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Suboperative functions and semi-groups of operators

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1. Let S be a locally compact space endowed with a composition law $(s, t) \to s \circ t$ defined for $(s, t) \in D \subset S \times S$, $\mathfrak{J}$ the set of open parts of S, $\mathfrak{B} \subset \mathfrak{J}$ a filter basis on S and μ a positive Radon measure[1] on S. Suppose that:

1) $(s, t) \to s \circ t$ is continuous on D;

2) for each $t \in S$ and $W \in \mathfrak{V}(t)$ there exist $U \in \mathfrak{B}$ and $V \in \mathfrak{V}(t)$ such that $U \circ V \subset W$;

3) $\mu(U) \neq 0$ for every non void set $U \in \mathfrak{J}$.

A function f on S to $[-\infty, \infty)$ is said to be suboperative if $f(s \circ t) \leqslant f(s) + f(t)$ for every $(s, t) \in D$. In this paper various properties of suboperative functions and various measurability and continuity properties of certain representations of S are given. Functions satisfying conditions less restrictive than the suboperative ones are also studied.

Most of the results proved below may be considered as extensions of those given in (5) p. 92–94 and (6) p. 741–748. In fact, the subject matter of this paper has been suggested by the reading of (5) p. 92–94 and (6) p. 741–748.

2. It will be supposed in this paragraph that the following condition is satisfied (we denote with $\mathfrak{K}$ the set of all compact parts of S):

(H_1) *For every* $t \in S$ *there exist* $K_t \in \mathfrak{K} \cap \mathfrak{V}(t)$, $V_t \in \mathfrak{B}$ *and a continuous mapping* $(s, u) \to p_s(u)$ *of* $K_t \times V_t$ *into* S *having the following properties:* (i) $u \circ p_s(u) = s$ *for every* $s \in K_t$ *and* $u \in V_t$; (ii) $\lim_{\mathfrak{B}} p_s = s$ *for every* $s \in K_t$; (iii) *there is a constant* $\lambda(K_t) > 0$ *which satisfies the inequality* $\mu(p_s^{-1}(U)) \leqslant \lambda(K_t)\,\mu(U)$ *for every* $s \in K_t$ *and* $U \in \mathfrak{J}$.

The condition (H_1) is a refinement of condition (P_9^*), (6) p. 741. Let us remark that if (H_1) is satisfied, then, for every $s \in K_t$ and μ-measurable set $A \subset S$, the set $p_s^{-1}(A)$ is μ-measurable and $\mu(p_s^{-1}(A)) \leqslant \lambda(K_t)\,\mu(A)$.

Proposition 1.—*Let* $t \in S$, $K_1 \subset V_t$, $K_1 \in \mathfrak{K}$ *and* f *a locally* μ*-integrable mapping of* S *into a Banach space* E. *Then for every* $t' \in K_t$

4) $$\lim_{s \to t',\, s \in K_t} \int \varphi_{K_1}(u)\, \| f(p_s(u)) - f(p_{t'}(u)) \|\, d\mu(u) = 0\,[2].$$

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[1] Various indications concerning the terminology are given in paragraph 6.

[2] We take $v(u) = 0$ if $v = h \circ p_s$ and $u \notin V_t$.

Let us remark first that if h is a locally μ-integrable mapping of S into a Banach space E and $C=\{p_s(u) \mid s \in K_t,\ u \in K_1\}$, then, for every $s \in K_t$, $h \circ p_s$ is μ-measurable and

$$5) \qquad \int \varphi_{K_1}(u) \, \| h(p_s(u)) \| \, d\mu(u) \leqslant \lambda(K_t) \int \varphi_C(u) \, \| h(u) \| \, d\mu(u)$$

(we use (H_1) and we verify first 5) for $h=\varphi_A x$, where $x \in E$ and A is a μ-integrable part of S). Take $\varepsilon>0$ and choose a function g, continuous and with compact support, such that

$$6) \qquad \int \varphi_C(u) \, \| f(u)-g(u) \| \, d\mu(u) \leqslant \varepsilon.$$

Now for $s, t' \in K_t$ we have

$$\int \varphi_{K_1}(u) \, \| f(p_s(u))-f(p_{t'}(u)) \| \, d\mu(u) \leqslant \int \varphi_{K_1}(u) \, \| f(p_s(u))-g(p_s(u)) \| \, d\mu(u)$$
$$+\int \varphi_{K_1}(u) \, \| g(p_s(u))-g(p_{t'}(u)) \| \, d\mu(u)+\int \varphi_{K_1}(u) \, \| g(p_{t'}(u))-f(p_{t'}(u)) \| \, d\mu(u).$$

Since $(s,u) \rightarrow p_s(u)$ is uniformly continuous on $K_t \times K_1$, there is a $U \in \mathfrak{V}(t')$ such that $\| g(p_s(u))-g(p_{t'}(u)) \| \leqslant \varepsilon$ for $s \in U \cap K_t$ and $u \in K_1$. We deduce, using 5) and 6), that for $s \in U \cap K_t$ the sum in the second term of the above inequality is bounded by $(2\lambda(K_t)+\mu(K_1))\,\varepsilon$; since $\varepsilon>0$ is arbitrary, the relation 4) is completely proved.

Proposition 2.—*Let $t \in S$, B a compact set contained in $p_t(V_t)$ and A a relatively compact set whose closure is contained in $p_t^{-1}(B)$. Suppose that $\mu^*(A)\,\mu(B) \neq 0$. Then the interior of the set*

$$\{s \circ t \mid (s,t) \in D,\ s \in A,\ t \in B\}$$

contains t.

Let $g(s)=\overset{*}{\int} \varphi_A(u)\,\varphi_B(p_s(u))\,d\mu(u)$ for every $s \in K_t$. If $s_1, s_2 \in K_t$ then $|g(s_1)-$ $-g(s_2)|$ is bounded above by

$$\overset{*}{\int} \varphi_A(u) \, |\varphi_B(p_{s_1}(u))-\varphi_B(p_{s_2}(u))| \, d\mu(u) \leqslant \int \varphi_{\bar{A}}(u) \, |\varphi_B(p_{s_1}(u))-\varphi_B(p_{s_2}(u))| \, d\mu(u).$$

Using proposition 1, we deduce that g is continuous on K_t. If $u \in A$ we have $\varphi_A(u)\,\varphi_B(p_t(u))=1$; hence $g(t) \neq 0$. But if $g(s) \neq 0$, then, there is a $u \in A$ for which $\varphi_A(u)\,\varphi_B(p_s(u))=1$. It follows that

$$\{s \circ t \mid (s,t) \in D,\ s \in A,\ t \in B\} \supset \{s \mid s \in K_t,\ g(s) \neq 0\} \ni t.$$

Hence the proposition is completely proved.

Remark.—Proposition 2 is somewhat similar to theorem 1, (1) p. 648.

Let S_f, S_g, S_h be three separated (=Hausdorff) topological spaces and G a mapping of $S_g \times S_h$ into S_f, continuous on every compact set $K \subset S_g \times S_h$.

Theorem 1.—*Let f, g, h be three mappings of S into S_f, S_g and S_h respectively. Suppose that g, h are μ-measurable and that, for every $(s,t)\in D$, $f(s\circ t)=G(g(s),\ h(t))$. Then, f is continuous on S.*

Let $t\in S$. Since g and h are μ-measurable, there exist compact sets $B\subset p_t(V_t)$ and $A\subset p_t^{-1}(B)$ such that g_A and h_B are continuous and $\mu(A)\,\mu(B)\neq 0$ (we may suppose that $V_t\times p_t(V_t)\subset D$). By proposition 2 the interior of $A\circ B$ contains t; hence to prove the theorem, it is enough to show that $f_{A\circ B}$ is continuous at t. Suppose that $f_{A\circ B}$ is not continuous at t. Then there exists a directed family $(s_i)_{i\in I}$ of elements belonging to $A\circ B$, which converges to t, and a $V\in\mathfrak{V}(f(t))$ such that $f(s_i)\notin V$, for each $i\in I$. For every $i\in I$, $s_i=a_i\circ b_i$ where $a_i\in A$, $b_i\in B$. Since A and B are compact, the family $((a_i,b_i))_{i\in I}$ has a cluster point $(a,b)\in A\times B$ and obviously $a\circ b=t$. Now let $W\in\mathfrak{V}((a,b))$ be such that $G(g(s'),h(s''))\in V$ if $(s',s'')\in W\cap(A\times B)$. Then if we take $j\in I$ such that $(a_j,b_j)\in W$ we obtain $f(s_j)=G(g(a_j),h(b_j))\in V$. Since this leads to contradiction it follows that $f_{A\circ B}$ is continuous at t.

Theorem 2.—*Let f, g, h be three mappings of S into $[-\infty,\infty)$. Suppose that h is μ-measurable and that, for each $(s,t)\in D$, $f(s\circ t)\leqslant g(s)+h(t)$. Then f is bounded above on every set $K\in\mathcal{K}$.*

We shall give two proofs of this theorem. The first is essentially based on proposition 2. The second is an adaptation of a proof given in (6) p. 742–743.

I) Let $t\in S$ (we may suppose that $V_t\times p_t(V_t)\subset D$). Take $A_1\subset V_t$, $A_1\in\mathcal{K}$ such that $\mu(A_1)\neq 0$ and let $B_1=p_t(A_1)\subset p_t(V_t)$; using $\{(H_1),\ (\text{iii})\}$ we deduce $\mu(B_1)\neq 0$. Since h is μ-measurable there exists a compact set $B\subset B_1$ on which h is bounded above and such that $\mu(p_t^{-1}(B)\cap A_1)\neq 0$. Now there is a set $A\subset p_t^{-1}(B)\cap A_1$ on which g is bounded above and such that $\mu^*(A)\neq 0$. It follows that f is bounded above on $A\circ B$. Since $t\in S$ is arbitrary and since by proposition 2, $A\circ B\in\mathfrak{V}(t)$ the theorem is completely proved.

II) Suppose that $K\in\mathcal{K}$ and that f is not bounded above on K. Then there exists a sequence $(t(n))_{n\in N}$ of elements belonging to K such that $f(t(n))\geqslant 2n$ for every $n\in N$. The set K being compact the considered sequence has a cluster point $t\in K$. Let $U\in\mathfrak{V}(t)$ and $K_1\subset V_t$, $K_1\in\mathcal{K}$, $\mu(K_1)\neq 0$ such that $p_s(u)\in K_t$ if $u\in K_1$ and $s\in U\cap K_t$. We may suppose that $t(n)\in U\cap K_t$ for every $n\in N$.

For each $n\in N$ define $A(n)=\{s\,|\,s\in K_1,\ g(s)\geqslant n\}$ and $B(n)=\{s\,|\,s\in K_t,\ h(s)\geqslant n\}$. We shall show that, for every $n\in N$,

$$A(n)\cup p_{t(n)}^{-1}(B(n))\supset K_1. \tag{7}$$

In fact, take $u\in K_1$. Then $u\circ p_{t(n)}(u)=t(n)$ for each $n\in N$; since

$$2n\leqslant f(t(n))\leqslant g(u)+h(p_{t(n)}(u))$$

we deduce that either $g(u)\geqslant n$ and then $u\in A(n)$ or $h(p_{t(n)}(u))\geqslant n$ and then $u\in p_{t(n)}^{-1}(B(n))$.

Now, for each $n\in N$, $B(n)$ is μ-measurable and $B(n)\supset B(n+1)$. Let $B(\infty)=\bigcap_{n\in N}B(n)$. Since $s\in B(\infty)$ implies $h(s)=\infty$ and since this leads to contradiction we must have $B(\infty)=\emptyset$. It follows then that $\lim_{n\in N}\mu(B(n))=0$; hence there is a $n_0\in N$ such that if $n\geqslant n_0$ we have $\mu(B(n))<\mu(K_1)/2\lambda(K_t)$, whence

$$\mu(p_{t(n)}^{-1}(B(n)))\leqslant\lambda(K_t)\,\mu(B(n))<\mu(K_1)/2.$$

But then $\mu(K_1 - p_{t(n)}^{-1}(B(n))) > \mu(K_1)/2$, whence, using 7), $\mu_*(A(n)) > \mu(K_1)/2$. Since for each $n \in N$, $A(n) \supset A(n+1)$, this implies that $A(\infty) = \bigcap_{n \in N} A(n) \neq \emptyset$, relation which leads again to contradiction. Hence the hypothesis that f is not bounded above on every $K \in \mathcal{K}$ leads to contradiction and so the theorem is completely proved.

Remarks.—1) A family $\mathcal{J} \subset \mathcal{K}$ is *μ-dense* in S if for every $K \in \mathcal{K}$ and $\varepsilon > 0$ there is $K \in \mathcal{J}$ such that $\mu(K - K_\varepsilon) \leqslant \varepsilon$. It is easy to see (using the method of proof I) that the conclusion of theorem 2 remains valid if instead of supposing that h is μ-measurable we suppose that the family of sets $K \in \mathcal{K}$ on which h is bounded above is μ-dense in S.

2) The conclusion of theorem 2 remains valid (use the method of proof II) if instead of supposing that h is μ-measurable we suppose that g is μ-measurable.

3) Let G be a mapping of $[-\infty, \infty] \times [-\infty, \infty]$ into $[-\infty, \infty]$ and suppose that G is bounded above on every set of the form $[-\infty, a') \times [-\infty, a'')$, where $a', a'' \in (-\infty, \infty)$. Let f, g, h be three mappings of S into $[-\infty, \infty)$. Suppose that h is μ-measurable and that $f(s \circ t) \leqslant G(g(s), h(t))$ for every $(s, t) \in D$. Then it can be shown that f is bounded above on every $K \in \mathcal{K}$ (use the method of proof I). The conclusion remains still valid if instead of supposing that h is μ-measurable we suppose that the family of sets $K \in \mathcal{K}$ on which h is bounded above is μ-dense in S.

For every Banach space E denote by $\mathcal{L}(E, E)$ the set of linear continuous mappings of E into E endowed with the usual structure.

Theorem 3.—*Let T: $s \to T(s)$ be a strongly μ-measurable mapping of S into $\mathcal{L}(E, E)$ such that $T(s \circ t) = T(s) \circ T(t)$ for every $(s, t) \in D$. Then T is strongly continuous on S.*

Let us define on S the functions $f_x = h_x \, (x \in E)$ and g by the equations $f_x(s) = h_x(s) = \log \|T(s)x\|$ and $g(s) = \log \|T(s)\|$. Theorem 2 implies that, for each $x \in E$, f_x is bounded above on every $K \in \mathcal{K}$. We deduce that T is (uniformly) bounded on every $K \in \mathcal{K}$ and in particular that $s \to T(s)x$ is locally μ-integrable for each $x \in E$.

Now let $t \in S$ and $A \subset V_t$, $A \in \mathcal{K}$ such that $\mu(A) \neq 0$. Since for every $u \in V_t$, $s \in K_t$ and $x \in E$ we have

$$T(s)x - T(t)x = T(u)\,T(p_s(u))\,x - T(u)\,T(p_t(u))\,x$$

it follows that $\|T(s)x - T(t)x\|$ is bounded above by

$$(\sup_{u \in A} \|T(u)\| / \mu(A)) \int \varphi_A(u) \, \|T(p_t(u))\,x - T(p_s(u))\,x\| \, d\mu(u).$$

The strong continuity of T follows then from proposition 1.

3. In this section three results (theorems 4, 5 and 6) are proved. The first two concern suboperative functions and the third representations of S into $\mathcal{L}(E, E)$. It is not necessarily supposed, here, that (H_1) is satisfied. Instead various other conditions $((H_2) - (H_5))$ are introduced. It will be mentioned in every case which of these conditions are supposed to be satisfied.

(H_2) *For every* $t \in S$ *there exist* $K \in \mathcal{K}$ *and* $V \in \mathcal{V}(t)$ *such that*

$$\inf \{\mu((s \circ K) \cup (K \circ s)) \mid s \in V\} > 0.$$

(H_3) *For every* $t \in S$ *there exist* $K^t \in \mathcal{K} \cap \mathcal{V}(t)$ *and* $V^t \in \mathcal{B}$ *such that, for each* $s \in K^t$, $u \to s \circ u$ *is an open mapping of* V^t *into* S *(condition* (H_3) *is understood as implying that* $K^t \times V^t \subset D$*).*

(H_4) *For every* $t \in S$ *there is a* $\tilde{V}_t \in \mathcal{B}$ *and a mapping* q_t *of* $\tilde{V}_t$ *into* S *such that* $\lim_{\mathcal{B}} q_t = t$ *and* $q_t(u) \circ u = t$ *for each* $u \in \tilde{V}_t$.

(H_5) $S \subset R^n$ $(n \geqslant 1)$, $\mathcal{B} = \{V_\varepsilon(0) \cap S \mid \varepsilon > 0\}$, μ *is the Lebesgue measure and:* (i) *for each* $t \in S$ $\liminf_{\varepsilon \to 0} \mu_*(V_\varepsilon(t) \cap S^t)/\mu(V_\varepsilon(t)) > 0$; (ii) *for each* $t \in S$, $\mathcal{B}_t$ *is a filter basis, weaker than the basis* $\mathcal{B}(t) = \{V_\varepsilon(t) \cap S^t \mid \varepsilon > 0\}$.

We have denoted with S^t, for every $t \in S$, the set $\{t \circ s \mid (t, s) \in D,\ s \in S\}$. By $\mathcal{B}_t$ we have denoted the set $\{U_t \mid U \in \mathcal{B}\}$, where, for each $U \in \mathcal{B}$,

$$U_t = \{t \circ u \mid (t, u) \in D,\ u \in U\}.$$

Theorem 4.—*Let* f *be a* μ*-measurable suboperative mapping of* S *into* $(-\infty, \infty)$. *If* (H_1) *and* (H_2) *are satisfied then* f *is bounded on every set* $K \in \mathcal{K}$.

Suppose that there exists a set $K \in \mathcal{K}$ on which f is unbounded. Then there is a sequence of elements belonging to K, $(t(n))_{n \in N}$, such that $f(t(n)) \leqslant -n$ for every $n \in N$. Let t be a cluster point of the considered sequence and let $V \in \mathcal{V}(t)$ and $K \in \mathcal{K}$ be such that

$$\inf \{\mu((s \circ K) \cup (K \circ s)) \mid s \in V\} = d > 0.$$

Let $M = \sup_{s \in K} f(s)$ and for every $n \in N$ let $E(n) = \{s \mid f(s) \leqslant -n + M\}$. We deduce $E(n) \supset (t(n) \circ K) \cup (K \circ t(n))$, whence

$$\mu(E(n)) \geqslant \mu((t(n) \circ K) \cup (K \circ t(n)) \geqslant d,$$

if $t(n) \in V$. Since t is a cluster point of the sequence $(t(n))_{n \in N}$, $t(n) \in V$ for infinitely many $n \in N$. It follows that $E(\infty) = \bigcap_{n \in N} E(n) \neq \emptyset$. But $f(s) = -\infty$ for $s \in E(\infty)$; hence the hypothesis that f is unbounded on K leads to contradiction.

Theorem 5.—*Let* f *be a* μ*-measurable suboperative mapping of* S *into* $(-\infty, \infty)$. *If* (H_1) *and* (H_2) *are satisfied then* $d = \liminf_{\mathcal{B}} f \in [0, \infty]$.

Suppose that $d = -\infty$. Take $t \in S$ and let $M = \sup_{s \in K_t} |f(s)|$. By 2) there exists a $U \in B$ such that $U \circ t \subset K_t$. Since $d = -\infty$ there exists a $u \in U$ for which $f(u) < -M - f(t)$. But then $-M \leqslant f(u \circ t) \leqslant f(u) + f(t) < -M$. It follows that the hypothesis $d = -\infty$ leads to contradiction. Hence $d \in (-\infty, \infty]$.

Suppose now that $d \in (-\infty, \infty)$, let $\varepsilon > 0$ and choose an $U \in \mathcal{B}$ such that $d - \varepsilon \leqslant f(s)$ if $s \in U$. Take $t \in U$ satisfying the inequality $f(t) \leqslant d + \varepsilon$. By 2) there is an $U' \in \mathcal{B}$ such that $U' \circ t \subset U$. In U' we may find a t' for which $f(t') \leqslant d + \varepsilon$. We deduce $d - \varepsilon \leqslant f(t' \circ t) \leqslant f(t') + f(t) \leqslant 2(d + \varepsilon)$, whence since $\varepsilon > 0$ is arbitrary, $d \in [0, \infty)$. Hence the theorem is completely proved.

Theorem 6.—*Let* $T\colon s \to T(s)$ *be a mapping of* S *into* $\mathcal{L}(E, E)$ *such that* $T(s \circ t) = T(s) \circ T(t)$ *for* $(s, t) \in D$. 1° *If* (H_3) *and* (H_4) *are satisfied and if*

$\lim_{\mathcal{B}} T = I$ *strongly (uniformly) then T is strongly (uniformly) continuous on S.* 2° *If (H_1), (H_3) and (H_5) are satisfied and if $\lim_{\mathcal{B}} T = I$ weakly, then T is strongly continuous on S.*

1° Let $t \in S$ and $x \in E$. Choose $V \in \mathcal{B}$ such that: (i) $V \subset V^t \cap \tilde{V}_t$, where V^t and $\tilde{V}_t$ are the sets considered in the statements of the conditions (H_3) and (H_4); (ii) $q_t(u) \in K^t$ for $u \in V$; (iii) $\sup_{u \in V} \| T(u)x \| < \infty$. Take $a \in V$ and let $b = q_t(a)$. By (H_3) and (H_4), $b \circ V \in \mathfrak{V}(t)$ and for $s \in b \circ V$

$$\| T(s)x \| \leqslant \| T(b) \| \sup_{u \in V} \| T(u)x \|.$$

It follows that T is (uniformly) bounded on every set belonging to $\mathcal{K}$.

Now let $\varepsilon > 0$ and let $V_1 \subset V$, $V_1 \in \mathcal{B}$ such that, for $u \in V_1$

$$\| T(u)x - x \| \leqslant \varepsilon / 2 \sup_{s \in K^t} \| T(s)x \|.$$

Take $a_1 \in V_1$ and let $b_1 = q_t(a_1)$. Then, for $s \in b_1 \circ V_1$ we have

$$\| T(s)x - T(t)x \| = \| T(b_1 \circ u)x - T(b_1 \circ a_1)x \| \leqslant \| T(b_1) \| \; \| T(u)x - T(a_1)x \| \leqslant \varepsilon.$$

Hence T is strongly continuous on S. The uniform continuity is proved exactly in the same way.

2° The hypotheses in the statement implies that, for every $t \in S$, $\lim_{\mathcal{B}(t)} T = T(t)$ weakly. We deduce, using (H_5), that T is weakly μ-measurable (see for instance (7)). If we take a denumerable set $D \subset S$ such that $\bar{D} \supset S$ it follows from (H_3) that, for every $t \in S$ and $x \in E$, $T(t)x$ belongs to the closed linear space spanned by $\{T(s)x \mid s \in D\}$. Hence T is strongly μ-measurable and using theorem 3 we deduce that T is strongly continuous on S.

4. The conditions (H_1), (H_3), (H_4) are refinements of the conditions (P_9), (P_9^*) and (P_{10}) given in (6). Let G be a locally compact group, $S \subset G$ an open semi-group such that $\bar{S} \ni e$ ($e =$ the unit element of G), $\mathcal{B} = \{U \cap S \mid U$ open neighborhood of $e\}$, μ the restriction to S of a left invariant Haar measure of G and $\circ$ the law of G. It is easy to see that in this case the conditions 1)–3) and (H_1)–(H_4) are satisfied by the system $\{S, \mathcal{B}, \mu, \circ\}$. Let us also remark that if $\{S, \mathcal{B}, \mu, \circ\}$ and $\{S', \mathcal{B}', \mu', \circ'\}$ satisfy one of the conditions 1)–3) or (H_1), (H_3), (H_4) then the system $\{S \times S', \mathcal{B} \otimes \mathcal{B}', \mu \otimes \mu', \circ''\}$ satisfies the same condition ($\mathcal{B} \otimes \mathcal{B}' = \{U \times U' \mid U \in \mathcal{B}, U' \in \mathcal{B}'\}$ and $\circ''$ is the composition law defined by the equation $(s, s') \circ (t, t') = (s \circ t, s' \circ t')$).

5. Theorem 2 is an extension of theorem 25.5.1, (6) p. 742–743 (we take $S = \pi_0 - \Phi$). Theorem 2 and the remark following it remain valid if instead of supposing that $f(s \circ t) \leqslant g(s) + h(t)$ for $(s, t) \in D$, we suppose that $f(s \circ t) \leqslant g(s) + h(t)$ for $(s, t) \in D - (N \times N)$, where N is a locally μ-negligible set (see (3) p. 618). Theorems 1, 3 and 6 are extensions of theorems 25.6.1, (6) p. 743–744, 25.7.1, (6) p. 747, 25.6.2, (6) p. 745–746 and 25.7.2, (6) p. 747–748. In connection with theorems 1, 2, 3 and 6 see also (4) and (5). For theorems 4 and 5 see (6), p. 241–242 and p. 253–254 and (8). Theorems 1–5 are taken from the author's Yale thesis.

6. See (2) for the definitions and the results concerning the integration theory. $\int$ means $\int_S$. For each $t \in S$ we denote by $\mathfrak{V}(t)$ the set of all neighborhoods of t. The notations $s \circ t$ and $A \circ B$ are understood as implying that $(s, t) \in D$ and $A \times B \subset D$. For every $t \in S$, K_t and V_t are the sets introduced in the statement of the condition (H_1).

If X is a set and $K \subset X$, then φ_K is the characteristic function of K. If f is a mapping of a set X into a set Y and $A \subset X$, then we denote with f_A the restriction of f to A. For each $t \in R^n$ and $\varepsilon > 0$, $V_\varepsilon(t) = \{s \mid d(t, s) < \varepsilon\}$, where d is the euclidian distance in R^n.

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