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On weak and strong extensions of partial differential operators with constant coefficients

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Introduction

Let Ω be a bounded domain in R^ν. We denote by $C^\infty(\Omega)$ the set of infinitely differentiable functions defined in Ω, and by $C_0^\infty(\Omega)$ the set of those functions in $C^\infty(\Omega)$ which have compact support in Ω.

With the notations of Hörmander [1], there is a one to one correspondence between the partial differential operators on the functions in R^ν

$$P(D)=\sum_\alpha a^{\alpha_1\cdots\alpha_k}\frac{1}{i}\frac{\partial}{\partial x^{\alpha_1}}\cdots\frac{1}{i}\frac{\partial}{\partial x^{\alpha_k}}=\sum_\alpha a^\alpha D_\alpha \quad (\alpha=(\alpha_1\ldots\alpha_k))$$

and the polynomials in the dual space C_ν

$$P(\xi)=\sum_\alpha a^{\alpha_1\cdots\alpha_k}\xi_{\alpha_1}\ldots\xi_{\alpha_k}=\sum_\alpha a^\alpha\xi_\alpha.$$

The algebraic adjoint of $P(D)$ is $\bar P(D)=\sum\bar a^\alpha D_\alpha$.

Definition. (Cf. [1] p. 168 and p. 241.) *The closure P_0 of the operator in L^2 with domain $C_0^\infty(\Omega)$ defined by $P(D)$ is called the minimal operator defined by $P(D)$.*

The adjoint $P_w(\Omega)$ of the minimal operator $\bar P_0(\Omega)$ defined by $\bar P(D)$ is called the maximal operator defined by $P(D)$ or the weak extension of $P(D)$.

The closure $P_s(\Omega)$ of the operator $P(D)$ defined in the set $\{u\,|\,u\in C^\infty(\Omega),\ u\in L^2(\Omega),\ P(D)u\in L^2(\Omega)\}$ is called the strong extension of $P(D)$.

It is natural to suppose that the weak and strong extensions should generally be equal. A result confirming this hypothesis was given by Hörmander ([1], Theorem 3.12) who proved that it is true when $P(D)$ is of local type and Ω is any domain. Another result of the same author [3] secures the assertion for any operator with constant coefficients as soon as the boundary of the domain Ω satisfies certain regularity properties.

It seems likely that, unless $P(D)$ is of local type, it is necessary to impose some condition on Ω in order that $P_s(\Omega)=P_w(\Omega)$. It is the aim of this paper to prove this under the additional assumption that $P(D)$ is homogeneous. This will be done by modifying an unpublished example given by L. Hörmander for $P(D)=\partial^2/\partial x\,\partial y$ (Theorem 1 below).

I want to take the opportunity to thank my teacher, Professor Lars Hörmander, for his constant interest and assistance.

1. Algebraic lemmas

Lemma 1. *The integral*

$$\int\int \frac{1}{1+\xi_1^2+\xi_2^2+|P(\xi_1,\xi_2)|}\,d\xi_1\,d\xi_2$$

is convergent if and only if the degree n of the polynomial $P(\xi_1,\xi_2)$ exceeds two.

Proof: First consider a sector between a characteristic of the polynomial $P(\xi_1,\xi_2)$ and a half-ray from the origin, leaving the sector free from other characteristics. We may assume that these lines are $\xi_2=0$ and $l(\xi_1,\xi_2)=0$. Consider a curve $\xi_2=r^{1/\alpha}$ $(\xi_1^2+\xi_2^2=r^2, \alpha>1)$ and a circle (C) with radius r_0, where α and r_0 will be chosen in (III) below. We now get the following estimate for the denominator $N(\xi_1,\xi_2)$ of the integrand in the sector:

(I) Inside the circle (C) we have $N\geqslant 1$.

(II) Outside the circle, between the line $\xi_2=0$ and the curve $\xi_2=r_{1/\alpha}$ we have $N\geqslant r^2$. If (r,φ) are polar coordinates of a point on the curve we get, setting $(1-\alpha^{-1})=\varepsilon$, that $\varphi=\arcsin\, r^{-\varepsilon}=O(r^{-\varepsilon})$.

(III) Between the line $l(\xi_1,\xi_2)=0$ and the curve $\xi_2=r^{1/\alpha}$ we have $N\geqslant C\,r^{n-\delta}$ where we can make δ arbitrarily small if we choose ε small enough. In fact, let the principal part of $P(\xi_1,\xi_2)$ be $p(\xi_1,\xi_2)=\xi_2^m\,q(\xi_1,\xi_2)$, $(m\geqslant 1)$, where $q(\xi_1,\xi_2)$ is a homogeneous polynomial of degree $(n-m)$, $q\neq 0$ in and on the boundary of the sector except at the origin. Let min $|q(\xi_1,\xi_2)|=\mu$ when the point (ξ_1,ξ_2) moves on the part of the unit circle lying in the sector. Then for a point in the sector with coordinates (r,φ) we have $|q(\xi_1,\xi_2)|\geqslant\mu\,r^{n-m}$. Within the domain (III) we have $\xi_2\geqslant r^{1/\alpha}$. Thus in (III) we get $|p(\xi_1,\xi_2)|\geqslant C\,r^{n-m\varepsilon}$. Choosing α so that $\delta=m\varepsilon<1$ we now conclude

$$N\geqslant|P(\xi_1,\xi_2)|\geqslant C\,r^{n-\delta}\,(1-O\,(r^{-(1-\delta)}))\geqslant C' r^{n-\delta}$$

when $r>r_0$. (C and C' denote different constants.)

We now estimate the integral in the sector, using that $n>2$.

$$\int\int\frac{1}{N(\xi_1,\xi_2)}\,d\xi_1\,d\xi_2\leqslant C\left(\int\int_{r<r_0} d\xi_1\,d\xi_2+\int_{r=r_0}^{\infty}\int_{\varphi=0}^{O(r^{-\varepsilon})}\frac{1}{r}\,d r\,d\varphi+\int\int_{\text{(III)}}\frac{1}{r^{n-1-\delta}}\,d r\,d\varphi\right)<\infty.$$

In the remaining sectors, i.e. those which are not bounded by characteristics, we get the estimates:

$$N\geqslant C \quad \text{when } r<r_0$$
$$N\geqslant C\,r^n \quad \text{when } r\geqslant r_0$$

Now the convergence follows in the whole plane.

From [1] we adopt the notation $\tilde{P}(\xi)^2=\sum_{\alpha}|P^{(\alpha)}(\xi)|^2$.

Lemma 2. *If $P(\xi)$ is a complete polynomial of two variables ξ_1 and ξ_2, the integral*

$$\int \frac{1}{\tilde{P}(\xi)^2} d\xi$$

is convergent unless with suitable coordinates

$$P(\xi) = C[(\xi_1 + i\alpha_1) + (\xi_2 + i\alpha_2)^2], \quad (\alpha_1 \text{ and } \alpha_2 \text{ denote real constants}).$$

Proof. If the principal part of $P(\xi)$ is complete we know from Lemma 2.13 in [1] that the set

$$\{\operatorname{Re} P^{(\alpha)}(\xi);\ \operatorname{Im} P^{(\alpha)}(\xi)\} \quad (|\alpha| = n-1)$$

contains two real linearly independent linear forms l_1 and l_2. We may assume the coordinates so chosen that $l_1 = \xi_1$, $l_2 = \xi_2$. This gives the estimate

$$\tilde{P}^2 \geqslant C(1 + \xi_1^2 + \xi_2^2 + |P(\xi_1, \xi_2)|^2).$$

Thus in this case the integral converges according to Lemma 1.

If the principal part of $P(\xi)$ is not complete we can set, with coordinates conveniently chosen,

$$P(\xi) = c\,\xi_2^n + Q(\xi_1, \xi_2),$$

where Q is a polynomial of degree $< n$ not independent of ξ_1 and c is a constant. Now

$$\frac{\partial^{n-1} P}{\partial \xi_2^{n-1}} = a_1(\xi_2 + a_2) \quad (a_1 \neq 0 \text{ and } a_2 \text{ are constants}),$$

and for some α we get $P^{(\alpha)}(\xi) = a_3(\xi_1 + R(\xi_2))$, $(a_3 \neq 0)$. With $S(\xi_2) = \operatorname{Re} R(\xi_2)$ we get the estimate

$$\tilde{P}^2 \geqslant C(1 + \xi_2^2 + |\xi_1 + S(\xi_2)|^2 + |P(\xi)|^2).$$

We now set

$$\left.\begin{aligned} \eta_1 &= \xi_1 + S(\xi_2) \\ \eta_2 &= \xi_2 \end{aligned}\right\} \quad \frac{D(\xi_1, \xi_2)}{D(\eta_1, \eta_2)} = 1.$$

We get from the above inequality

$$\tilde{P}^2 \geqslant C(1 + \eta_1^2 + \eta_2^2 + |P(\eta_1 - S(\eta_2), \eta_2)|^2).$$

Now if the degree of the polynomial $P(\eta_1 - S(\eta_2), \eta_2)$ is $\geqslant$ two, the integral will converge according to Lemma 1.

If this is not the case we have

$$P(\eta_1 - S(\eta_2), \eta_2) = A\eta_1 + B\eta_2 + E \quad (A,\ B,\ E \text{ are constants}).$$

This gives

$$P(\xi_1, \xi_2) = A(\xi_1 + S(\xi_2)) + B\xi_2 + E.$$

Since $P(\xi)$ is complete we conclude that $A \neq 0$. Now if $S(\xi_2)$ is of degree exceeding two we have

$$\tilde{P}^2 \geqslant C\left(1 + \xi_2^2 + \left|\xi_1 + S(\xi_2) + \operatorname{Re}\left(\frac{B\xi_2 + E}{A}\right)\right|^2 + \left|\frac{\partial}{\partial \xi_2} S(\xi_2) + \operatorname{Re}\frac{B}{A}\right|^2\right).$$

The non-singular transformation

$$\zeta_1 = \xi_1 + S(\xi_2) + \mathrm{Re}\left(\frac{B\xi_2 + E}{A}\right)$$
$$\zeta_2 = \xi_2$$

now gives
$$\tilde{P}^2 \geqslant C(1 + \zeta_1^2 + \zeta_2^2 + T(\zeta_2)),$$

where T denotes a positive polynomial of degree $\geqslant 4$. So the integral converges. Finally if the degree of $S(\xi_2)$ is two we have the exceptional case.

It is easily verified that the integral diverges in this case. We get

$$\tilde{P}^2 \leqslant C(1 + (\xi_1 + \xi_2^2)^2 + \xi_2^2).$$

The transformation $\eta_1 = \xi_1 + \xi_2^2, \eta_2 = \xi_2$ gives

$$\tilde{P}^2 \leqslant C(1 + \eta_1^2 + \eta_2^2).$$

Thus we have logarithmical divergence.

Lemma 3. *If $P(\xi)$ is a homogeneous polynomial in C_ν where ξ_1 and ξ_2 occur with total degree $m \geqslant 2$, and if the lineality space $\Lambda(P)$ is defined by $\xi_1 = \xi_2 = \dots = \xi_\varkappa = 0$, the function $1/\tilde{P}$ is uniformly square integrable in the varieties Σ parallel to the $\xi_1\xi_2$-plane, i.e. $\int 1/\tilde{P}^2 d\xi_1 d\xi_2 < C_P$, where the constant C_P depends on P but is independent of $(\xi_3 \dots \xi_\nu)$.*

Proof: Suppose the degree of $P(\xi)$ is $= n$. In view of the fact that $P(\xi) = P(\xi_1 \dots \xi_\varkappa)$ is complete and homogeneous in the variables $(\xi_1 \dots \xi_\varkappa)$ we infer from Lemma 2.13 in [1] that the set

$$\{\mathrm{Re}\, P^{(\alpha)}(\xi), \mathrm{Im}\, P^{(\alpha)}(\xi)\} \quad (|\alpha| = n-1)$$

contains $\varkappa$ linearly independent linear homogeneous forms $(l_1 \dots l_\varkappa)$ of the variables $(\xi_1 \dots \xi_\varkappa)$. Hence there are constants $C_1 > 0$ and $C_2 > 0$ so that

$$C_1 < \frac{\sum_{i=1}^{\varkappa} l_i^2}{\sum_{i=1}^{\varkappa} \xi_i^2} < C_2.$$

Now select in $P(\xi)$ one term, where ξ_1 and ξ_2 occur with maximal total degree $m \geqslant 2$, say

$$a \cdot \xi_1^l \cdot \xi_2^{m-l} \cdot \prod_{i=3}^{\varkappa} \xi_i^{l_i}.$$

We study instead of $P(\xi)$ the weaker polynomial

$$Q(\xi) = \prod_3^{\varkappa} \left(\frac{\partial}{\partial \xi_i}\right)^{l_i} P(\xi).$$

We set $Q(\xi)=Q_0(\xi)+Q_1(\xi)+\cdots+Q_m(\xi)$, where $Q_i(\xi)$ is homogeneous of degree i in (ξ_1,ξ_2) and of degree $(m-i)$ in $(\xi_3\cdots\xi_\varkappa)$.

After these remarks we get the estimate:

$$\tilde{P}^2(\xi)\geqslant C\,(|Q(\xi)|^2+\sum_{i=1}^{\varkappa}\xi_i^2+1).$$

We use the notation $\xi_3^2+\xi_4^2+\cdots+\xi_\varkappa^2=t^2$. Let $\tilde{P}^2_\Sigma(\xi)$ denote the restriction of $\tilde{P}^2(\xi)$ to Σ.

If $t\geqslant 1$ we transform by the formulas $\xi_i=t\cdot\eta_i\,(i=1\ldots\varkappa)$. Setting $\eta_1^2+\eta_2^2=r^2$ we get

$$\tilde{P}^2_\Sigma(\xi)\geqslant C\,(|R_{2m-1}(\eta_1,\eta_2)+|Q_m(\eta_1\eta_2)|^2|\,t^{2m}+(r^2+1)\,t^2+1).$$

Here $R_{2m-1}(\eta_1,\eta_2)$ denotes a real polynomial of degree $2m-1$ at most, the coefficients depending on $\eta_3\ldots\eta_\varkappa$. We now get the estimates:

$$\int_\Sigma\frac{1}{\tilde{P}^2_\Sigma(\xi_1,\xi_2)}\,d\xi_1\,d\xi_2\leqslant C\int\frac{1}{t^{2(m-1)}|R_{2(m-1)}+|Q_m|^2|+r^2+1+t^{-2}}\,d\eta_1\,d\eta_2$$

$$\leqslant C\int\frac{1}{|R_{2m-1}+|Q_m|^2|+r^2+1}\,d\eta_1\,d\eta_2.$$

We now use the method of proof of Lemma 1. We can determine constants C and r_0 and an arbitrarily small fixed number δ such that, if $r>r_0$, the term

$$|R_{2m-1}+|Q_m|^2|\quad\text{will be}\quad\geqslant C\,r^{2m-\delta}$$

except in narrow domains enclosing the characteristics of the polynomial $Q_m(\eta_1,\eta_2)$. The constants in these inequalities can be determined independently of $(\xi_3\ldots\xi_\varkappa)$ because the coefficients of R_{2m-1} are bounded when $\sum_{i=3}^{\varkappa}\eta_i^2=1$, while the coefficients of Q_m are independent of $(\eta_3\ldots\eta_\varkappa)$. In the narrow domains just referred to we replace the denominator by the smaller quantity r^2 and inside the circle $r\leqslant r_0$ by $+1$. In this way we get the estimate

$$\int_\Sigma\frac{1}{\tilde{P}^2_\Sigma(\xi)}\,d\xi_1\,d\xi_2\leqslant C'_P\quad(t\geqslant 1).$$

Now suppose $t<1$. We get

$$\tilde{P}^2_\Sigma(\xi)\geqslant C\,(|R_{2m-1}(\xi_1,\xi_2)+|Q_m(\xi_1,\xi_2)|^2|+\xi_1^2+\xi_2^2+1).$$

Here we can apply the same method as before without any preliminary transformation getting the estimate

$$\int_\Sigma\frac{1}{\tilde{P}^2_\Sigma(\xi)}\,d\xi_1\,d\xi_2\leqslant C''_P\quad(t<1).$$

Combining these results, the proof is complete.

2. Homogeneous operators of two variables

Let $\hat{\Omega}$ denote a bounded domain containing the origin and Ω the same domain with the origin left out.

Theorem 1. *$P_s(\Omega) \neq P_w(\Omega)$ if $P(D) = \partial^2/\partial x^1 \partial x^2$.*

Proof: We are going to prove the equivalent assertion

$$P_s^*(\Omega) \neq P_w^*(\Omega) = \bar{P}_0(\Omega)$$

by indicating an example of a function $u \in \mathcal{D}_{P_s^*}, u \notin \mathcal{D}_{\bar{P}_0}$.

From direct computation or Lemma 2 or 3 we get

$$\int \frac{1}{\tilde{\bar{P}}(\xi)^2} d\xi < \infty.$$

Applying [1] Theorem 2.6 with $Q(\xi) \equiv 1$ we realize that, after correction on a null set, every function $v \in \mathcal{D}_{\bar{P}_0}$ is continuous and vanishes on the boundary of Ω, thus in particular at the origin. Accordingly the theorem follows from the following

Lemma 4. *If $\hat{u} \in C_0^\infty(\hat{\Omega})$ and $\hat{u} = constant = c \neq 0$ in a neighbourhood ω of the origin, and if u is the restriction of $\hat{u}$ to Ω, we have $u \in \mathcal{D}_{P_s^*}$.*

Proof: We have to prove the relation

$$(\bar{P}(D)u, v)_\Omega - (u, P(D)v)_\Omega = 0 \quad \text{for every} \quad v \in \mathcal{D}_{P_s}.$$

In view of the definition of $P_s(\Omega)$ it is sufficient to show this for any $v \in C^\infty(\Omega)$, satisfying the conditions $v \in L^2(\Omega)$ and $P(D)v \in L^2(\Omega)$. In the present case the relation can be written

$$\iint_\Omega \left(\frac{\partial^2 u}{\partial x^1 \partial x^2} \cdot \bar{v} - u \cdot \frac{\partial^2 \bar{v}}{\partial x^1 \partial x^2}\right) dx^1 dx^2 = 0.$$

Let $\Omega_\varepsilon^\delta$ denote the domain obtained from Ω by excluding the rectangle $|x^1| < \frac{1}{2}\varepsilon$, $|x^2| < \frac{1}{2}\delta$ and let R_ε^δ be the boundary of this rectangle. Suppose that the quantities ε and δ are small enough to make R_ε^δ lie entirely inside ω. Set

$$I_\varepsilon^\delta = \iint_{\Omega_\varepsilon^\delta} \left(\frac{\partial^2 u}{\partial x^1 \partial x^2} \cdot \bar{v} - u \cdot \frac{\partial^2 \bar{v}}{\partial x^1 \partial x^2}\right) dx^1 dx^2.$$

Rewriting and integrating by parts we get

$$I_\varepsilon^\delta = \iint_{\Omega_\varepsilon^\delta} \left[\frac{\partial}{\partial x^2}\left(\frac{\partial u}{\partial x^1} \cdot \bar{v}\right) - \frac{\partial}{\partial x^1}\left(u \cdot \frac{\partial \bar{v}}{\partial x^2}\right)\right] dx^1 dx^2 = -\int_{R_\varepsilon^\delta} \left(\frac{\partial u}{\partial x^1} \cdot \bar{v}\, dx^1 + u \cdot \frac{\partial \bar{v}}{\partial x^2} dx^2\right)$$

since $\hat{u} \in C_0^\infty(\hat{\Omega})$ and hence vanishes in a neighbourhood of the boundary of $\hat{\Omega}$. If we now introduce the condition $u(x) = c \neq 0$ in ω we get:

$$I_\varepsilon^\delta = -c\,[\bar{v}(\varepsilon,\delta) - \bar{v}(\varepsilon,-\delta) + \bar{v}(-\varepsilon,-\delta) - \bar{v}(-\varepsilon,\delta)].$$

We finally use that $v \in C^\infty(\Omega)$, which gives, with ε fixed,

$$\lim_{\delta \to 0} I_\varepsilon^\delta = 0.$$

Since $\Omega - \Omega_\varepsilon^0$ is a null set and I_0^0 is absolutely convergent we get

$$\lim_{\delta \to 0} I_\varepsilon^\delta = 0 = (\bar{P}(D)\,u, v) - (u, P(D)\,v)$$

which completes the proof.

Theorem 2. $P_s(\Omega) \neq P_w(\Omega)$ *if the corresponding polynomial* $P(\xi_1, \xi_2)$ *is complete, homogeneous and non-elliptic.*

Proof: We again prove $P_s^*(\Omega) \neq P_w^*(\Omega) = \bar{P}_0(\Omega)$.

Since the polynomial has at least one real characteristic we can, with suitable coordinates write

$$P(\xi) = \xi_1 \cdot Q(\xi) \quad \text{and} \quad P(D) = \frac{1}{i} \cdot \frac{\partial}{\partial x^1}\, Q(D).$$

Application of Lemma 2 or 3 combined with [1] Theorem 2.6 shows as in the proof of Theorem 1 that we need only prove Lemma 4 for the present operator $P(D)$.

Let $\hat{u}$ and u satisfy the conditions of Lemma 4. Let v be any function in $C^\infty(\Omega)$ satisfying $v \in L^2(\Omega)$ and $P(D)\,v \in L^2(\Omega)$. We are going to show the equation

$$(\bar{P}(D)\,u, v) - (u, P(D)\,v) = 0$$

i.e.

$$\iint_\Omega \left[\left(\bar{Q}(D)\frac{1}{i}\frac{\partial}{\partial x^1}\,u\right)\cdot \bar{v} - u \cdot \overline{\frac{1}{i}\frac{\partial}{\partial x^1}\,Q(D)\,v}\right] dx^1\,dx^2 = 0.$$

We use the same notations $\Omega_\varepsilon^\delta$ and R_ε^δ as in the proof of Theorem 1. We have

$$\iint_{\Omega_\varepsilon^\delta} \left(\bar{Q}(D)\frac{1}{i}\frac{\partial}{\partial x^1}\,u\right)\cdot \bar{v}\,dx^1\,dx^2 = \iint_{\Omega_\varepsilon^\delta} \frac{1}{i}\frac{\partial}{\partial x^1}\,u \cdot \overline{Q(D)\,v}\,dx^1\,dx^2$$

for $(1/i)(\partial u/\partial x^1) \in C_0^\infty(\Omega_\varepsilon^\delta)$ in view of the fact that $\hat{u} \in C_0^\infty(\hat{\Omega})$ and is constant in a neighbourhood ω of the origin enclosing R_ε^δ and this means that the boundary integrals which arise all vanish.

If we make another partial integration we get

$$\iint_{\Omega_\varepsilon^\delta} \left(\bar{Q}(D)\frac{1}{i}\frac{\partial}{\partial x^1}\,u\right)\cdot \bar{v}\,dx^1\,dx^2 = \int_{R_\varepsilon^\delta} \frac{1}{i}\,u \cdot \overline{Q(D)\,v}\,dx^2 + \iint_{\Omega_\varepsilon^\delta} u \cdot \overline{\frac{1}{i}\frac{\partial}{\partial x^1}\,Q(D)\,v} \cdot dx^1\,dx^2.$$

Since $v \in C^\infty(\Omega)$ the function $|Q(D)v|$ is bounded on R_ε^δ. Let M_Q denote the maximal value on the sides parallel to the x^2-direction. Then the curve integral in the last equation is bounded by the quantity $2|c|\cdot\delta\cdot M_Q \to 0$ if $\delta \to 0$ with ε fixed. We now set

$$I_\varepsilon^\delta = \iint_{\Omega_\varepsilon^\delta} (\overline{P}(D)u\cdot\bar{v} - u\cdot\overline{P(D)v})\,dx^1\,dx^2.$$

Since $\Omega - \Omega_\varepsilon^0$ is a null set and the integral I_0^0 is absolutely convergent according to the assumptions, we get when $\delta \to 0$ with ε fixed

$$\lim_{\delta\to 0} I_\varepsilon^\delta = 0 = (\overline{P}(D)u, v) - (u, P(D)v)$$

and the proof is complete.

Let Ω be defined as before. As a supplement to the results of Theorems 1–2 we prove

Theorem 3. *Let the polynomial $P(\xi_1, \xi_2)$ be complete, homogeneous and non-elliptic. If E is any proper fundamental solution (cf. [2]) of the operator $P(D)$ and if e is the restriction of E to Ω, then $e \in \mathcal{D}_{P_w}(\Omega)$, $e \notin \mathcal{D}_{P_s}(\Omega)$.*

Proof: According to [2] Theorem 2.2 we have $E \in L^2_{\text{loc}}$ if and only if

$$\int \frac{1}{\tilde{P}^2(\xi)}\,d\xi < \infty.$$

In view of Lemma 2 this condition is satisfied. Now since $P(D)E = \delta_0$ by definition, we have $P(D)E = 0$ in Ω, and consequently $P(D)e$ belongs to $L^2(\Omega)$. Therefore $e \in \mathcal{D}_{P_w}(\Omega)$.

Since $\hat{u} \in C_0^\infty(\hat{\Omega})$ we get in the sense of distribution theory

$$\int_\Omega e\cdot\overline{\overline{P}(D)u}\cdot dx = \int_{R^2} E\cdot\overline{\overline{P}(D)\hat{u}}\cdot dx = \langle P(D)E, \bar{\hat{u}}\rangle = \langle \delta_0, \bar{\hat{u}}\rangle = \bar{\hat{u}}(0) = \bar{c} \neq 0.$$

On the other hand we know from Lemma 4 that $u \in \mathcal{D}_{P_s^*}$ so that

$$\int_\Omega e\cdot\overline{\overline{P}(D)u}\,dx = \int_\Omega e\cdot\overline{P_s^* u}\cdot dx.$$

Now if it were true that $e \in \mathcal{D}_{P_s}(\Omega)$ we would get

$$\int_\Omega e\cdot\overline{P_s^* u}\,dx = \int_\Omega P_s e\cdot\bar{u}\,dx = \int_\Omega P_w e\cdot\bar{u}\,dx = 0$$

because the integrand equals zero. This gives a contradiction, i.e.

$$e \in \mathcal{D}_{P_w}(\Omega) \quad\text{but}\quad e \notin \mathcal{D}_{P_s}(\Omega).$$

3. Homogeneous operators of ν variables

We shall now extend Theorem 2.

Theorem 4. *If the polynomial $P(\xi)$ is homogeneous and non-elliptic modulo the lineality space $\Lambda(P)$ there is an open set Ω such thnt $P_s(\Omega) \neq P_w(\Omega)$.*

Proof: Let the polynomial be defined in C_ν. According to the assumptions we can find a real vector $e_2 \neq 0$ not in $\Lambda(P)$, such that

$$P(t e_2) = 0 \quad \text{for every real } t. \tag{1}$$

In view of the definition of $\Lambda(P)$ ([1], Definition 2.2) there exists a second real vector e_1 so that

$$P(e_1 + t e_2) \neq P(e_1) \quad \text{for some real } t. \tag{2}$$

This shows that e_1 and e_2 are not proportional. We further conclude that $P(\xi_1 e_1 + \xi_2 e_2)$ has ξ_1 as a factor and is not independent of ξ_2, hence is a complete polynomial of the two variables ξ_1 and ξ_2.

We now introduce a coordinate system such that the lineality space is defined by the equations $\xi_1 = \xi_2 = \cdots = \xi_\varkappa = 0$ and with the ξ_1- and ξ_2-axes along e_1 and e_2 respectively. In this system we write

$$P(\xi) = P_0(\xi) + P_1(\xi) + \cdots + P_m(\xi). \tag{3}$$

Here $P_i(\xi)$ denotes a polynomial that is homogeneous of degree i in (ξ_1, ξ_2) and of degree $(m-i)$ in $(\xi_3 \ldots \xi_\varkappa)$. In particular we notice that $P_m(\xi) = P(\xi_1 e_1 + \xi_2 e_2)$.

Let $\hat{\Omega}$ denote a bounded domain in R^ν containing the origin, and let Ω be the domain obtained by excluding from $\hat{\Omega}$ a cut Γ defined by $x^1 = x^2 = 0, |x^i| \leq \frac{1}{2}\varrho$ $(i = 3, \ldots, \nu)$, where ϱ is so small that the cut is enclosed in Ω.

Applying [1] Theorem 2.8 and Lemma 3 above we conclude that if $u \in \mathcal{D}_{\bar{P}_0}(\Omega)$ is distinguished ([1] p. 195) the restriction of u to any variety Γ' parallel to Γ is in $L^2(\Gamma')$ and converges strongly to zero when $\Gamma' \to \Gamma$. (We observe at this stage that the cut need not always be taken of dimension $(\nu - 2)$. In fact it is sufficient for our purpose that the function $1/\tilde{P}$ is uniformly square integrable in the varieties Σ orthogonal to Γ.)

As before we are going to prove that $P_s^*(\Omega) \neq P_w^*(\Omega) = \bar{P}_0(\Omega)$. To this end we adapt Lemma 4 to the new situation:

Lemma 5. *If $\hat{u} \in C_0^\infty(\hat{\Omega})$ and $\hat{u} = c \neq 0$ in a neighbourhood ω of the cut Γ, and if u is the restriction of $\hat{u}$ to Ω we have $u \in \mathcal{D}_{P_s^*}$.*

Proof: As before we have to prove the equation

$$\int_\Omega (\bar{P}(D) u \cdot v - u \cdot \overline{P(D) v}) \cdot dx = 0$$

for every $v \in C^\infty(\Omega)$ satisfying the conditions $v \in L^2(\Omega)$ and $P(D) v \in L^2(\Omega)$.

Let $\Omega_{\varepsilon\varrho'}^\delta$ denote the domain obtained from $\hat{\Omega}$ by excluding the "parallelepiped" $R_{\varepsilon\varrho'}^\delta$ defined by $|x^1| \leq \frac{1}{2}\varepsilon, |x^2| \leq \frac{1}{2}\delta, |x^i| \leq \frac{1}{2}\varrho'$ $(i = 3 \ldots \nu)$. We suppose the parameters ε, δ and $\varrho' > \varrho$ are so determined as to let $R_{\varepsilon\varrho'}^\delta$ lie entirely in ω.

Consider the operator $P(D) - P_m(D) = P_0(D) + P_1(D) + \cdots + P_{m-1}(D)$ corresponding to the decomposition of $P(\xi)$ given by (3). An arbitrary term may be written:

$$C\frac{1}{i}\frac{\partial}{\partial x^l}\prod_{j=1}^{m-1}\frac{1}{i}\frac{\partial}{\partial x^{l_j}} = C\frac{1}{i}\frac{\partial}{\partial x^l}D_\alpha \qquad \begin{pmatrix} |\alpha| = m-1 \\ C = \text{constant} \\ l \neq 1, 2 \end{pmatrix}.$$

We have
$$\int_{\Omega^\delta_{\varepsilon\varrho'}}\left(\bar{C}\cdot D_\alpha\frac{1}{i}\frac{\partial}{\partial x^l}u\right)\cdot\bar{v}\cdot dx = \int_{\Omega^\delta_{\varepsilon\varrho'}}\left(\bar{C}\frac{1}{i}\frac{\partial}{\partial x^l}u\right)\cdot\overline{D_\alpha v}\,dx$$

for $(1/i)(\partial u/\partial x^l)\in C_0^\infty(\Omega^\delta_{\varepsilon\varrho'})$ in view of the fact that $\hat{u}\in C_0^\infty(\hat{\Omega})$ and is constant in a neighbourhood of the cut Γ containing $R^\delta_{\varepsilon\varrho'}$. If we make another partial integration we get

$$\int_{\Omega^\delta_{\varepsilon\varrho'}}\left(\bar{C}\cdot\frac{1}{i}\frac{\partial}{\partial x^l}D_\alpha u\right)\cdot\bar{v}\cdot dx = \int_{\Delta_l}\bar{C}\frac{1}{i}u\cdot\overline{D_\alpha v}\,d\sigma + \int_{\Omega^\delta_{\varepsilon\varrho'}}u\cdot\overline{C\frac{1}{i}\frac{\partial}{\partial x^l}D_\alpha v}\cdot dx.$$

The surface integral is here extended over the two faces Δ_l of $R^\delta_{\varepsilon\varrho'}$, orthogonal to the x^l-direction. The absolute value of this integral is thus bounded by the quantity $2|c|\cdot|C|\cdot\varepsilon\cdot\delta\cdot\varrho'^{\nu-3}\cdot M_\alpha$, where M_α denotes the maximum of the function $|D_\alpha v(x)|$ on Δ_l, and hence $\to 0$ when $\delta\to 0$ with ε and ϱ' fixed.

When the method used in the proof of Theorem 2 is applied to the operator $P_m(D) = (1/i)\ (\partial/\partial x^1)Q(D)$ corresponding to $P_m(\xi) = \xi_1 Q(\xi)$, we get the result that the absolute value of

$$\int_{\Omega^\delta_{\varepsilon\varrho'}}(\bar{P}_m(D)u)\bar{v}\,dx - \int_{\Omega^\delta_{\varepsilon\varrho'}}u\cdot\overline{P_m(D)v}\,dx$$

is smaller than $2|c|\delta\cdot\varrho'^{\nu-2}\cdot M_Q$, where M_Q denotes the greatest value of the function $|Q(D)v(x)|$ on the faces of $R^\delta_{\varepsilon\varrho'}$ orthogonal to the x^1-direction.

If we now set

$$I^\delta_{\varepsilon\varrho'} = \int_{\Omega^\delta_{\varepsilon\varrho'}}(\bar{P}(D)u\cdot\bar{v} - u\cdot\overline{P(D)v})\,dx$$

we get, since $\Omega - \Omega^0_{\varepsilon\varrho'}$ is a null set and the integral $I^0_{0\varrho}$ is absolutely convergent according to the assumptions,

$$\lim_{\delta\to 0} I^\delta_{\varepsilon\varrho'} = 0 = (\bar{P}(D)u, v) - (u, P(D)v)$$

which completes the proof.

We finally generalize Theorem 3. Let the coordinate system be defined in relation to $P(\xi)$ as in Theorem 4 and let $\hat{\Omega}$, Ω and Γ denote the same sets as before. We set $\check{\Omega} = -\hat{\Omega} = \{x \mid -x\in\hat{\Omega}\}$ and similarly, for the sake of symmetry of notations, $\check{\Gamma} = -\Gamma\ (=\Gamma)$. We further denote by Σ' the subspace of R^ν defined by the equations $x^1 = x^2 = 0$. Let ψ be a function in Σ' satisfying $\psi\in C_0^\infty(\Gamma)$ and

$\int \psi\, d\sigma \neq 0$, where $d\sigma$ is the element of surface of Σ'. Denote by μ the measure $\psi\, d\sigma$ in R^ν with support in Γ.

Theorem 5. *Let $P(\xi)$ be homogeneous and non-elliptic modulo the lineality space $\Lambda(P)$. If E is any proper fundamental solution of $P(D)$ and if e is the restriction to Ω of $E * \mu$, we have $e \in \mathcal{D}_{P_w}(\Omega)$ and $e \notin \mathcal{D}_{P_s}(\Omega)$.*

Proof: We first prove that the restriction of $E * \mu$ to $\hat{\Omega}$ is in $L^2(\hat{\Omega})$. This will follow from a theorem of F. Riesz if we prove that $\langle E * \mu, u \rangle$ is a linear functional of $u \in L^2(\hat{\Omega})$ or, equivalently, an inequality of the form

$$|\langle E * \mu, u \rangle| \leq C \|u\|_{L^2(\hat{\Omega})} \quad \text{if} \quad u \in C_0^\infty(\hat{\Omega}).$$

Let $\check{\varphi}$ denote a function in $C_0^\infty(\check{\Omega})$ for which the restriction to Σ' is $\equiv +1$ in $\check{\Gamma}$ and let u be any function in $C_0^\infty(\hat{\Omega})$. We set $\check{u}(x) = u(-x)$. Applying Leibniz' formula we get

$$P(D)(\check{\varphi}(E * \check{u})) = \sum_\alpha (P^{(\alpha)}(D)(E * \check{u}))\, D_\alpha \check{\varphi}/|\alpha|!.$$

Since E is proper we have (cf. [2] formula (1.11))

$$\|P^{(\alpha)}(D)(E * \check{u})\|_{L^2(\check{\Omega})} \leq C \|\check{u}\|_{L^2(\check{\Omega})} = C \|u\|_{L^2(\hat{\Omega})}.$$

Hence with $v = \check{\varphi}(E * \check{u})$

$$\|P(D) v\|_{L^2(\check{\Omega})} \leq C \|u\|_{L^2(\hat{\Omega})}.$$

From Lemma 3 we know that the function $1/\tilde{P}_\Sigma$ is uniformly square integrable in the varieties Σ parallel to the $\xi_1 \xi_2$-plane. Since $v = \check{\varphi}(E * \check{u}) \in C_0^\infty(\check{\Omega})$ ([4] VI, Théorème XI) we can apply [1] Theorem 2.8 getting

$$\|E * \check{u}\|_{L^2(\check{\Gamma})} = \|v\|_{L^2(\check{\Gamma})} \leq \|v\|_{L^2(\Sigma')} \leq C \|P(D) v\|_{L^2(\check{\Omega})} \leq C' \|u\|_{L^2(\hat{\Omega})}.$$

Now we get by Schwarz' inequality

$$|\langle E * \check{u}, \check{\mu} \rangle| \leq C \|E * \check{u}\|_{L^2(\check{\Gamma})}$$

and making use of the associativity and commutativity of the convolution when all but one of the components have compact support ([4] VI, Théorème VII)

$$\langle E * \mu, u \rangle = E * \mu * \check{u}(0) = \langle E * \check{u}, \check{\mu} \rangle.$$

Combining these results we get

$$|\langle E * \mu, u \rangle| \leq C \|u\|_{L^2(\hat{\Omega})}.$$

Now we can reproduce the method of Theorem 3 almost word for word. We infer from what was just proved that $e \in L^2(\Omega)$. Since $P(D) E * \mu = \delta * \mu = \mu$ we have $P(D) E * \mu = 0$ in Ω and consequently $P(D) e$ belongs to $L^2(\Omega)$. Therefore we have $e \in \mathcal{D}_{P_w}(\Omega)$.

If $\hat{u} \in C_0^\infty(\hat{\Omega})$, $\hat{u} = c \neq 0$ in a neighbourhood of Γ we get in the sense of distribution theory

$$\int_\Omega e \cdot \overline{\overline{P}(D) u} \cdot dx = \int_{R^n} (E * \mu) \overline{\overline{P}(D) \hat{u}}\, dx = \langle P(D) E * \mu, \bar{\hat{u}} \rangle =$$

$$= \langle \delta * \mu, \bar{\hat{u}} \rangle = \int_\Gamma \psi \cdot \bar{\hat{u}} \cdot d\sigma = \bar{c} \int_\Gamma \psi\, d\sigma \neq 0.$$

On the other hand we know from Lemma 5 that $u \in \mathcal{D}_{P_s^*}(\Omega)$ so that

$$\int_\Omega e \cdot \overline{\overline{P}(D) u}\, dx = \int_\Omega e \cdot \overline{P_s^* u} \cdot dx.$$

Now if it were true that $e \in \mathcal{D}_{P_s}(\Omega)$ we would get

$$\int_\Omega e \cdot \overline{P_s^* u}\, dx = \int_\Omega P_s e \cdot \bar{u} \cdot dx = \int_\Omega P_w e \cdot \bar{u}\, dx = 0$$

because the integrand equals zero. This gives a contradiction, i.e.

$$e \in \mathcal{D}_{P_w}(\Omega) \quad \text{but} \quad e \notin \mathcal{D}_{P_s}(\Omega).$$

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