-\lambda+1} + \cdots + l_\nu)^{\frac{1}{k}},$$

then $S_R(x_\nu)$ contains a point $x_{\nu+1}$, where

$$u(x_{\nu+1}) \ge e^{\nu+1}.$$

Proof of Lemma 1. Suppose that $u(x) < e^{\nu+1}$ in $S_R(x_\nu)$. We are going to show that this implies a contradiction.

(2) and (1) give
$$e^{\nu} \le u(x_\nu) \le \frac{1}{S_R} \int\limits_{S_R(x_\nu)} u(x)\,dx. \tag{3}$$

Now let us denote by S' the set of all points in $S_R(x_\nu)$ which do not belong to the set

$$\bigcup_{\nu-\lambda}^{\nu} E_n.$$

Apparently $u(x) \le e^{\nu-\lambda}$, if $x \in S'$. Hence the right-hand member of (3) is

$$\le \frac{1}{S_R} \int\limits_{\bigcup_{\nu-\lambda}^{\nu} E_n} u(x)\,dx + \frac{1}{S_R} \int\limits_{S'} u(x)\,dx$$
$$\le \frac{1}{S_R} e^{\nu+1} (l_{\nu-\lambda} + \cdots + l_\nu) + \frac{1}{S_R} e^{\nu-\lambda} S_R < e^{\nu} \left(\frac{e R^k}{D^k S_R} + \frac{1}{e^{\lambda}} \right) \le e^{\nu},$$

which is impossible.

Proof of Theorem 2. Let D and λ be chosen in the way described in Lemma 1. Suppose that for some point $x_n \in E$ and for some $u \in \{F^+\}$

$$u(x_n) \ge e^n,$$

where n is an integer satisfying $n > \lambda$. Then according to Lemma 1 every sphere around x_n with a radius

$$> D \sum_{\nu=n}^{m} (l_{\nu-\lambda} + \cdots + l_\nu)^{\frac{1}{k}}$$

contains either a boundary point of E or a point $x_m \in E$ where

$$u(x_m) \ge e^m.$$

However, since $u(x)$ is bounded on every compact subset of E, this implies that the distance between x_n and the boundary of E is

$$\le D \sum_{n}^{\infty} (l_{\nu-\lambda} + \cdots + l_\nu)^{\frac{1}{k}} \le D \sum_{n}^{\infty} (l_{\nu-\lambda}^{\frac{1}{k}} + \cdots + l_\nu^{\frac{1}{k}})$$
$$\le D(\lambda+1) \sum_{n-\lambda}^{\infty} l_\nu^{\frac{1}{k}} = D(\lambda+1) \sum_{n-\lambda}^{\infty} \frac{1}{\nu^{\frac{k-1+\varepsilon}{k}}} \nu^{\frac{k-1+\varepsilon}{k}} l_\nu^{\frac{1}{k}}.$$

Using Hölder's inequality, we see that the right-hand member is

$$\le D(\lambda+1)\left[\sum_{n-\lambda}^{\infty}\frac{1}{\nu^{\frac{k-1+\varepsilon}{k-1}}}\right]^{\frac{k-1}{k}}\left[\sum_{n-\lambda}^{\infty}\nu^{k-1+\varepsilon}\,l_\nu\right]^{\frac{1}{k}}$$

$$\le \delta_n\left[\sum_{1}^{\infty}\nu^{k-1+\varepsilon}\,l_\nu\right]^{\frac{1}{k}}\le \delta_n\left\{\int_E[\log^+ u(x)]^{k-1+\varepsilon}\,dx\right\}^{\frac{1}{k}}\le \delta_n\left\{\int_E[\log^+ F(x)]^{k-1+\varepsilon}\,dx\right\}^{\frac{1}{k}},$$

where δ_n is independent of the choice of $u(x)$ and tends to 0, when $n\to\infty$. Hence if we have. at some point in E,

$$M(x) > e^n,$$

we may conclude that the distance between this point and the boundary of E is

$$\le \delta_n\left[\int_E[\log^+ F(x)]^{k-1+\varepsilon}\,dx\right]^{\frac{1}{k}}.$$

This shows that $M(x)$ is bounded on any subset of E which is situated at a positive distance from the boundary of E, and hence the theorem is proved.

Remark. The theorem is apparently still true under a weaker assumption, namely if for every compact set $C\subset E$ there exists a number $\varepsilon>0$ such that

$$\int_C[\log^+ F(x)]^{k-1+\varepsilon}\,dx<\infty.$$

3. A similar theorem under more restrictive assumptions

If $k=2$, the case when

$$F(x^1,x^2)=F(x^1),$$

where x^1 and x^2 are the Cartesian coordinates, and when E is a rectangle with its sides parallel to the coordinate axes, is of particular interest. In that case the exponent $1+\varepsilon$ in Theorem 2 may be exchanged for 1. This is an easy consequence of Theorème III in Sjöberg [4]. We thall now study the corresponding problem for an arbitrary k.

We assume that $x^1, x^2, \ldots, x^k$ are Cartesian coordinates of the space. Let p be a positive integer $<k$, and let O be an open set in the p-dimensional Euclidean space which has the Cartesian coordinates $x^1, x^2, \ldots, x^p$. a_i and b_i, where $i=p+1, p+2, \ldots, k$, are assumed to be finite real numbers.

We define E as the set of all points which satisfy

$$(x^1, x^2, \ldots x^p)\in O$$

and

$$a_i<x^i<b_i \qquad i=p+1,\ p+2,\ \ldots.\ k.$$

Then we let F be any function, defined in E, which only depends of the first p coordinates, i.e.

$$F(x) = F(x^1, x^2, \ldots, x^p). \tag{4}$$

Let mE denote the Lebesgue measure of E. There exists a function $y(m)$, defined and non-increasing in the interval $O \leq m < mE$ and such that, for every y_0, the one-dimensional measure of the set where $y(m) > y_0$ is the same as the k-dimensional measure of the subset of E, where $F(x) > y_0$.

Theorem 3. *$M(x)$ is bounded on every compact subset of E if*

$$\int_0^{mE} \log^+ y(m)\, d(m^{\frac{1}{p}}) < \infty.$$

For the proof we need the following lemma, where $u(x)$ is an arbitrary function in the class $\{F^+\}$ and where m_ν is the measure of the set F_ν, where $F(x) \geq e^\nu$. Obviously m_ν is finite if $\nu > 0$, and if the condition in the theorem is fulfilled.

Lemma 2. *Let D be a positive constant and λ a positive integer, both so large that*

$$\frac{e\, 2^{k-p}}{D^k S_1 \prod_{p+1}^{k} (b_i - a_i)} + \frac{1}{e^\lambda} \leq 1.$$

Then the following is true:

If for some integer ν and some point $x_\nu \in E$

$$u(x_\nu) \geq e^\nu \tag{5}$$

and

$$S_R(x_\nu) \subset E,$$

where

$$R > D\, m_{\nu-\lambda}^{\frac{1}{p}},$$

then $S_R(x_\nu)$ contains a point $x_{\nu+1}$ where

$$u(x_{\nu+1}) \geq e^{\nu+1}.$$

Proof of Lemma 2. The condition (4) and our special choice of the set E imply that $F_{\nu-\lambda}$ consists of the points, which satisfy

$$(x^1, x^2, \ldots, x^p) \in F'_{\nu-\lambda}$$

and

$$a_i < x^i < b_i, \qquad i = p+1,\, p+2, \ldots, k,$$

where $F'_{\nu-\lambda}$ is a certain p-dimensional set. Moreover, the set

$$S = S_R(x_\nu) \cap F_{\nu-\lambda}$$

is included in the set, which is characterized by the conditions

$$(x^1, x^2, \ldots, x^p) \in F'_{\nu-\lambda}$$

and $$x_\nu^i - R < x^i < x_\nu^i + R, \quad i = p+1, p+2, \ldots, k,$$

if we let $(x_\nu^1, \ldots, x_\nu^k)$ denote the coordinates of x_ν. Hence the measure of the set S is

$$\leq \frac{(2R)^{k-p}}{\prod\limits_{p+1}^{k} (b_i - a_i)} m_{\nu-\lambda}.$$

We suppose that $u(x) < e^{\nu+1}$ in $S_R(x_\nu)$, and are going to show that this implies a contradiction. By (5) and (1) we have

$$e^\nu \leq u(x_\nu) \leq \frac{1}{S_R} \int\limits_{S_R(x_\nu)} u(x)\, dx. \tag{6}$$

We denote by S' the set of all points in $S_R(x_\nu)$ which do not belong to the set S. Apparently $u(x) < e^{\nu-\lambda}$ if $x \in S'$. Hence the right-hand member of 6 is

$$= \frac{1}{S_R} \int\limits_{S} u(x)\, dx + \frac{1}{S_R} \int\limits_{S'} u(x)\, dx$$

$$\leq \frac{1}{S_R} e^{\nu+1} \frac{(2R)^{k-p}}{\prod\limits_{p+1}^{k} (b_i - a_i)} m_{\nu-\lambda} + \frac{1}{S_R} e^{\nu-\lambda} S_R < e^\nu \left[\frac{e\, 2^{k-p} R^k}{D^k S_R \prod\limits_{p+1}^{k} (b_i - a_i)} + \frac{1}{e^\lambda} \right] \leq e^\nu.$$

This gives a contradiction, and hence the lemma is true.

Proof of Theorem 3. Let D and λ be chosen in the way described in Lemma 2. Suppose that for some point $x_n \in E$ and for some $u \in \{F^+\}$

$$u(x_n) \geq e^n,$$

where n is an integer satisfying $n > \lambda$. Then, using Lemma 2, we may as in the proof of Theorem 2 conclude that the distance from x_n to the boundary of E is

$$\leq D \sum_{n-\lambda}^{\infty} m_\nu^{\frac{1}{p}} = D \sum_{n-\lambda}^{\infty} \int\limits_{0}^{m_\nu} d\,(m^{\frac{1}{p}})$$

$$\leq D \sum_{n-\lambda}^{\infty} \int\limits_{m_{\nu+1}}^{m_\nu} \nu\, d\,(m^{\frac{1}{p}}) \leq D \sum_{n-\lambda}^{\infty} \int\limits_{m_{\nu+1}}^{m_\nu} \log^+ y(m)\, d\,(m^{\frac{1}{p}}) = D \int\limits_{0}^{m_{n-\lambda}} \log^+ y(m)\, d\,(m^{\frac{1}{p}}),$$

and since this tends to 0, when $n \to \infty$, the theorem is proved.

Remark 1. If $p=1$, the condition in the theorem is equivalent to

$$\int_E \log^+ F(x)\,dx<\infty.$$

If $p>1$, the condition is fulfilled if for some $\varepsilon>0$

$$\int_E [\log^+ F(x)]^{p+\varepsilon}\,dx<\infty.$$

Remark 2. As Theorem 2 this theorem is true even if the condition is only fulfilled on every compact subset of E. The theorem may furthermore be extended to certain cases where $x^1, \ldots, x^k$ do not mean Cartesian coordinates. One example of this is the case when $k=2$ and $p=1$, and when x^1 and x^2 denote angle and radius, respectively, in polar coordinates. Sjöberg proved the theorem this case.

4. On the necessity of the conditions in Theorem 2 and Theorem 3

We shall now show that the exponents $k-1+\varepsilon$ and $1/p$ in Theorem 2 and Theorem 3, respectively, cannot be improved.

To this end we shall introduce, for every $A>0$, a function $u_A(\sigma, t)$ in the set Ω:

$$0<\sigma<1, \qquad -\infty<t<\infty,$$

(Cartesian coordinates).

In the set Ω_A: $\qquad 0<\sigma<1, \qquad |t|\leq \frac{1}{A} \operatorname{arc\,sin} e^{-A\sigma},$

we define $u_A(\sigma, t)$ as

$$\operatorname{Re}\{e^{\frac{\pi}{2}e^{A(\sigma+it)}}\}=e^{\frac{\pi}{2}e^{A\sigma}\cos At}\cos\left(\frac{\pi}{2}e^{A\sigma}\sin At\right).$$

This function is harmonic, ≥ 0, and vanishes if

$$|t|=\frac{1}{A}\operatorname{arc\,sin} e^{-A\sigma}.$$

Then we define $u_A(\sigma, t)$ as 0 if

$$0<\sigma<1, \qquad |t|>\frac{1}{A}\operatorname{arc\,sin} e^{-A\sigma}.$$

We shall state some simple lemmas concerning the functions $u_A(\sigma, t)$.

Lemma 3. *For any* $n>0$

$$\iint_\Omega [\log^+ u_A(\sigma, t)]^n\,d\sigma\,d(t^n)\leq 2\left(\frac{\pi^2}{4A}\right)^n.$$

Proof. In Ω we have $$u_A(\sigma, t) \le e^{\frac{\pi}{2} e^{A\sigma}}, \tag{7}$$

and hence

$$\iint_{\Omega} [\log^+ u_A(\sigma, t)]^n \, d\sigma \, d(t^n) \le \iint_{\Omega_A} \left(\frac{\pi}{2}\right)^n e^{nA\sigma} \, d\sigma \, d(t^n)$$

$$= 2 \int_0^1 \left(\frac{\pi}{2}\right)^n e^{nA\sigma} \left(\frac{1}{A}\right)^n (\arcsin e^{-A\sigma})^n \, d\sigma \le 2 \left(\frac{\pi^2}{4A}\right)^n.$$

Lemma 4. *There exists an even function $G(t)$, non-increasing for $t > 0$, such that*

$$\log G(t) \sim \frac{\text{const.}}{|t| \log \frac{1}{|t|}},$$

when $t \to 0$, and such that, for every $A > 0$,

$$u_A(\sigma, t) \le G(t),$$

if $(\sigma, t) \in \Omega$.

Proof. In Ω_A we have $$e^{A\sigma} |\sin At| \le 1,$$

hence by (7) $$u_A(\sigma, t) \le e^{\frac{\pi}{2} \frac{1}{|\sin At|}}.$$

And (7) gives moreover $$u_A(\sigma, t) \le e^{\frac{\pi}{2} e^A}.$$

Since e^A and $1/|\sin At|$ ($|At| < \pi/2$) are increasing and decreasing, respectively, considered as functions of A, it is easy to see that we may choose for $G(t)$ the expression

$$e^{\frac{\pi}{2} \frac{1}{|\sin A_0(t) t|}},$$

where $A_0(t)$ is the smallest solution of the equation

$$e^A = \frac{1}{|\sin At|}.$$

Then a simple estimation shows that

$$\log G(t) \sim \frac{\pi/2}{|t| \log \frac{1}{|t|}},$$

when $t \to 0$.

Lemma 5. *Let $\sigma, x_1, x_2, \ldots, x_{n-1}$ be the Cartesian coordinates of the n-dimensional Euclidean space. ($n \geq 2$). Put*

$$t = (x_1^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}}.$$

Then the function $$v = u_A(\sigma, t)^{2n-3}$$

is subharmonic, considered as a function on the subset $0 < \sigma < 1$ of the n-dimensional space.

Proof. It is well known that a continuous, non-negative function v is subharmonic if it satisfies

$$\Delta v \geq 0$$

whenever $v > 0$. Here Δ denotes the Laplace operator. Hence in our case we have to prove that

$$\Delta v = \frac{\delta^2 v}{\delta \sigma^2} + \frac{\delta^2 v}{\delta t^2} + \frac{n-2}{t} \frac{\delta v}{\delta t} \geq 0$$

if $(\sigma, t) \in \Omega_A$. A direct computation gives

$$\Delta v = (2n-3)(2n-4)\, u_A(\sigma, t)^{2n-5} \left\{ e^{\pi e^{A\sigma}} \left(\frac{\pi}{2}\right)^2 e^{2A\sigma} A^2 - \right.$$

$$\left. - e^{\pi e^{A\sigma} \cos t} \frac{\pi}{2} e^{A\sigma} A \cos\left(\frac{\pi}{2} e^{A\sigma} \sin At\right) \frac{\sin\left(At + \frac{\pi}{2} e^{A\sigma} \sin At\right)}{2t} \right\}.$$

Using the inequality

$$\left| \sin\left(At + \frac{\pi}{2} e^{A\sigma} \sin At\right) \right| \leq \pi e^{A\sigma} A\, |t|$$

we see that $\Delta v \geq 0$. The above arguments fail if $t = 0$, but since the second derivatives of $u_A(\sigma, t)$ are continuous, we must have $\Delta v \geq 0$ also in that case.

Now we are in a position to discuss the exponents in Theorem 2 and Theorem 3.

Theorem 4. *The exponent $k - 1 + \varepsilon$ in Theorem 2, $k \geq 2$, cannot be exchanged for $k - 1$.*

Proof. Let us choose E as the set

$$0 < \sigma < 1, \qquad -\infty < x_\nu < \infty, \quad \nu = 1, 2, \ldots, k-1,$$

where $\sigma, x_1, \ldots, x_{k-1}$ denote Cartesian coordinates. We put

$$t = (x_1^2 + x_2^2 + \cdots + x_{k-1}^2)^{\frac{1}{2}}$$

and study the functions $$v_A(\sigma, t) = u_A(\sigma, t)^{2k-3},$$

which are subharmonic, according to Lemma 5. By Lemma 3, if $A \geq 1$,

$$\int_E [\log^+ v_A(\sigma, t)]^{k-1}\, d\sigma\, dx_1 \ldots dx_{k-1} \leq \frac{\text{const.}}{A}, \tag{8}$$

where the constant is independent of A.

Let us choose a sequence of positive numbers $A_1, A_2, \ldots, A_n, \ldots$, such that

$$\sum_{\nu=1}^{\infty} 1/A_\nu$$

converges. Then we define the function F in the theorem as

$$F(\sigma, t) = \sup_\nu v_{A_\nu}(\sigma, t).$$

(8) gives
$$\int_E [\log^+ F(\sigma, t)]^{k-1}\, d\sigma\, dx_1 \ldots dx_{k-1} < \infty,$$

and hence the condition in Theorem 2 is fulfilled with the exponent $k-1$.

But the subharmonic functions $v_{A_\nu}(\sigma, t)$ satisfy

$$v_{A_\nu}(\sigma, t) \leq F(\sigma, t),$$

and nevertheless they are not uniformly bounded if $t=0$, $0<\sigma<1$. Hence Theorem 2 is not true with this change in the exponent.

Theorem 5. *The exponent $1/p$ in Theorem 3 cannot be exchanged for a larger number.*

Proof. In the definition of E which precedes Theorem 3 we choose for O the p-dimensional unit sphere, and put

$$t = ((x^1)^2 + (x^2)^2 + \cdots + (x^p)^2)^{\frac{1}{2}}.$$

Let σ denote one of the remaining $k-p$ coordinates.

We choose as F the function $G(t)^{2p-1}$, where $G(t)$ is defined in Lemma 4. It is easy to show that if we put

$$m = c\, t^p$$

for a suitable choice of the constant c, then we obtain

$$y(t) = G(t)^{2p-1}.$$

Hence, if the exponent in the theorem is larger than $1/p$, the integral has the value

$$\text{const.} \int_0^1 \log^+ G(t)\, d(t^\alpha),$$

where $\alpha > 1$, and then Lemma 4 implies that the integral converges; thus the modified condition of Theorem 3 is fulfilled.

On the other hand, the functions $u_A(\sigma, t)^{2p-1}$ are subharmonic in E according to Lemma 5, and they satisfy

$$u_A(\sigma, t)^{2p-1} \leq G(t)^{2p-1}$$

by Lemma 4, but they are not uniformly bounded. Hence Theorem 3 cannot be true with this change in the exponent.

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