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The modulus of convexity in normed linear spaces

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With 1 figure in the text

The modulus of convexity $\delta(\varepsilon)$ of a normed linear space is defined as

$$\delta(\varepsilon)=\inf_{\substack{\|x\|=\|y\|=1\\ \|x-y\|=\varepsilon}}\left(1-\left\|\frac{x+y}{2}\right\|\right)$$

(Clarkson, Day (1), Lemma 5.1). For example, the modulus of convexity $\delta_E(\varepsilon)$ of an abstract Euclidean space is easily determined from the parallelogram identity and is found to be

$$\delta_E(\varepsilon)=1-\sqrt{1-\frac{\varepsilon^2}{4}}.$$

Theorem. *The inequality*

$$\delta(\varepsilon)\leqslant\delta_E(\varepsilon)$$

holds for every modulus of convexity $\delta(\varepsilon)$ in a space with real scalars.

Proof. It is sufficient to prove the theorem in the two-dimensional case. In the usual way such a space may be thought of as a plane in which the norm is determined by a central-symmetric convex curve Γ with midpoint in O. The norm of a vector is determined by the ratio of the length of the vector to the length of the parallel vector starting at O and ending at Γ. Thus Γ is the locus of the endpoints of the vectors from O of norm one.

If the two unit vectors $x=\overrightarrow{OA}$ and $y=\overrightarrow{OB}$ are rotated around Γ, while their difference $x-y=\overrightarrow{BA}$ has constantly the norm ε, the endpoint M of the vector $\overrightarrow{OM}=\frac{1}{2}(x+y)$ describes a curve Γ_ε (Fig. 1).

Γ_ε is a simple closed curve, as can be inferred from Lemma 2.4 of Day (1). Let Y be the area of the region inside Γ and Y_ε the area of the region inside Γ_ε.

Lemma.

$$Y_\varepsilon=\left(1-\frac{\varepsilon^2}{4}\right)Y.$$

The lemma is a simple extension of a theorem of Holditch (de La Vallée Poussin, p. 318), and is proved by the same method.

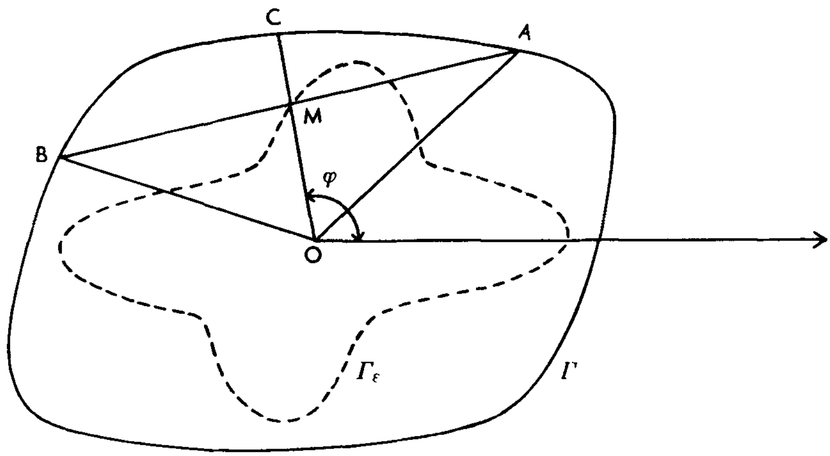

Fig. 1.
(The polar angle φ is measured from an arbitrarily chosen direction.)

Let (x_1, y_1) and (x_2, y_2) be the coordinates of A and B in an arbitrary rectangular coordinate system. Then

$$Y = \int_\alpha^\beta y_1\, d x_1 = \int_\alpha^\beta y_2\, d x_2$$

where the variables x_1, y_1, x_2 and y_2 are thought of as dependent of a parameter t, varying from α to β as the curve is described once in the negative direction. The coordinates of M are then $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and

$$Y_\varepsilon = \tfrac{1}{4}\int_\alpha^\beta (y_1+y_2)\, d(x_1+x_2).$$

Hence
$$Y - Y_\varepsilon = \tfrac{1}{4}\int_\alpha^\beta (y_1-y_2)\, d(x_1-x_2).$$

The integral in the right member expresses the area of the region inside the curve described by the endpoint of the vector $\overrightarrow{AB}$, if this vector is laid off from a fixed point. This curve is similar to Γ in the ratio ε. Hence

$$Y - Y_\varepsilon = \tfrac{1}{4}\cdot \varepsilon^2 \cdot Y,$$

which proves the lemma.

The norm of the part $\overrightarrow{MC}$ of the unit vector $\overrightarrow{OC}$ through M is denoted $\Delta(\varepsilon, \varphi)$. We have

$$\delta(\varepsilon) = \inf_\varphi \Delta(\varepsilon, \varphi).$$

If the areas Y and Y_ε are expressed in polar coordinates the lemma takes the form

$$\int_0^{2\pi} (1-\Delta(\varepsilon,\varphi))^2\, r^2(\varphi)\, d\varphi = \left(1-\frac{\varepsilon^2}{4}\right)\int_0^{2\pi} r^2(\varphi)\, d\varphi,$$

where $r(\varphi)$ is the length of the vector $\overrightarrow{OC}$. Hence

$$\int_0^{2\pi}\left[(1-\Delta(\varepsilon,\varphi))^2-\left(1-\frac{\varepsilon^2}{4}\right)\right] r^2(\varphi)\, d\varphi = 0,$$

which implies

$$\delta(\varepsilon)=\inf_{\varphi}\Delta(\varepsilon,\varphi)\leqslant 1-\sqrt{1-\frac{\varepsilon^2}{4}}=\delta_E(\varepsilon),$$

because $r^2(\varphi)>0$.

Theorem 4.1 of Day (2) shows that if there is equality in our theorem for all ε, $0<\varepsilon\leqslant 2$, the space is abstract Euclidean. I have not been able to decide whether the same conclusion holds under the sole assumption of equality for a certain ε_0, $0<\varepsilon_0<2$.

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