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Approximate Fourier analysis of distribution functions

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NOTATIONS

$F(x)$ = distribution function = non-decreasing function, continuous to the right, and which fulfills the conditions $F(-\infty)=0$ and $F(+\infty)=1$

$$F_1(x) * F_2(x) = \int_{-\infty}^{+\infty} F_1(x-y)\, dF_2(y).$$

$$\bar{F}(x) = \frac{F(x+0)+F(x-0)}{2}.$$

$f(x)$ $= F'(x) =$ frequency function if $F(x)$ is absolutely continuous.

$\varphi(t)$ = characteristic function

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x).$$

Principal value. The general definition of an integral from $-\infty$ to $+\infty$ is

$$\int_{-\infty}^{+\infty} g(x)\, dx = \lim_{x'\to\infty} \int_{-x'}^{0} g(x)\, dx + \lim_{x''\to\infty} \int_{0}^{x''} g(x)\, dx,$$

supposing that these limits exist separately. In the present paper we will throughout define such an integral as its principal value, that is

$$\int_{-\infty}^{+\infty} g(x)\, dx = \lim_{x'\to\infty} \int_{-x'}^{x'} g(x)\, dx.$$

Analogously we define for sums

$$\sum_{-\infty}^{+\infty} a_\nu = \lim_{n\to\infty} \sum_{-n}^{n} a_\nu.$$

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SUMMARY

In this paper we discuss the numerical integration of an integral of the type

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt,$$

where $\varphi(t)$ is a characteristic function in the sense of probability theory. If $\varphi(t)$ is a characteristic function, the same is true of the integrand of

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\,\varphi(t)\,dt$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin h t}{h t}\cdot e^{-ixt}\,\varphi(t)\,dt.$$

These two integrals, which give the frequency function and the distribution function respectively, are thus of the type considered in this paper.

Chapter 1. We investigate the class of characteristic functions which are equal to zero outside a finite interval $(-T, T)$. If $\varphi(t)$ is multiplied by such a function $\varphi_1(t)$, the result will be a new characteristic function and the interval of integration will be reduced from $(-\infty, +\infty)$ to $(-T, T)$. We deduce an inequality, which makes it possible to estimate the error obtained by integrating $\varphi\cdot\varphi_1$ instead of φ. We also deduce a characteristic function $C(t)$, which is zero for $|t|\geqslant 1$ and which from a certain point of view may be considered the best tool to use for this truncation purpose.

Chapter 2. Let $\varphi(t)$ be a characteristic function and suppose that is has been multiplied by a function of the type considered in chapter 1. The corresponding frequency function is

$$f(x)=\frac{1}{2\pi}\int_{-T}^{T}e^{-itx}\,\varphi(t)\,dt.$$

It seems natural to try to approximate $f(x)$ by first approximating $\varphi(t)$ by $\varphi_A(t)$ and then performing the integration with φ replaced by φ_A. The approximation φ_A should of course be chosen so simple that an explicit expression for the integrated function is obtainable. The interval $(-T, T)$ is divided into sub-intervals by equally spaced points. Two approximations φ_A are investigated. (1) In each sub-interval $\varphi(t)$ is approximated by a straight line which takes on the same values as $\varphi(t)$ at the end-points. (2) In each sub-interval $\varphi(t)$ is approximated by a second degree parabola which takes on the same values as $\varphi(t)$ at the end-points and at the mid-point of the interval.

Note the difference between this technique and the usual numerical integration methods If Simpson's rule is applied to the above integral for example, the whole of the integrand

$$e^{-itx}\varphi(t)$$

is first approximated by parabolas. After that the integration is performed.

Chapter 3. The main result of this chapter is the following theorem:

If 1) $\varphi(t)$ is a characteristic function,
2) There is a finite interval $(-T, T)$ such that $\varphi(t)=0$ for $|t|\geqslant T$,
3) $\varphi'(t)$ and $\varphi''(t)$ exist and are continuous for all t,

then the integral

$$I=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

is approximated by

$$I_A=T_\lambda(\varphi)-R$$

and

$$|I-I_A|\leqslant R$$

where

$$T_\lambda(\varphi)=\frac{\lambda}{2\pi}\sum_\nu\varphi(\lambda\nu)$$

$$R=-\frac{\lambda^3}{16\pi^3}\sum_\nu\varphi''(\lambda\nu).$$

As R might be fairly complicated to calculate we deduce the following approximate expression for it

$$R\sim\frac{\lambda}{4\pi^3}\sum_\nu(-1)^\nu\varphi(\lambda\nu).$$

Chapter 4. Let $f(x)$ be a frequency function and $\varphi(t)$ its characteristic function. Consider the frequency function

$$g(x)=\sum_k f\left(\frac{2k\pi+x}{\lambda}\right)\quad\text{for } |x|\leqslant\pi.$$

The Fourier series of $g(x)$ reads

$$g(x)\sim\frac{\lambda}{2\pi}\sum e^{i\nu x}\varphi(-\lambda\nu).$$

We are in this chapter concerned with the problem of approximating $g(x)$ by applying a summation method on its Fourier series. We consider the well-known $\sigma_n(x)$, obtained by summation by arithmetic means, and a corresponding sum $g_n(x)$, obtained from the characteristic function $C(t)$ in the same way as $\sigma_n(x)$ is obtained from the function

$$h(t)=\begin{cases}1-|t| & \text{for } |t|<1,\\ 0 & \text{for } |t|\geqslant 1.\end{cases}$$

The asymptotic properties of σ_n and g_n are deduced.

Chapter 5. In this chapter we chose a certain set of characteristic functions and calculate the corresponding distribution functions, using the methods developed in chapter 3.

There are no references in the text. They have been brought together in the appendix at the end of the paper.

Chapter 1

A class of characteristic functions

In the present paper we are going to discuss various methods of integrating numerically an integral of the type

$$I(\varphi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt,$$

where $\varphi(t)$ is a characteristic function in the sense of probability theory, and where it is assumed that this integral exists.

The first difficulty which must be tackled concerns the "tails" of the integral. As a numerical integration formula cannot make use of an infinite number of values of the integrand the integral must be "truncated". We are going to use the following procedure for this truncation.

Let $\varphi_1(t)$ be a real characteristic function which fulfills the condition

$$\varphi_1(t)=0$$

for $|t|\geqslant T$ and $T>0$. Put

$$\varphi_2(t)=\varphi_1(t)\cdot\varphi(t).$$

Then φ_2 is also a characteristic function which fulfills the same condition as φ_1. Instead of $I(\varphi)$ we consider the integral $I(\varphi_2)$

$$I(\varphi_2)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi_2(t)\,dt=\frac{1}{2\pi}\int_{-T}^{T}\varphi_2(t)\,dt.$$

Since this integral is extended over a finite interval $(-T, T)$ the desired truncation has thus been arrived at. Instead of the original distribution function $F(x)$ corresponding to $\varphi(t)$ we are, however, now studying the distribution function $F_2(x)$ corresponding to $\varphi_2(t)$. In order that the procedure shall have any meaning it will thus be necessary to estimate the error introduced by considering φ_2 instead of φ. This is done with the aid of the following inequality:

1.1. *Let $F(x)$ and $F_1(x)$ be distribution functions and suppose that $F_1(x)$ is continuous and corresponds to a symmetrical distribution, i.e. $F_1(x)+F_1(-x)=1$ for all x. Put*

$$F_2(x)=F(x)\ast F_1(x).$$

Then
$$F(y)-F(x)\leqslant\frac{F_2(y+\varepsilon)-F_2(x-\varepsilon)}{2F_1(\varepsilon)-1}$$

and
$$F(y)-F(x)\geqslant 1-\frac{1-F_2(y-\varepsilon)+F_2(x+\varepsilon)}{F_1(\varepsilon)}$$

where $\varepsilon>0$ and $y>x$.

To prove this let X, Y and $(X+Y)$ be random variables with distribution functions $F(x)$, $F_1(x)$ and $F_2(x)$ respectively. Then

$$P\{x-\varepsilon<X+Y\leqslant y+\varepsilon\}\geqslant P\{x<X\leqslant y\}\cdot P\{|Y|<\varepsilon\}$$

$$F(y)-F(x)\leqslant\frac{F_2(y+\varepsilon)-F_2(x-\varepsilon)}{2F_1(\varepsilon)-1}.$$

The second inequality is deduced from

$$\begin{aligned}1-P\{x+\varepsilon<X+Y\leqslant y-\varepsilon\}&=P\{X+Y\leqslant x+\varepsilon\}+\\&+P\{X+Y>y-\varepsilon\}\geqslant P\{X\leqslant x\}\cdot P\{Y<\varepsilon\}+\\&+P\{X>y\}\cdot P\{Y>-\varepsilon\}=F(x)\cdot F_1(\varepsilon)+[1-F(y)][1-F_1(-\varepsilon)]\\1-F_2(y-\varepsilon)+F_2(x+\varepsilon)&\geqslant F_1(\varepsilon)[F(x)+1-F(y)]\end{aligned}$$

from which the inequality follows.

In this connection we also prove a similar inequality.

1.2. *If $F_1(x)$ and $F_2(x)$ are distribution functions with characteristic functions $\varphi_1(t)$ and $\varphi_2(t)$ respectively, and*

$$|\varphi_1(t)-\varphi_2(t)|\leqslant\lambda\cdot|t|$$

then
$$F_2(x-a)-\frac{2\lambda}{a}\leqslant F_1(x)\leqslant F_2(x+a)+\frac{2\lambda}{a}.$$

To prove this let

$$g(x)=\begin{cases}1-|x| & \text{for } |x|<1,\\ 0 & \text{for } |x|\geqslant 1,\end{cases}$$

in which case

$$\frac{1}{\alpha} g\left(\frac{x}{\alpha}\right)$$

is a frequency function with the characteristic function

$$\frac{1}{\alpha}\int_{-\infty}^{+\infty} e^{itx} g\left(\frac{x}{\alpha}\right) dx = 2\,\frac{1-\cos\alpha t}{\alpha^2 t^2}.$$

The function

$$F_1(y+x) - F_2(y+x),$$

considered as a function of x for fixed y, has the "characteristic" function

$$\int_{-\infty}^{+\infty} e^{itx}\{F_1(x+y) - F_2(x+y)\}\,dx =$$

$$= -\frac{1}{it}\int_{-\infty}^{+\infty} e^{itx}\{d\,F_1(x+y) - d\,F_2(x+y)\} = e^{-ity}\,\frac{\varphi_2(t)-\varphi_1(t)}{it}.$$

According to Parseval's theorem

$$\int_{-\infty}^{+\infty}\frac{1}{\alpha} g\left(\frac{x}{\alpha}\right)\{F_1(x+y) - F_2(x+y)\}\,dx = \frac{2}{2\pi i}\int_{-\infty}^{+\infty} e^{-ity}\,\frac{1-\cos\alpha t}{\alpha^2 t^2}\cdot\frac{\varphi_2(t)-\varphi_1(t)}{t}\,dt.$$

As $|\varphi_2 - \varphi_1| \leqslant \lambda\cdot|t|$ the right hand side has an absolute value less than

$$\frac{2\lambda}{2\pi}\int_{-\infty}^{+\infty}\frac{1-\cos\alpha t}{\alpha^2 t^2}\,dt = \frac{\lambda}{\alpha}.$$

Since F_1 and F_2 are distribution functions we get on the left hand side

$$I = \int_{-\infty}^{+\infty}\frac{1}{\alpha} g\left(\frac{x}{\alpha}\right)\{F_1(x+y) - F_2(x+y)\}\,dx$$

$$= \int_{-\alpha}^{\alpha}\frac{\alpha-|x|}{\alpha^2}\{F_1(x+y) - F_2(x+y)\}\,dx$$

$$\leqslant \int_{-\alpha}^{\alpha}\frac{\alpha-|x|}{\alpha^2}\{F_1(y+\alpha) - F_2(y-\alpha)\}\,dx = F_1(y+\alpha) - F_2(y-\alpha).$$

Hence
$$F_1(y+\alpha) - F_2(y-\alpha) \geqslant -\frac{\lambda}{\alpha}.$$

Putting
$$y+\alpha = x$$
$$a = 2\alpha,$$

we get
$$F_1(x) \geqslant F_2(x-a) - \frac{2\lambda}{a}.$$

This is the left hand side of the inequality. Analogously we get

$$I \geqslant \int_{-\alpha}^{+\alpha} \frac{\alpha - |x|}{\alpha^2} \{F_1(y-\alpha) - F_2(y+\alpha)\}\, dx = F_1(y-\alpha) - F_2(y+\alpha)$$

and the proof of the right hand side of the inequality is immediate. This proves 1.2

We now come upon the question how to make the best choice of $\varphi_1(t)$ for this truncation purpose from the class of characteristic functions having the property of being equal to zero outside the finite interval $(-T, T)$. Broadly speaking the answer is: That function φ_1 which deforms the given distribution as little as possible. This in turn implies that the distribution corresponding to φ_1 shall be as concentrated as possible near the value zero. Taking the second moment as a measure of concentration we then have the following problem:

1.3. *To find a characteristic function $\varphi(t)$, if it exists, which is equal to zero for $|t| \geqslant 1$ and for which*

$$\varlimsup_{h\to 0} \frac{2 - \varphi(h) - \varphi(-h)}{h^2}$$

is as small as possible.

Note that if $\varphi(t)$ is a characteristic function and $\varphi(t) = 0$ for $|t| \geqslant T$ then $\varphi(tT)$ is also a characteristic function and $\varphi(tT) = 0$ for $|t| \geqslant 1$.

Note also that if $\varphi''(0)$ exists, then

$$\lim_{h\to 0} \frac{2 - \varphi(h) - \varphi(-h)}{h^2} = -\varphi''(0).$$

As a preparation to the solution of problem 1.3 we start by solving the following problem:

1.4. *To find a frequency function $P_n(x)$ which fulfills the following conditions:*

a) $P_n(x) = 0$ *for* $|x| > \pi$,

b) $P_n(x)$ *is a non-negative trigonometric polynomial of degree n for* $|x| \leqslant \pi$,

c) $\alpha_2' = \int_{-\pi}^{\pi} (2-2\cos x)\, P_n(x)\, dx$ *is as small as possible,*

assuming that such a function exists.

To solve this problem we use the fact that $P_n(x)$ being non-negative, there are constants $u_0, u_1, \ldots, u_n$ such that

$$P_n(x) = \frac{1}{2\pi} \left| u_0 + u_1 e^{ix} + \cdots + u_n e^{inx} \right|^2 = \frac{1}{2\pi} \sum_{\nu=0}^{n} \sum_{\mu=0}^{n} u_\nu \bar{u}_\mu e^{ix(\nu-\mu)}.$$

$P_n(x)$ being a frequency function, we get

$$\int_{-\pi}^{\pi} P_n(x)\, dx = 1 = \sum_{\nu=0}^{n} |u_\nu|^2.$$

We are thus interested in the minimum of

$$\alpha_2' = \int_{-\pi}^{\pi} (2-2\cos x)\, P_n(x)\, dx,$$

under the side condition

$$\sum |u_\nu|^2 = 1.$$

The choice of α_2' as a measure of concentration instead of the usual measure

$$\alpha_2 = \int_{-\pi}^{\pi} x^2 P_n(x)\, dx$$

is merely a question of simplicity. It is easily seen that

$$\alpha_2' \leqslant \alpha_2 \leqslant \frac{\pi^2}{4} \alpha_2'.$$

We note that for $f(x)$ real and integrable in the interval $(-\pi, \pi)$

$$T_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left| u_0 + u_1 e^{ix} + \cdots + u_n e^{inx} \right|^2 dx$$

defines for $n = 0, 1, \ldots$ the Toeplitz forms associated with $f(x)$. To find the smallest value of $T_n(f)$ under the side condition

$$\sum |u_\nu|^2 = 1$$

means finding the smallest eigen-value of $T_n(f)$.

As
$$\alpha_2' = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2-2\cos x) \sum_\nu \sum_\mu u_\nu \bar{u}_\mu e^{ix(\nu-\mu)} dx =$$

$$= 2\sum_\nu |u_\nu|^2 - \sum_{\nu=0}^{n-1} \bar{u}_\nu u_{\nu+1} + u_\nu \bar{u}_{\nu+1} = 2 - \sum_{\nu=0}^{n-1} \bar{u}_\nu u_{\nu+1} + u_\nu \bar{u}_{\nu+1},$$

we have to find the largest eigen-value of the Hermitian form

$$2 - \alpha_2' = \sum_{\nu=0}^{n-1} \bar{u}_\nu u_{\nu+1} + u_\nu \bar{u}_{\nu+1}.$$

The characteristic equation of this form is

$$\Delta_n(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 1 & -\lambda & 1 & \cdots & 0 \\ 0 & 1 & -\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix} = 0.$$

Expanding this determinant we obtain the recurrent relation

$$\Delta_n(\lambda) = -\lambda \Delta_{n-1}(\lambda) - \Delta_{n-2}(\lambda).$$

As
$$0 \leqslant 2 - 2\cos x \leqslant 4,$$

all eigen-values of the form α_2' lie in the interval $(0, 4)$, and so the eigen-values of $(2-\alpha_2')$ lie in the interval $(-2, 2)$. Then put

$$\lambda = -2\cos v, \text{ with } 0 \leqslant v \leqslant \pi.$$

Then
$$\Delta_n = 2\cos v \cdot \Delta_{n-1} - \Delta_{n-2}.$$

To solve this difference equation consider the equation

$$x^2 = 2x\cos v - 1$$

$$x = \cos v \pm \sqrt{\cos^2 v - 1},$$

$$x = e^{\pm iv}.$$

From this follows that the general solution to the difference equation is

$$\Delta_n = A\, e^{inv} + B\, e^{-inv},$$

where A and B are constants.

For $\qquad n=0, \qquad \Delta_0 = -\lambda = 2\cos v,$

For $\qquad n=1, \qquad \Delta_1 = \lambda^2 - 1 = 4\cos^2 v - 1,$

$$A + B = 2\cos v,$$

$$(A+B)\cos v + i\,(A-B)\sin v = 4\cos^2 v - 1,$$

$$i\,(A-B)\sin v = \cos 2v,$$

$$\Delta_n = (A+B)\cos nv + i\,(A-B)\sin nv$$

$$= 2\cos v \cos nv + \frac{\cos 2v}{\sin v}\sin nv = \frac{\sin (n+2)\,v}{\sin v}.$$

The $(n+1)$ zeros of Δ_n are then

$$v = \frac{k}{n+2}\pi, \qquad k = 1, 2, \ldots, (n+1).$$

Since $\lambda = -2\cos v$, the largest eigen-value of the form $(2-\alpha_2')$ is

$$-2\cos\frac{n+1}{n+2}\pi = 2\cos\frac{\pi}{n+2},$$

and thus the smallest eigen-value of α_2' is

$$2 - 2\cos\frac{\pi}{n+2}.$$

As we are interested not only in the minimum value of α_2' but also in the corresponding trigonometric polynomial $P_n(x)$ we have now to determine the eigenvector corresponding to this minimum eigenvalue. Put

$$g(x) = u_0 + u_1 e^{ix} + \cdots + u_n e^{inx},$$

so that

$$P_n(x) = \frac{|g|^2}{2\pi}.$$

The determinant of the following system of equations is $\Delta_n(\lambda)$

$$
\begin{aligned}
-\lambda u_0 + u_1 \qquad\qquad\qquad\qquad\qquad\qquad &= 0\\
u_0 - \lambda u_1 + u_2 \qquad\qquad\qquad\qquad &= 0\\
u_1 - \lambda u_2 + u_3 \qquad\qquad\quad &= 0\\
\cdots\cdots\cdots\cdots\cdots &\\
u_{n-1} - \lambda u_n &= 0
\end{aligned}
$$

Putting $\lambda = 2\cos\frac{\pi}{n+2}$, we have $\Delta_n(\lambda)=0$ and the system will have solutions differing from the trivial one. Adding the side condition

$$\sum |u_\nu|^2 = 1$$

the eigen-vector in question will be uniquely determined.

Multiply the second equation by e^{ix}, the third by e^{2ix}, and so on. Finally, adding the equations we obtain

$$-\lambda g + e^{-ix}(g-u_0) + e^{ix}(g - u_n e^{inx}) = 0,$$

$$(2\cos x - \lambda)\cdot g = u_0 e^{-ix} + u_n e^{i(n+1)x}.$$

Now u_0 is proportionate to the cofactor of the first element of the first row of $\Delta_n(\lambda)$, i.e. proportionate to $\Delta_{n-1}(\lambda)$. But

$$\Delta_{n-1}(-2\cos v) = \frac{\sin (n+1)v}{\sin v}$$

and putting $v = \frac{n+1}{n+2}\pi$ this is equal to

$$\frac{\sin \frac{(n+1)^2}{n+2}\pi}{\sin\frac{n+1}{n+2}\pi} = \frac{\sin\left(n\pi + \frac{\pi}{n+2}\right)}{\sin\frac{\pi}{n+2}} = (-1)^n.$$

Furthermore u_n is proportionate to the cofactor of the last element of the first row of $\Delta_n(\lambda)$, i.e. proportionate to $(-1)^n$.

Since u_0 and u_n are both proportionate to $(-1)^n$ we have

$$g(x) = K\frac{e^{ix(n+1)} + e^{-ix}}{\cos x - \cos\frac{\pi}{n+2}},$$

where K is a constant to be determined later. Since

$$g(x) = u_0 + u_1 e^{ix} + \cdots + u_n e^{inx},$$

we have the equation

$$2K\frac{e^{ix(n+1)}+e^{-ix}}{e^{ix}+e^{-ix}-2\cos\frac{\pi}{n+2}}=\sum_0^n u_\nu e^{i\nu x},$$

from which we get

$$u_0=u_n=2K,$$

$$u_{\nu-1}+u_{\nu+1}-2u_\nu\cos\frac{\pi}{n+2}=0 \qquad \text{for } 0<\nu<n.$$

The equation

$$1+\varrho^2-2\varrho\cos\frac{\pi}{n+2}=0$$

has the roots

$$\varrho=e^{\pm\frac{i\pi}{n+2}},$$

and so

$$u_\nu=A\,e^{\frac{i\pi\nu}{n+2}}+B\,e^{-\frac{i\pi\nu}{n+2}}$$

or

$$u_\nu=(A+B)\cos\frac{\pi\nu}{n+2}+i(A-B)\sin\frac{\pi\nu}{n+2}.$$

$\nu=0$ gives $\quad A+B=2K,$

$\nu=n$ gives $\quad (A+B)\cos\frac{\pi n}{n+2}+i(A-B)\sin\frac{\pi n}{n+2}=2K,$

$$i(A-B)\sin\frac{\pi n}{n+2}=2K\left[1+\cos\frac{2\pi}{n+2}\right]=4K\cos^2\frac{\pi}{n+2},$$

$$i(A-B)=2K\cot\frac{\pi}{n+2},$$

$$u_\nu=2K\cos\frac{\pi\nu}{n+2}+2K\cot\frac{\pi}{n+2}\cdot\sin\frac{\pi\nu}{n+2},$$

$$u_\nu=\frac{2K}{\sin\frac{\pi}{n+2}}\sin\frac{\nu+1}{n+2}\pi.$$

Now $\quad g(x)=K'\sum e^{i\nu x}\sin\frac{\nu+1}{n+2}\pi$

$$|g|^2=|K'|^2\sum_{\nu=0}^{n}\sum_{\mu=0}^{n}e^{ix(\nu-\mu)}\sin\frac{\nu+1}{n+2}\pi\cdot\sin\frac{\mu+1}{n+2}\pi$$

For $k\geqslant 0$ we have

$$\sum_{\nu-\mu=k}\sin\frac{\nu+1}{n+2}\pi\cdot\sin\frac{\mu+1}{n+2}\pi=\tfrac{1}{2}\sum\cos\frac{\nu-\mu}{n+2}\pi-\cos\frac{\nu+\mu+2}{n+2}\pi$$

$$=\frac{n-k+1}{2}\cos\frac{k\pi}{n+2}-\tfrac{1}{2}\left\{\cos\frac{k+2}{n+2}\pi+\cos\frac{k+4}{n+2}\pi+\cdots+\cos\frac{2n-k+1}{n+2}\pi\right\}.$$

Now we make use of the following identity:

$$-\sum_{n=0}^{N} \cos(\pi + Nx - 2nx) = \tfrac{1}{2} \sum_{n=0}^{N} e^{ix(N-2n)} + \tfrac{1}{2} \sum_{n=0}^{N} e^{-ix(N-2n)}$$

$$= \sum_{n=0}^{N} e^{ix(N-2n)} = e^{ixN} \frac{e^{-2ix(N+1)}-1}{e^{-2ix}-1} = \frac{\sin(N+1)x}{\sin x},$$

which we apply to the sum

$$\cos\frac{k+2}{n+2}\pi + \cos\frac{k+4}{n+2}\pi + \cdots + \cos\frac{2n-k+2}{n+2}\pi$$

with $x=\dfrac{\pi}{n+2}$ and $N=n-k$. This sum is therefore equal to

$$-\frac{\sin\dfrac{n-k+1}{n+2}\pi}{\sin\dfrac{\pi}{n+2}} = -\frac{\sin\dfrac{k+1}{n+2}\pi}{\sin\dfrac{\pi}{n+2}}.$$

Hence for $k \geqslant 0$

$$\sum_{\nu-\mu=k} \sin\frac{\nu+1}{n+2}\pi \cdot \sin\frac{\mu+1}{n+2}\pi = \frac{n-k+1}{2}\cos\frac{k\pi}{n+2} + \frac{\sin\dfrac{k+1}{n+2}\pi}{2\sin\dfrac{\pi}{n+2}}$$

$$= \frac{n-k+2}{2}\cos\frac{k\pi}{n+2} + \tfrac{1}{2}\cot\frac{\pi}{n+2}\cdot\sin\frac{k\pi}{n+2}.$$

As $|g|^2=|\bar g|^2$ the coefficient of e^{ikx} is the same as the coefficient of e^{-ikx}.

$$|g(x)|^2 = \sum_{-n}^{n} C_k e^{ikx},$$

where
$$C_k = |K'|^2\left\{\frac{n-|k|+2}{2}\cos\frac{k\pi}{n+2} + \tfrac{1}{2}\cot\frac{\pi}{n+2}\cdot\sin\frac{|k\pi|}{n+2}\right\}.$$

The condition $C_0=1$ gives

$$1 = |K'|^2\cdot\frac{n+2}{2},$$

and the final result is

$$\begin{cases} P_n(x) = \dfrac{1}{2\pi}\displaystyle\sum_{-n}^{n} C_k e^{ikx} \\[2ex] C_k = \dfrac{n-|k|+2}{n+2}\cos\dfrac{k\pi}{n+2} + \dfrac{\cot\dfrac{\pi}{n+2}}{n+2}\sin\dfrac{|k\pi|}{n+2}. \end{cases}$$

Thus the problem 1.4 is solved. The polynomial has the property

$$\int_{-\pi}^{\pi} (2-2\cos x)\, P_n(x)\, dx = 2C_0 - C_1 - C_{-1} = 2 - C_1 - C_{-1} = 2 - 2\cos\frac{\pi}{n+2}$$

and our solution of the problem gives us the following theorem:

1.5. *If*

$$\sum_{-n}^{n} a_\nu e^{i\nu x}$$

is a non-negative trigonometric polynomial with $a_0 = 1$ *then*

$$2 - a_1 - a_{-1} \geq 2 - 2\cos\frac{\pi}{n+2}.$$

We will now allow n to tend to ∞ in the formula for C_k. Put $k = t\cdot n$;

$$C_{tn} = \frac{n+2-|tn|}{n+2}\cos\frac{tn\pi}{n+2} + \frac{\cot\frac{\pi}{n+2}}{n+2}\sin\frac{|tn\pi|}{n+2}.$$

When $n\to\infty$ a function of t is obtained which we will denote by $C(t)$:

$$C(t) = \begin{cases} (1-|t|)\cos\pi t + \frac{1}{\pi}\sin|\pi t| & \text{for } |t|<1 \\ 0 & \text{for } |t|\geq 1. \end{cases}$$

We deduce some properties of this function

1.6. $C(t)$ *is a characteristic function and it has continuous first and second derivatives.*

To prove the first part of the proposition put

$$f(x) = \frac{1}{2\pi}\int_{-1}^{1} e^{-ixt}\, C(t)\, dt$$

and we have to prove that $f(x) \geq 0$.

Integrating by part gives

$$f(x) = \frac{1}{2\pi}\int_{-1}^{1} e^{-ixt}\, C(t)\, dt = \frac{1}{\pi}\int_{0}^{1}\cos xt\cdot C(t)\, dt =$$

$$= \left[\frac{\sin xt}{\pi x}\, C(t)\right]_0^1 - \frac{1}{\pi x}\int_0^1 \sin xt\cdot C'(t)\, dt =$$

$$=\frac{1}{x}\int_0^1 \sin xt\cdot(1-t)\cdot\sin\pi t\,dt=$$

$$=\frac{1}{2x}\int_0^1 (1-t)\{\cos(x-\pi)t-\cos(x+\pi)t\}\,dt=$$

$$=\left[\frac{1-t}{2x}\left\{\frac{\sin(x-\pi)t}{x-\pi}-\frac{\sin(x+\pi)t}{x+\pi}\right\}\right]_0^1+\frac{1}{2x}\int_0^1\left\{\frac{\sin(x-\pi)t}{x-\pi}-\frac{\sin(x+\pi)t}{x+\pi}\right\}dt=$$

$$=\frac{1-\cos(x-\pi)}{2x(x-\pi)^2}-\frac{1-\cos(x+\pi)}{2x(x+\pi)^2}=\frac{2\pi}{(x^2-\pi^2)^2}(1+\cos x).$$

which is $\geqslant 0$. This proves the first part of the proposition. As for the second part we have for $0<t<1$

$$C(t)=(1-t)\cos\pi t+\frac{1}{\pi}\sin\pi t,$$

$$C'(t)=-\pi(1-t)\sin\pi t,$$

$$C''(t)=-\pi^2(1-t)\cos\pi t+\pi\sin\pi t,$$

$$C(1)=0 \qquad C(0)=1$$

$$C'(1-0)=0 \qquad C'(0+0)=0$$

$$C''(1-0)=0 \qquad C''(0+0)=-\pi^2.$$

As $C(t)=C(-t)$ implies

$$-C'(-t)=C'(t)$$

$$C''(-t)=C''(t),$$

we have

$$C'(0-0)=0$$

$$C''(0-0)=-\pi^2.$$

This proves the second part of the proposition. Next follows a proposition which will be used in the solution of problem 1.3.

1.7. *If $\varphi(t)$ is a characteristic function and*

$$S=\sum_{n=-\infty}^{+\infty}\varphi(\lambda n)$$

is convergent, then $S\geqslant 0$.

To prove this put $\theta = r\,e^{i\lambda x}$ with $r < 1$. Then

$$r^n \varphi(\lambda n) = r^n \int_{-\infty}^{+\infty} e^{i\lambda n x}\, dF(x) = \int_{-\infty}^{+\infty} \theta^n\, dF(x),$$

$$\sum_1^{\infty} r^n \varphi(\lambda n) = \int_{-\infty}^{+\infty} \sum_1^{\infty} \theta^n\, dF(x) = \int_{-\infty}^{+\infty} \frac{\theta}{1-\theta}\, dF(x),$$

$$\sum_1^{\infty} r^n \varphi(-\lambda n) = \int_{-\infty}^{+\infty} \frac{\bar\theta}{1-\bar\theta}\, dF(x),$$

$$\sum_{-\infty}^{+\infty} r^{|n|} \varphi(\lambda n) = \int_{-\infty}^{+\infty} \left\{1 + \frac{\theta}{1-\theta} + \frac{\bar\theta}{1-\bar\theta}\right\} dF(x) = \int_{-\infty}^{+\infty} \frac{1-|\theta|^2}{|1-\theta|^2}\, dF(x) \geqslant 0.$$

Since

$$\sum_{-\infty}^{+\infty} \varphi(\lambda n)$$

is convergent it now follows from Abel's theorem that

$$\lim_{r\to 1} \sum_{-\infty}^{+\infty} r^{|n|} \varphi(\lambda n) = \sum_{-\infty}^{+\infty} \varphi(\lambda n) \geqslant 0$$

which was to be proved.

1.5 and 1.7 will now be used to prove the following theorem:

1.8. *If $\varphi(t)$ is a characteristic function and $\varphi(t) = 0$ for $|t| \geqslant 1$ then*

$$\varliminf_{h\to 0} \frac{2-\varphi(h)-\varphi(-h)}{h^2} \geqslant \pi^2.$$

To prove this note that $\varphi(h t)\, e^{ixt}$ is a characteristic function and that

$$T = \sum_{\nu} \varphi(h\nu)\, e^{ix\nu}$$

is convergent since it has only a finite number of terms. According to 1.7 T is $\geqslant 0$. Putting $N = \left[\frac{1}{h}\right]$ it is evident that T is a trigonometric polynomial of degree N or $N-1$. As $\varphi(t)$ is a characteristic function $\varphi(0)$ is equal to 1. According to 1.5

$$2-\varphi(h)-\varphi(-h)\geqslant 2-2\cos\frac{\pi}{N+2},$$

$$\frac{2-\varphi(h)-\varphi(-h)}{h^2}\geqslant\frac{2-2\cos\dfrac{\pi}{N+2}}{h^2},$$

$$\varliminf_{h\to 0}\frac{2-\varphi(h)-\varphi(-h)}{h^2}\geqslant\pi^2$$

as was to be proved.

We are now in a position to state that $C(t)$ is a solution to problem 1.3. We have shown that $C''(0)=-\pi^2$, i.e.

$$\lim_{h\to 0}\frac{2-C(h)-C(-h)}{h^2}=\pi^2,$$

and 1.8 tells us that this is the smallest possible value.

We have not proved that $C(t)$ is the unique solution of problem 1.3 even if this seems probable.

The fact that $C(t)$ has continuous first and second derivatives turns out to be essential for the way in which we are going to use $C(t)$ in chapter 3. In that chapter we will deduce approximation formulas with error bounds whose validity will require that $\varphi(t)$ has continuous first and second derivatives. As has been pointed out earlier we suppose that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

is transformed into an integral from $-T$ to T by multiplying $\varphi(t)$ by a characteristic function $\varphi_1(t)$ which is equal to zero for $|t|\geqslant T$. In order that the error bounds of chapter 3 should be valid it will then be necessary that $\varphi_1(t)$ has continuous first and second derivatives. Note that the well-known characteristic function

$$g(t)=\begin{cases}1-\left|\dfrac{t}{T}\right| & \text{for } |t|<T\\ 0 & \text{for } |t|\geqslant T,\end{cases}$$

which is often used for truncation purposes, does not have a continuous first derivative.

In the following table we give some values of the distribution function corresponding to $C(t)$. For comparison some values of the normal distribution function have been added. They show that the agreement between the two distributions is fairly good.

Table of $\{F(x)-F(-x)\}$.

Column II: Distribution corresponding to $C(t/\pi)$, i.e. having mean 0 and standard deviation 1.

Column III: Normal distribution having mean 0 and standard deviation 1.

x	II	III
$\frac{1}{3}$	0.26558	
$\frac{2}{3}$	0.50491	
1	0.69840	0.68269
$1\frac{1}{3}$	0.83732	
$1\frac{2}{3}$	0.92426	
2	0.97009	0.95450
$2\frac{1}{3}$	0.98912	
$2\frac{2}{3}$	0.99440	
3	0.99494	0.99730
$3\frac{1}{3}$	0.99520	
$3\frac{2}{3}$	0.99624	
4	0.99756	0.99994
$4\frac{1}{3}$	0.99849	
$4\frac{2}{3}$	0.99887	
5	0.99892	1.00000

We conclude this chapter by demonstrating a method of constructing characteristic functions which are equal to zero outside a finite interval $(-T, T)$.

1.9. *Let* $f(x)$ *be a function, satisfying the following conditions:*

$$1)\ f(x)=0 \quad \text{for } |x|>\frac{T}{2},$$

$$2)\ f(x)\in L,$$

$$3)\ f(t)=\bar{f}(-t);$$

then

$$\varphi(t)=\frac{\int_{-\infty}^{+\infty} f(t-x)\,f(x)\,dx}{\int_{-\infty}^{+\infty} |f|^2\,dx}$$

is a characteristic function and $\varphi(t)=0$ *for* $|t|\geq T$.

To prove this put

$$g(x)=\int_{-\infty}^{+\infty} e^{-itx} f(t)\,dt.$$

Then $g(x)=\bar{g}(x)$ and hence $g(x)^2 \geqslant 0$.

$$g(x)^2 = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} e^{-ix(u+v)} f(u) f(v)\, dv =$$

$$= \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} e^{-ixt} f(u) f(t-u)\, dt = \int_{-\infty}^{+\infty} e^{-ixt} \left\{ \int_{-\infty}^{+\infty} f(u) f(t-u)\, du \right\} dt.$$

That $g^2 \in L$ follows from Parseval's theorem. Except for a suitable factor the function

$$\int_{-\infty}^{+\infty} f(u) f(t-u)\, du$$

is thus a characteristic function. The factor is to be chosen in such a way that the function becomes equal to 1 for $t=0$. Thus

$$\varphi(t) = \frac{\int_{-\infty}^{+\infty} f(u) f(t-u)\, du}{\int_{-\infty}^{+\infty} f(u) f(-u)\, du}$$

is a characteristic function and since $\varphi(t)$ is evidently equal to zero for $|t| \geqslant T$ the result follows.

As an example we chose

$$f(x) = \begin{cases} \cos \pi x & \text{for } |x| < \frac{1}{2} \\ 0 & \text{for } |x| \geqslant \frac{1}{2}. \end{cases}$$

Then for $0 \leqslant t \leqslant 1$

$$\int_{-\infty}^{+\infty} f(t-x) f(x)\, dx = \int_{t-\frac{1}{2}}^{\frac{1}{2}} \cos \pi (t-x) \cdot \cos \pi x\, dx =$$

$$= \tfrac{1}{2} \int_{t-\frac{1}{2}}^{\frac{1}{2}} \{\cos \pi t + \cos \pi (2x-t)\}\, dx = \tfrac{1}{2}(1-t) \cos \pi t + \frac{1}{2\pi} \sin \pi t.$$

That is, we obtain the characteristic function

$$C(t) = (1-|t|) \cos \pi t + \frac{1}{\pi} \sin |\pi t| \qquad \text{for } |t| \leqslant 1.$$

CHAPTER 2

Approximation First—Integration Afterwards

Suppose that $\varphi(t)$ is a characteristic function and that there is a finite interval $(-T, T)$, such shat $\varphi(t)=0$ for $|t|\geqslant T$. We want to determine the frequency function $f(x)$, corresponding to $\varphi(t)$, where $f(x)$ is given by

$$f(x)=\frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\varphi(t)\,dt.$$

In 1928 L. N. G. Filon suggested in a paper that such an integral should be dealt with in a manner different from the ordinary methods of numerical integration.

Divide the interval $(-T, T)$ into $4N$ equal parts and put $T/2N=\lambda$. Then put

$$t_\nu=-T+\nu\cdot\lambda \qquad 0\leqslant\nu\leqslant 4N.$$

In each sub-interval $(t_{2\nu}, t_{2\nu+2})$ we approximate $\varphi(t)$ by a parabola which for $t=t_{2\nu}$, $t_{2\nu+1}$ and $t_{2\nu+2}$ takes on the same values as $\varphi(t)$. The approximation arrived at in this manner will thus consist of $2N$ parabolic arcs. It is substituted in the integral for $\varphi(t)$ and an explicit expression for the value of the integral may then be set forth. This method will be discussed in the present chapter, the title of which indicates that $\varphi(t)$ is first approximated by a suitable function $\varphi_A(t)$. The integral

$$\frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\varphi_A(t)\,dt$$

is then evaluated and is taken as the approximation $f_A(x)$ of $f(x)$.

We will here consider two cases. First we are going to investigate the result of approximating $\varphi(t)$ in each sub-interval $(t_\nu, t_{\nu+1})$ by a straight line. After that we turn to Filon's method of approximating $\varphi(t)$ by parabolic arcs.

Our investigation of the first-mentioned method will be based on the assumption that φ' and φ'' are continuous for $|t|<T$. We define the auxiliary function $g(t)$:

$$g(t)=\begin{cases}1-|t| & \text{for } |t|<1\\ 0 & \text{for } |t|\geqslant 1.\end{cases}$$

Then $g(t)$ is a characteristic function, corresponding to the frequency function

$$\frac{1-\cos x}{\pi x^2}.$$

The approximation $\varphi_A(t)$ of $\varphi(t)$ by straight lines can be written

$$\varphi_A(t)=\sum_{\nu=1}^{4N-1}\varphi(t_\nu)\,g\left(\frac{t-t_\nu}{\lambda}\right),$$

for if $t = t_n$ then

$$\varphi_A(t_n) = \sum_{\nu=1}^{4N-1} \varphi(t_\nu)\, g\left(\frac{t_n - t_\nu}{\lambda}\right) = \varphi(t_n)$$

and in each interval $(t_\nu, t_{\nu+1})$ $\varphi_A(t)$ is a linear function.

The approximation $f_A(x)$ of $f(x)$ is thus equal to

$$f_A(x) = \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\varphi_A(t)\,dt = \sum_{\nu=1}^{4N-1}\varphi(t_\nu)\frac{1}{2\pi}\int_{-T}^{T} e^{-itx} g\left(\frac{t-t_\nu}{\lambda}\right)dt$$

$$= \sum_{\nu=1}^{4N-1}\varphi(t_\nu)\frac{1}{2\pi}\int_{t_\nu-\lambda}^{t_\nu+\lambda} e^{-itx} g\left(\frac{t-t_\nu}{\lambda}\right)dt =$$

$$= \sum_{\nu=1}^{4N-1}\varphi(t_\nu)\frac{\lambda}{2\pi}\int_{-1}^{1} e^{-it_\nu x}\cdot e^{-ix\lambda t} g(t)\,dt = \frac{1-\cos\lambda x}{\pi\lambda x^2}\sum_{\nu=1}^{4N-1}\varphi(t_\nu)\, e^{-it_\nu x},$$

and this may, of course, be written

$$\frac{1-\cos\lambda x}{\pi\lambda x^2}\sum \varphi(\lambda\nu)\, e^{-i\lambda x\nu}.$$

We have now to estimate the error $|f_A(x) - f(x)|$. We need the following inequality.

2.1. *For real x and t and $a \leqslant t \leqslant b$*

$$H(t) = \left| e^{itx} - \frac{t-a}{b-a} e^{ibx} - \frac{b-t}{b-a} e^{iax}\right| \leqslant \frac{x^2}{2}(t-a)(b-t).$$

Proof: For real y we have

$$e^{iy} = 1 + iy + \frac{y^2}{2}\vartheta,$$

where

$$|\vartheta| \leqslant 1.$$

$$H(t) = \left|1 - \frac{t-a}{b-a} e^{i(b-t)x} - \frac{b-t}{b-a} e^{i(a-t)x}\right| = \left|1 - \frac{t-a}{b-a}\left(1 + i(b-t)x + \frac{(b-t)^2}{2} x^2\vartheta_1\right) -\right.$$

$$\left. - \frac{b-t}{b-a}\left(1 + i(a-t)x + \frac{(a-t)^2}{2}x^2\vartheta_2\right)\right| =$$

$$= \left|\frac{t-a}{b-a}\cdot\frac{(b-t)^2}{2}x^2\vartheta_1 + \frac{b-t}{b-a}\frac{(a-t)^2}{2}x^2\vartheta_2\right|$$

$$\leqslant \frac{(t-a)(b-t)^2 + (b-t)(a-t)^2}{2(b-a)}x^2 = \frac{(t-a)(b-t)}{2}x^2$$

as was to be proved.

In each interval $(t_\nu, t_{\nu+1})$ $\{\varphi(t)-\varphi_A(t)\}$ is equal to

$$\varphi(t)-\varphi_A(t)=\int_{-\infty}^{+\infty}\left\{e^{itx}-\frac{t-t_\nu}{\lambda}e^{ixt_{\nu+1}}-\frac{t_{\nu+1}-t}{\lambda}e^{ixt_\nu}\right\}dF,$$

$$|\varphi(t)-\varphi_A(t)|\leqslant\int_{-\infty}^{+\infty}\frac{x^2}{2}(t-t_\nu)(t_{\nu+1}-t)\,dF(x)=-\frac{\varphi''(0)}{2}(t-t_\nu)(t_{\nu+1}-t).$$

Now we can use this inequality to estimate the difference $f(x)-f_A(x)$:

$$|f(x)-f_A(x)|=\left|\frac{1}{2\pi}\int_{-T}^{T}e^{itx}\{\varphi(t)-\varphi_A(t)\}\,dt\right|\leqslant$$

$$\leqslant\frac{1}{2\pi}\int_{-T}^{T}|\varphi(t)-\varphi_A(t)|\,dt\leqslant-\frac{\varphi''(0)}{2\pi}4N\int_0^{\lambda}\frac{t(\lambda-t)}{2}\,dt$$

$$=-\frac{\varphi''(0)}{2\pi}4N\frac{\lambda^3}{12}=-\lambda^2\frac{T\varphi''(0)}{12\pi}.$$

Summing up:

2.2. *Exact value:* $$f(x)=\frac{1}{2\pi}\int_{-T}^{T}e^{-itx}\varphi(t)\,dt.$$

Approximate value: $$f_A(x)=\frac{1-\cos\lambda x}{\pi\lambda x^2}\sum_\nu\varphi(\lambda\nu)\,e^{-i\lambda x\nu}.$$

Error bound: $$|f(x)-f_A(x)|\leqslant-\frac{T\lambda^2}{12\pi}\varphi''(0).$$

We now turn to Filon's approximation of $\varphi(t)$ by parabolic arcs. Here we assume that φ', φ'' and φ''' are continuous for $|t|<T$.

We define two auxiliary functions

$$g_1(t)=\begin{cases}1-t^2 & \text{for } |t|<1\\ 0 & \text{for } |t|\geqslant 1.\end{cases}$$

$$g_2(t)=\begin{cases}(1-|t|)\left(1-\left|\frac{t}{2}\right|\right) & \text{for } |t|<2\\ 0 & \text{for } |t|\geqslant 2.\end{cases}$$

Then

$$\frac{1}{2\pi}\int_{-1}^{1}e^{-itx}g_1(t)\,dt=\frac{1}{\pi}\int_0^1\cos tx\,(1-t^2)\,dt=\frac{2}{\pi x}\int_0^1\sin tx\cdot t\,dt=\frac{2}{\pi}\,\frac{\sin x-x\cdot\cos x}{x^3},$$

and
$$\frac{1}{2\pi}\int_{-2}^{2} e^{-itx} g_2(t)\,dt = \frac{1}{\pi}\int_0^2 \cos tx\cdot(1-t)\left(1-\frac{t}{2}\right)dt =$$
$$= \frac{1}{\pi x}\int_0^2 \sin tx\left(\frac{3}{2}-t\right)dt = \frac{3(1-\cos 2x)}{2\pi x^2} -$$
$$-\frac{\sin 2x - 2x\cos 2x}{\pi x^3} = \frac{3x + x\cdot\cos 2x - 2\sin 2x}{2\pi x^3}.$$

By the aid of g_1 and g_2 the approximation $\varphi_A(t)$ may be expressed as follows

$$\varphi_A(t) = \sum{}' \varphi(t_\nu)\, g_1\left(\frac{t-t_\nu}{\lambda}\right) + \sum{}'' \varphi(t_\nu)\, g_2\left(\frac{t-t_\nu}{\lambda}\right),$$

where $\sum'$ and $\sum''$ denote summation with respect to odd and even values of ν respectively. That this is the desired approximation can be proved in the following way.

For $t=t_{2k}$ we have $g_2\left(\frac{t-t_{2k}}{\lambda}\right)=1$ and all the other functions g_1 and g_2 equal to zero.

For $t=t_{2k+1}$ we have $g_1\left(\frac{t-t_{2k+1}}{\lambda}\right)=1$ and all the other functions g_1 and g_2 equal to zero.

In the interval (t_{2k}, t_{2k+2}) $\varphi_A(t)$ is composed of three parabolas, namely

$$\varphi(t_{2k+1})\cdot g_1\left(\frac{t-t_{2k+1}}{\lambda}\right), \text{ and}$$

the right hand half of

$$\varphi(t_{2k})\cdot g_2\left(\frac{t-t_{2k}}{\lambda}\right), \text{ and}$$

the left hand half of

$$\varphi(t_{2k+2})\cdot g_2\left(\frac{t-t_{2k+2}}{\lambda}\right).$$

Thus $\varphi_A(t)$ has the desired properties. The approximation $f_A(x)$ of $f(x)$ is equal to

$$f_A(x) = \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\varphi_A(t)\,dt =$$
$$= \sum{}' \varphi(t_\nu)\frac{1}{2\pi}\int_{-T}^{T} e^{-itx} g_1\left(\frac{t-t_\nu}{\lambda}\right)dt + \sum{}'' \varphi(t_\nu)\frac{1}{2\pi}\int_{-T}^{T} e^{-itx} g_2\left(\frac{t-t_\nu}{\lambda}\right)dt =$$
$$= \sum{}' \varphi(t_\nu)\, e^{-ixt_\nu}\frac{\lambda}{2\pi}\int_{-1}^{1} e^{-i\lambda xt} g_1(t)\,dt + \sum{}'' \varphi(t_\nu)\, e^{-ixt_\nu}\frac{\lambda}{2\pi}\int_{-2}^{2} e^{-i\lambda xt} g_2(t)\,dt =$$

$$= \frac{2 \sin \lambda x - 2 \lambda x \cos \lambda x}{\pi \lambda^2 x^3} \sum' \varphi(t_\nu) e^{-ixt_\nu} +$$

$$+ \frac{3 \lambda x + \lambda x \cos 2 \lambda x - 2 \sin 2 \lambda x}{2 \pi \lambda^2 x^3} \sum'' \varphi(t_\nu) e^{-ixt_\nu}.$$

To estimate the error we first note that if $P_2(t)$ is a second degree polynomial, $f(t)$ is real and $P_2(t) = f(t)$ for $t = a$, b and c then

$$f(t) = P_2(t) + \frac{(t-a)(t-b)(t-c)}{6} f'''(\theta),$$

where θ is a point in the smallest interval including a, b, c and t. Consider now the difference

$$\varphi(t) - \varphi_A(t).$$

In each sub-interval (t_{2k}, t_{2k+2}) the real and imaginary parts of this difference are both equal to a function minus its approximation by a second degree polynomial. Putting $\varphi(t) = u(t) + i v(t)$ we then have

$$\varphi(t) = \varphi_A(t) + \frac{(t - t_{2k})(t - t_{2k+1})(t - t_{2k+2})}{6} \{u'''(\theta_1) + i v'''(\theta_2)\}$$

in each sub-interval (t_{2k}, t_{2k+2}). Since

$$|u'''(\theta_1) + i v'''(\theta_2)| \leqslant 2 \operatorname*{Max}_t \{|\varphi'''(t)|\},$$

we get $$|\varphi(t) - \varphi_A(t)| \leqslant \frac{|(t - t_{2k})(t - t_{2k+1})(t - t_{2k+2})|}{3} \operatorname*{Max}_t \{|\varphi'''(t)|\}.$$

Now

$$|f(x) - f_A(x)| = \left| \frac{1}{2\pi} \int_{-T}^{T} e^{-itx} \{\varphi(t) - \varphi_A(t)\} dt \right| \leqslant$$

$$\leqslant \frac{1}{2\pi} \int_{-T}^{T} |\varphi(t) - \varphi_A(t)| dt \leqslant \frac{N}{3 \cdot \pi} \operatorname*{Max}_t \{|\varphi'''(t)|\} \cdot \int_0^{2\lambda} |t(t-\lambda)(t-2\lambda)| dt =$$

$$= \frac{N \lambda^4}{6\pi} \operatorname*{Max}_t \{|\varphi'''(t)|\} = \frac{T \lambda^3}{12\pi} \operatorname*{Max}_t \{|\varphi'''(t)|\}.$$

Chapter 3

The Trapezoidal Rule

Let $\varphi(t)$ be a characteristic function and suppose that the integral

$$I(\varphi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

exists. In this chapter we will discuss to what extent the trapezoidal rule is suitable for the calculation of an approximate value of this integral.

Let us first note that if $\varphi(t)$ is a characteristic function then the same is true of

$$\varphi_1(t)=e^{-itx}\varphi(t)$$

and

$$\varphi_2(t)=\frac{\sin ht}{ht}e^{-itx}\varphi(t).$$

The frequency function $f(x)$ corresponding to $\varphi(t)$ is given by the integral

$$f(x)=I(\varphi_1)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi_1(t)\,dt$$

if this integral exists. The increment of the distribution function is given by

$$\bar{F}(x+h)-\bar{F}(x-h)=2h\cdot I(\varphi_2)=\frac{2h}{2\pi}\int_{-\infty}^{+\infty}\varphi_2(t)\,dt.$$

It follows that these integrals are both of the type considered in this chapter. The so-called Poisson's formula will play an essential rôle in this chapter. According to this formula

$$\sum_{-\infty}^{+\infty}g(\nu)=\sum_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{2\pi i\nu t}g(t)\,dt,$$

where $g(t)$ is a real or complex-valued function. Before we state any conditions under which this formula holds we note that if

$$g(t)=\varphi(\lambda t),$$

where $\lambda>0$ and $\varphi(t)$ is a characteristic function corresponding to the frequency function $f(x)$ and if Poisson's formula holds

$$\frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty}\varphi(\lambda\nu)=\sum_{-\infty}^{+\infty}f\left(\frac{2\pi\nu}{\lambda}\right).$$

We will then say that Poisson's formula applies to (f, φ) with parameter value λ.

We start by stating two well-known theorems, providing sufficient conditions for the validity of Poisson's formula.

3.1. *If* 1) $|g(t)| \in L(-\infty, +\infty)$
2) $g(t)$ *is of bounded variation in* $(-\infty, +\infty)$
3) $2g(t) = g(t+0) + g(t-0)$ *for all* t *then Poisson's formula holds.*

3.2. *If* 1) $G(t) = \sum_{-\infty}^{+\infty} g(\nu + t)$ *is uniformly convergent for* $|t| \leqslant \frac{1}{2}$
2) $G(t)$ *is of bounded variation in* $(-\frac{1}{2}, \frac{1}{2})$
3) $2g(t) = g(t+0) + g(t-0)$ *for all* t *then Poisson's formula holds.*

Proof of (3.1) *and* (3.2): In both cases

$$G(t) = \sum_{-\infty}^{+\infty} g(\nu + t)$$

is integrable over $(-\frac{1}{2}, \frac{1}{2})$ and of bounded variation. It follows that

$$G(0) = \sum_{-\infty}^{+\infty} g(\nu)$$

is equal to the sum of the Fourier coefficients of $G(t)$. That is

$$G(0) = \sum_{-\infty}^{+\infty} \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i n t}\, G(t)\, dt.$$

But

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i n t}\, G(t)\, dt = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{\nu=-\infty}^{+\infty} e^{2\pi i n t}\, g(\nu + t)\, dt =$$

$$= \sum_{\nu=-\infty}^{+\infty} \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i n t}\, g(\nu + t)\, dt = \sum_{\nu=-\infty}^{+\infty} \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} e^{2\pi i n t}\, g(t)\, dt = \int_{-\infty}^{+\infty} e^{2\pi i n t}\, g(t)\, dt$$

the inversion of the order of integration and summation being justified in (3.2) by uniform convergence and in (3.1) by $|g(t)|$ belonging to $L(-\infty, +\infty)$.

Hence
$$\sum_{-\infty}^{+\infty} g(\nu) = \sum_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi i \nu t}\, g(t)\, dt$$
as was to be proved.

When we are dealing with a frequency function f and its characteristic function φ the theorems (3.1) and (3.2) may be applied to either f or φ in order to make certain that Poisson's formula holds for the pair of functions (f, φ). The following (new) theorem applies only to characteristic functions.

3.3. *If* 1) $g(t)$ *is a characteristic function*

2) $G(t)=\sum_{-\infty}^{+\infty} g(\nu+t)$ *is uniformly convergent for* $|t|\leqslant\frac{1}{2}$ *then Poisson's formula holds.*

Proof: Since $g(t)$ is a characteristic function it is continuous and since the series which defines $G(t)$ converges uniformly $G(t)$ is continuous. It follows that the Fourier series corresponding to $G(t)$ is summable $(C, 1)$ to the sum $G(t)$.

Hence
$$G(0)=\lim_{n\to\infty}\sum_{m=-n}^{n}\left(1-\frac{|m|}{n}\right)\int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i m t}\, G(t)\, dt.$$

But
$$G(0)=\sum_{-\infty}^{+\infty} g(\nu)$$

and
$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i m t}\, G(t)\, dt=\int_{-\infty}^{+\infty} e^{2\pi i m t}\, g(t)\, dt$$

as in the proof of the foregoing theorem. Hence

$$\sum_{-\infty}^{+\infty} g(\nu)=\lim_{n\to\infty}\sum_{m=-n}^{n}\left(1-\frac{|m|}{n}\right)\int_{-\infty}^{+\infty} e^{2\pi i m t}\, g(t)\, dt.$$

Since $g(t)$ is a characteristic function all integrals on the right hand side are non-negative. Hence summation by arithmetic means leaves the same result as ordinary summation. That is

$$\sum_{-\infty}^{+\infty} g(\nu)=\sum_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\pi i m t}\, g(t)\, dt$$

as was to be proved.

Our next task will be to deduce three corollaries to the theorems we have just proved.

3.1′. *If* $g(t)$ *fulfills the conditions* (3.1) *then*

$$\sum_{-\infty}^{+\infty} g(\lambda\nu)=\sum_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\pi i\nu t}\, g(\lambda t)\, dt$$

for all $\lambda>0$.

3.2′. *If* $\lambda>0$ *and* $g(\lambda t)$, *considered as a function of* t, *fulfills the conditions* (3.2) *then*

$$\sum_{\nu=-\infty}^{+\infty} g\left(\frac{\lambda\nu}{p}\right)=\sum_{\nu=-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\pi i\nu t}\, g\left(\frac{\lambda t}{p}\right) dt$$

for all integers $p\geqslant 1$.

3.3′. *If* $\lambda > 0$ *and* $g(\lambda t)$, *considered as a function of* t, *fulfills the conditions* (3.3) *then*

$$\sum_{\nu=-\infty}^{+\infty} g\left(\frac{\lambda \nu}{p}\right) = \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi i \nu t} g\left(\frac{\lambda t}{p}\right) dt$$

for all integers $p \geq 1$.

If $g(t)$ fulfills the conditions (3.1) the same is true of $g(\lambda t)$. This proves (3.1′).
If $g(\lambda t)$ fulfills the conditions (3.2) then

$$G(t) = \sum_{-\infty}^{+\infty} g(\lambda \nu + \lambda t)$$

is equal to the sum of its Fourier series.

$$G(t) = \sum_{-\infty}^{+\infty} e^{2\pi i n t} \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-2\pi i n u} G(u)\, du.$$

The sum of the $G(t)$ values for $t = 0, \frac{1}{p}, \ldots, \frac{p-1}{p}$ is equal to

$$\sum_{m=0}^{p-1} G\left(\frac{m}{p}\right) = \sum_{\nu=-\infty}^{+\infty} \sum_{m=0}^{p-1} g\left(\lambda \nu + \frac{\lambda m}{p}\right) = \sum_{\nu=-\infty}^{+\infty} g\left(\frac{\lambda \nu}{p}\right) =$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{p-1} e^{\frac{2\pi i m n}{p}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-2\pi i n u} G(u)\, du = \sum_{n=-\infty}^{+\infty} p \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-2\pi i n p u} G(u)\, du =$$

$$= \sum_{n=-\infty}^{+\infty} p \int_{-\infty}^{+\infty} e^{-2\pi i n p u} g(\lambda u)\, du = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi i n t} g\left(\frac{\lambda t}{p}\right) dt,$$

where we make use of the fact that

$$\sum_{m=0}^{p-1} e^{\frac{2\pi i m n}{p}} = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{p} \\ p & \text{if } n \equiv 0 \pmod{p} \end{cases}.$$

This proves (3.2′).

The proof of (3.3′) is almost the same as the proof of (3.2′), the only difference being that we have to use summation by arithmetic means instead of direct summation. Thus we get

$$\sum_{\nu=-\infty}^{+\infty} g\left(\frac{\lambda \nu}{p}\right) = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) p \int_{-\infty}^{+\infty} e^{-2\pi i k p u} g(\lambda u)\, du.$$

Since $g(\lambda u)$ is a characteristic function all terms on the right hand side are non-negative. Hence

$$\sum_{\nu=\infty}^{+\infty} g\left(\frac{\lambda\nu}{p}\right)=\sum_{n=-\infty}^{+\infty} p\int_{-\infty}^{+\infty} e^{-2\pi i n p u}\, g(\lambda u)\, du$$

and the proof follows as in (3.2′).

In (3.3′) we assumed that there was a $\lambda>0$ such that

$$\sum_{-\infty}^{+\infty} \varphi(\lambda\nu+t)$$

was uniformly convergent, and from this assumption we could conclude that Poisson's formula was valid for a certain set of λ-values. It is impossible to prove that the formula is valid for all λ-values if it is valid for one λ-value. We will show this with the aid an example.

According to Polya's well-known theorem the function

$$\frac{1}{1+|t|}$$

is a characteristic function. Thus the same is true of

$$\varphi(t)=\frac{\cos 2\pi t}{1+|t|}.$$

With $\lambda=1$

$$\sum_{-\infty}^{+\infty} \varphi(\nu)=+\infty.$$

This means that Poisson's formula does not hold for $\lambda=1$. It holds, however, for $\lambda=\frac{1}{2}$:

$$\sum_{-\infty}^{+\infty}\varphi\left(\frac{\nu}{2}+t\right)=\sum_{-\infty}^{+\infty}\frac{\cos(\nu\pi+2\pi t)}{1+\left|\frac{\nu}{2}+t\right|}=\cos 2\pi t\sum_{-\infty}^{+\infty}\frac{(-1)^{\nu}}{1+\left|\frac{\nu}{2}+t\right|}.$$

This series is easily seen to be uniformly convergent for $|t|\leqslant\frac{1}{4}$ by considering the sum of two consecutive terms i.e.

$$(-1)^{\nu}\cos 2\pi t\left\{\frac{1}{1+\frac{\nu}{2}+t}-\frac{1}{1+\frac{\nu+1}{2}+t}\right\}=(-1)^{\nu}\cos 2\pi t\frac{\frac{1}{2}}{\left(1+\frac{\nu}{2}+t\right)\left(1+\frac{\nu+1}{2}+t\right)}$$

if $\nu>0$, and a similar result holds if $\nu<0$. The result then follows from (3.3).

We are now going to prove some theorems with the aid of (3.1)–(3.3′).

3.4. *If $F(x)$ is a distribution function and $\varphi(t)$ its characteristic function, then*

$$\frac{\lambda}{\pi}\sum_{\nu=-\infty}^{+\infty}\frac{\sin h\lambda\nu}{\lambda\nu}e^{-i\lambda x\nu}\varphi(\lambda\nu)=\sum_{\nu=-\infty}^{+\infty}\bar{F}\left(x+h+\frac{2\pi\nu}{\lambda}\right)-\bar{F}\left(x-h+\frac{2\pi\nu}{\lambda}\right).$$

This is an immediate consequence of (3.1) and (3.1').

$$\frac{\bar{F}(x+h+y)-\bar{F}(x-h+y)}{2h},$$

considered as a function of y, is a frequency function of bounded variation in $(-\infty, +\infty)$. The characteristic function corresponding to this frequency function is

$$\frac{1}{2h}\int_{-\infty}^{+\infty} e^{ity}\{\bar{F}(x+h+y)-\bar{F}(x-h+y)\}\,dy=$$

$$=\frac{1}{2hti}\int_{-\infty}^{+\infty} e^{ity}\,d\bar{F}_y(x-h+y)-\frac{1}{2hti}\int_{-\infty}^{+\infty} e^{ity}\,d\bar{F}_y(x+h+y)=$$

$$=\frac{e^{ith}-e^{-ith}}{2hti}\,e^{-itx}\varphi(t)=\frac{\sin ht}{ht}\,e^{-itx}\varphi(t),$$

and the result follows.

3.5. *If Poisson's formula applies to (f, φ) with parameter value λ, then*

$$\frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty}\varphi(\lambda\nu)\geqslant f(0).$$

This is an immediate consequence of Poisson's formula. We can also express this result in a way that better illustrates its connection with our approximation problem

$$f(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt.$$

If we calculate an approximate value of this integral according to the trapezoidal rule using values of the integrand at equally spaced points, starting at $t=0$, we get

$$\frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty}\varphi(\lambda\nu),$$

and (3.5) says that this approximate value is larger than or equal to the exact value.

3.6. *If Poisson's formula applies to (f, φ) with parameter values λ and $p\lambda$ (p integer > 1), then*

$$\frac{p\lambda}{2\pi}\sum_{\nu}\varphi(\nu p\lambda)\geqslant\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\nu\lambda).$$

Proof:

$$\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\nu\lambda)=\sum_{\nu}f\left(\frac{2\pi\nu}{\lambda}\right),$$

$$\frac{p\lambda}{2\pi}\sum_{\nu}\varphi(\nu p\lambda)=\sum_{\nu}f\left(\frac{2\pi\nu}{p\lambda}\right).$$

From this we get

$$\frac{\lambda p}{2\pi}\sum_{\nu}\varphi(\nu\lambda p)-\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\nu\lambda)=\sum f\left(\frac{2\pi\nu}{\lambda p}\right)\geqslant 0,$$

where the last sum extends over ν-values $\nu\not\equiv 0 \pmod p$. This completes the proof.

3.7. *If $\varphi''(0)$ exists and Poisson's formula applies to $[f(x),\varphi(t)]$ and to $[x^2 f(x), -\varphi''(t)]$ with parameter value λ, then*

$$\frac{\lambda}{2\pi}\sum_{\nu=-\infty}^{+\infty}\varphi(\nu\lambda)+\frac{\lambda^2}{4\pi^2}\varphi''(\nu\lambda)\leqslant f(0).$$

Proof:

$$\varphi(t)=\int_{-\infty}^{+\infty}e^{itx}f(x)\,dx,$$

$$\varphi''(t)=-\int_{-\infty}^{+\infty}e^{itx}x^2 f(x)\,dx,$$

$$|\varphi''(t)|\leqslant|\varphi''(0)|.$$

This means that $\varphi''(t)/\varphi''(0)$ is a characteristic function, corresponding to the frequency function

$$-\frac{x^2 f(x)}{\varphi''(0)}.$$

We apply Poisson's formula:

$$\frac{\lambda}{2\pi\varphi''(0)}\sum_{\nu}\varphi''(\nu\lambda)=-\frac{1}{\varphi''(0)}\sum_{\nu}\left(\frac{2\pi\nu}{\lambda}\right)^2 f\left(\frac{2\pi\nu}{\lambda}\right),$$

$$\frac{\lambda^3}{8\pi^3}\sum_{\nu}\varphi''(\nu\lambda)=-\sum_{\nu}\nu^2 f\left(\frac{2\pi\nu}{\lambda}\right).$$

Since

$$\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\nu\lambda)=\sum_{\nu}f\left(\frac{2\pi\nu}{\lambda}\right)$$

we get

$$\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\nu\lambda)+\frac{\lambda^2}{4\pi^2}\varphi''(\nu\lambda)=f(0)-\sum_{\nu\neq 0}(\nu^2-1)f\left(\frac{2\pi\nu}{\lambda}\right).$$

Since the last sum is $\geqslant 0$ the result follows.

The inequalities (3.5) and (3.6) are based on the assumption that Poisson's formula holds. We are now going to deduce the same inequalities from other assumptions.

3.8. *If*

$$T_\lambda = \frac{\lambda}{2\pi} \sum_{-\infty}^{+\infty} \varphi(\nu\lambda)$$

and

$$I = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(t)\, dt$$

are both convergent, then $T_\lambda \geqslant I.$

To prove this we apply theorem (3.3). As $e^{-\varepsilon|t|}$ is a characteristic function, so is

$$\psi(t) = e^{-\varepsilon|t|}\varphi(t).$$

According to condition 2 of (3.3)

$$\sum_{-\infty}^{+\infty} \psi(\lambda\nu + t) = \sum_{-\infty}^{+\infty} \varphi(\lambda\nu + t)\, e^{-\varepsilon|\lambda\nu + t|}$$

should be uniformly convergent for $|t| \leqslant \lambda/2$.

Since this series is absolutely less then

$$\sum_\nu e^{-\varepsilon|\lambda\nu + t|} < e^{\varepsilon|t|} \sum_\nu e^{-\varepsilon\lambda|\nu|} < e^{\frac{1}{2}\varepsilon\lambda} \sum_\nu e^{-\varepsilon\lambda|\nu|} < \infty$$

for $|t| \leqslant \lambda/2$, uniform convergence holds and (3.3) may be applied. Thus we get

$$\frac{\lambda}{2\pi} \sum_\nu e^{-\varepsilon\lambda|\nu|} \varphi(\lambda\nu) \geqslant \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\varepsilon|t|} \varphi(t)\, dt.$$

Since $$\sum_\nu \varphi(\lambda\nu)$$

is convergent it follows from Abel's theorem that the left hand side tends to this limit as $\varepsilon \to 0$.

Since further

$$\int_{-\infty}^{+\infty} \varphi(t)\, dt$$

is convergent, it follows that

$$\int_{-\infty}^{+\infty} e^{-\varepsilon|t|} \varphi(t)\, dt = \int_0^\infty e^{-\varepsilon t} \{\varphi(t) + \varphi(-t)\}\, dt =$$

$$= \varepsilon \int_0^\infty dt \int_0^t e^{-\varepsilon t} \{\varphi(u) + \varphi(-u)\}\, du \to \int_{-\infty}^{+\infty} \varphi(t)\, dt$$

as $\varepsilon \to 0$. Thus
$$\frac{\lambda}{2\pi}\sum_{\nu}\varphi(\lambda\nu) \geqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

as was to be proved.

3.9. *If*
$$T_{\lambda} = \frac{\lambda}{2\pi}\sum_{\nu}\varphi(\lambda\nu)$$

and
$$T_{p\lambda} = \frac{p\lambda}{2\pi}\sum_{\nu}\varphi(p\lambda\nu)$$

are both convergent and p is a positive integer, then
$$T_{p\lambda} \geqslant T_{\lambda}.$$

As in the proof of (3.8) it follows from (3.3) that Poisson's formula applies to the characteristic function
$$\psi(t) = e^{-\varepsilon|t|}\varphi(t)$$

and the corresponding frequency function. Applying (3.6) we get
$$\frac{p\lambda}{2\pi}\sum_{\nu}\psi(p\lambda\nu) \geqslant \frac{\lambda}{2\pi}\sum_{\nu}\psi(\lambda\nu).$$

But this is equivalent to
$$\frac{p\lambda}{2\pi}\sum_{\nu}\varphi(p\lambda\nu)\,e^{-\varepsilon p\lambda|\nu|} \geqslant \frac{\lambda}{2\pi}\sum_{\nu}\varphi(\lambda\nu)\,e^{-\varepsilon\lambda|\nu|}.$$

When $\varepsilon \to 0$ it follows from Abel's theorem that
$$\frac{p\lambda}{2\pi}\sum_{\nu}\varphi(p\lambda\nu) \geqslant \frac{\lambda}{2\pi}\sum_{\nu}\varphi(\lambda\nu),$$

as was to be proved.

The theorems (1.7), (3.3) and (3.8) may be epitomized as follows. If the series
$$g(t) = \frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty}\varphi(\lambda\nu + t)$$

is convergent for $t = 0$ then its sum $g(0) \geqslant 0$. If the frequency function $f(x)$ exists for $x = 0$ in the sense that
$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt,$$

then $g(0) \geqslant f(0)$. If finally $g(t)$ converges uniformly for $|t| \leqslant \lambda/2$ then we can also estimate how much larger $g(0)$ is than $f(0)$.

As a first application of the theorems concerning the Poisson formula we will consider the following problem.

Suppose we wish to calculate an approximate value of the integral

$$I=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

using Simpson's rule. Suppose furthermore that we use the abscissas $\nu\lambda$, with spacing λ so that there will be two different ways of applying Simpson's rule:

$$S_1=\frac{\lambda}{3\pi}\sum_\nu\varphi(2\nu\lambda+\lambda)+\frac{2\lambda}{3\pi}\sum_\nu\varphi(2\nu\lambda),$$

$$S_2=\frac{\lambda}{3\pi}\sum_\nu\varphi(2\nu\lambda)+\frac{2\lambda}{3\pi}\sum_\nu\varphi(2\nu\lambda+\lambda),$$

According to the trapezoidal rule we get

$$T_\lambda=\frac{\lambda}{2\pi}\sum_\nu\varphi(\nu\lambda)$$

$$T_{2\lambda}=\frac{\lambda}{\pi}\sum_\nu\varphi(2\nu\lambda)$$

with spacings λ and 2λ respectively.

We assume that these sums are both convergent. Then

$$S_1=\tfrac{2}{3}T_\lambda+\tfrac{1}{3}T_{2\lambda}=T_\lambda+\tfrac{1}{3}(T_{2\lambda}-T_\lambda),$$

$$S_2=\tfrac{4}{3}T_\lambda-\tfrac{1}{3}T_{2\lambda}=T_\lambda-\tfrac{1}{3}(T_{2\lambda}-T_\lambda).$$

According to (3.8) and (3.9)

$$T_{2\lambda}\geqslant T_\lambda\geqslant I.$$

Thus $$S_1\geqslant T_\lambda\geqslant I$$

and $$S_2\leqslant T_\lambda.$$

Hence it is evident that T_λ is a better approximation than S_1. If S_2 is a better or worse approximation than T_λ cannot be ascertained is this way.

We now turn to the main problem of this chapter, which is the numerical integration of

$$I=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(t)\,dt$$

by use of the trapezoidal rule.

We have already pointed out that the numerical integration of this integral involves fixing, sooner on later during the calculations, a finite interval $(-T, T)$, such that the approximate value of I will be based on values $\varphi(t_\nu)$ with $|t_\nu| < T$. The tails of the integral from $-\infty$ to $-T$ and from T to $+\infty$ are disregarded and an estimation of the tails must thus be undertaken in order to control the error introduced in this way. In chapter 1 we decided to truncate the integral by multiplying the integrand by a characteristic function which is equal to zero outside the finite interval $(-T, T)$.

Throughout the rest of this chapter we will assume that this method has been applied upon the integrand. We will further assume that the given characteristic function as well as the characteristic function by which it has been multiplied have continuous first and second derivatives. We may thus reformulate our problem in the following way.

The integral

$$I = \frac{1}{2\pi} \int_{-T}^{T} \varphi(t)\, dt$$

is to be integrated numerically, where

$V1)$ $\varphi(t)$ is a characteristic function,

$V2)$ $\varphi(t) = 0$ for $|t| \geqslant T$,

$V3)$ $\varphi'(t)$ and $\varphi''(t)$ exist and are continuous.

Inspection of theorems (3.1)–(3.9) easily shows that these theorems all hold for a characteristic function fulfilling the conditions $V2$ and $V3$. The parameter λ may then take any value > 0 and the inequality (3.6) holds for every integer $p \geqslant 1$.

According to the inequalities (3.5) and (3.7) we get

$$\frac{\lambda}{2\pi} \sum_\nu \varphi(\lambda\nu) + \frac{\lambda^2}{4\pi^2} \varphi''(\lambda\nu) \leqslant I \leqslant \frac{\lambda}{2\pi} \sum_\nu \varphi(\lambda\nu).$$

If we take the arithmetic mean of the upper and lower bounds we get the following approximation formula.

3.10. *Exact value:* $$I = \frac{1}{2\pi} \int_{-T}^{T} \varphi(t)\, dt.$$

Approximate value: $I_A = T_\lambda(\varphi) + \frac{\lambda^2}{8\pi^2} T_\lambda(\varphi'')$.

Error bound: $|I - I_A| \leqslant -\frac{\lambda^2}{8\pi^2} T_\lambda(\varphi'')$.

$$T_\lambda(\varphi) = \frac{\lambda}{2\pi} \sum_\nu \varphi(\lambda\nu).$$

$$T_\lambda(\varphi'') = \frac{\lambda}{2\pi} \sum_\nu \varphi''(\lambda\nu).$$

In practical application this formula has the disadvantage that $T_\lambda(\varphi'')$ will as a rule be fairly complicated to calculate. Suppose for example that the characteristic function φ fulfills condition $V3$ but not $V2$. By multiplying by the function $C(t/T)$ of chapter 1 the product will again be a characteristic function, which fulfills condition $V2$ and $V3$. We want to calculate the increment of the corresponding distribution function over the interval $(a-h, a+h)$. This increment is given by the integral

$$2h\cdot I(\varphi_1)=2h\cdot\frac{1}{2\pi}\int_{-T}^{T}\varphi_1(t)\,dt,$$

where

$$\varphi_1(t)=\frac{\sin ht}{ht}e^{-ita}\varphi(t)\,C\left(\frac{t}{T}\right).$$

This integral is thus of the type considered here. Applying formula (3.10) means calculating inter alia $T_\lambda(\varphi_1'')$. We will not enter upon the details but it is evident from the expression of φ_1 alone that φ_1'' will be of fairly cumbersome form. From a practical point of view it seems desirable to replace $T_\lambda(\varphi'')$ in (3.10) by some approximate expression, if possible.

Since $\varphi''(t)/\varphi''(0)$ is a characteristic function, theorem (3.9) gives

$$-T_\lambda(\varphi'')\leqslant -T_{2\lambda}(\varphi'')=-\frac{2\lambda}{2\pi}\sum_\nu\varphi''(2\lambda\nu).$$

In this expression we replace $\varphi''(2\lambda\nu)$ by the approximation

$$-\varphi''(2\lambda\nu)\sim\frac{2\varphi(2\lambda\nu)-\varphi(2\lambda\nu+\lambda)-\varphi(2\lambda\nu-\lambda)}{\lambda^2}.$$

Hence

$$-\frac{2\lambda}{2\pi}\sum_\nu\varphi''(2\lambda\nu)\sim\frac{2}{\pi\lambda}\sum_\nu(-1)^\nu\varphi(\lambda\nu).$$

We are thus led to presume that

$$\frac{\lambda}{4\pi^3}\sum_\nu(-1)^\nu\varphi(\lambda\nu)$$

is an approximation on the "safe side" of the error bound

$$-\frac{\lambda^2}{8\pi^2}T_\lambda(\varphi'').$$

Our next task will be to discuss to which extent this presumption is correct. We will base this discussion upon the assumption that $\varphi(t)$ fulfills not only conditions $V1-V3$ but also

$V\,4)$ The function $\psi(t)=\varphi(t)+\varphi(-t)$ has continuous derivatives up to order $2p$ inclusive for $0\leqslant t\leqslant T$. At the endpoints the derivatives are defined as

$$\psi^{(\nu)}(0)=\psi^{(\nu)}(0+0),$$
$$\psi^{(\nu)}(T)=\psi^{(\nu)}(T-0).$$

$V\,5)$ $\psi^{(\nu)}(T)-\psi^{(\nu)}(0)=0$ for odd values of ν less than $(2p-1)$, while

$$\psi^{(2p-1)}(T)-\psi^{(2p-1)}(0)\neq 0.$$

We will also assume that $T/2\lambda$ is an integer. Let us now deduce the Euler-Maclaurin sum formula.

$$\frac{y}{e^y-1}e^{xy}=\sum\frac{y^n}{n!}B_n(x),$$

where $B_n(x)$ is the n:th Bernoulli polynomial, $B_n(0)=B_n=$ the n:th Bernoulli number, and

$$B_0(x)=1,$$
$$B_n'(x)=n\,B_{n-1}(x).$$

Put $j_n(x)=B_n(x-[x])$ so that $j_n(x)$ is periodic with period 1.
The integral

$$\frac{1}{2\pi}\int_{-T}^{T}\varphi(t)\,dt=\frac{1}{2\pi}\int_{0}^{T}\{\varphi(t)+\varphi(-t)\}\,dt=\frac{1}{2\pi}\int_{0}^{T}\psi(t)\,dt$$

will be evaluated according to the Euler-Maclaurin sum formula. Divide the interval $(0, T)$ into sub-intervals of length λ.

$$\int_{\nu\lambda}^{\nu\lambda+\lambda}\psi(t)\,dt=\left[j_1\left(\frac{t}{\lambda}\right)\lambda\,\psi(t)\right]_{\nu\lambda}^{\nu\lambda+\lambda}-\lambda\int_{\nu\lambda}^{\nu\lambda+\lambda}j_1\left(\frac{t}{\lambda}\right)\psi'(t)\,dt.$$

Successive partial integration gives

$$\int_{\nu\lambda}^{\nu\lambda+\lambda}\psi\,dt=-\left[\sum_{m=1}^{2p}\frac{(-\lambda)^m}{m!}j_m\left(\frac{t}{\lambda}\right)\psi^{(m-1)}(t)\right]_{\nu\lambda}^{\nu\lambda+\lambda}+\frac{(-\lambda)^{2p}}{(2p)!}\int_{\nu\lambda}^{\nu\lambda+\lambda}j_{2p}\left(\frac{t}{\lambda}\right)\psi^{(2p)}(t)\,dt.$$

Summing with respect to ν and using the fact that

$$j_1(\nu-0)=-j_1(\nu+0)=\tfrac{1}{2},$$
$$j_k(t)\text{ is continuous for }k>1,$$
$$j_k(0)=B_k\text{ for }k>1,$$

we obtain

$$\int_0^T \psi\, dt = \frac{\lambda}{2}\psi(0) + \lambda \sum_{\nu>0} \psi(\lambda\nu) -$$

$$- \sum_{m=2}^{2p} \frac{(-\lambda)^m}{m!} B_m [\psi^{(m-1)}(T) - \psi^{(m-1)}(0)] + \frac{\lambda^{2p}}{(2p)!} \int_0^T j_{2p}\left(\frac{t}{\lambda}\right) \psi^{(2p)}(t)\, dt.$$

Now $B_m = 0$ for odd $m > 1$ and

$$\psi^{(m-1)}(T) - \psi^{(m-1)}(0) = 0$$

for even $m < 2p$ according to assumption $V5$.

Making use of the definition of ψ we may then write

$$\lambda \sum \varphi(\nu\lambda) = \int_{-T}^{T} \psi(t)\, dt + \lambda^{2p} \frac{B_{2p}}{(2p)!} [\psi^{(2p-1)}(T) - \psi^{(2p-1)}(0)] -$$

$$- \frac{\lambda^{2p}}{(2p)!} \int_0^T j_{2p}\left(\frac{t}{\lambda}\right) \psi^{(2p)}(t)\, dt.$$

It is easily seen that the last integral tends to zero as $\lambda \to 0$. Thus, using the notation of (3.10)

$$T_\lambda(\varphi) = \frac{1}{2\pi} \int_{-T}^{T} \varphi(t)\, dt + \lambda^{2p} \frac{B_{2p}}{2\pi(2p)!} [\psi^{(2p-1)}(t)]_0^T + o(\lambda^{2p}).$$

The same formula with spacing 2λ instead of λ gives

$$T_{2\lambda}(\varphi) = \frac{1}{2\pi} \int_{-T}^{T} \varphi(t)\, dt + (2\lambda)^{2p} \frac{B_{2p}}{2\pi(2p)!} [\psi^{(2p-1)}(t)]_0^T + o(\lambda^{2p}).$$

Subtracting these expressions gives

$$\frac{\lambda}{2\pi} \sum (-1)^\nu \varphi(\nu\lambda) = T_{2\lambda}(\varphi) - T_\lambda(\varphi) = \lambda^{2p}(2^{2p}-1) \frac{B_{2p}}{2\pi(2p)!} [\psi^{(2p-1)}(t)]_0^T + o(\lambda^{2p}).$$

Using the expansion of $T_\lambda(\varphi)$ with φ'' instead of φ and $(p-1)$ instead of p gives

$$T_\lambda(\varphi'') = \frac{1}{2\pi} \int_{-T}^{T} \varphi''(t)\, dt + \lambda^{2p-2} \frac{B_{2p-2}}{2\pi(2p-2)!} [\psi^{(2p-1)}(t)]_0^T + o(\lambda^{2p-2}),$$

but as

$$\int_{-T}^{T} \varphi''(t)\, dt = \int_0^T \psi''(t)\, dt = \psi'(T) - \psi'(0) = 0,$$

we have $$T_\lambda(\varphi'')=\lambda^{2p-2}\frac{B_{2p-2}}{2\pi(2p-2)!}[\psi^{(2p-1)}(t)]_0^T+o(\lambda^{2p-2}).$$

Now we recall that our presumption was that

$$R_1=\frac{\lambda}{4\pi^3}\sum(-1)^\nu\varphi(\nu\lambda)=\frac{1}{2\pi^2}\{T_{2\lambda}(\varphi)-T_\lambda(\varphi)\}$$

was an approximation on the "safe side" of

$$R_2=-\frac{\lambda^2}{8\pi^2}T_\lambda(\varphi'').$$

According to the above expansions R_1/R_2 tends to the following limit as $\lambda\to 0$:

$$Q_p=\frac{1}{2\pi^2}\cdot(2^{2p}-1)\frac{B_{2p}}{2\pi(2p)!}\cdot 8\pi^2\cdot\frac{2\pi(2p-2)!}{-B_{2p-2}},$$

$$Q_p=-\frac{B_{2p}}{B_{2p-2}}\cdot\frac{2}{p(2p-1)}\cdot(2^{2p}-1).$$

The following table gives some values of Q_p

$p=$	2	3	4	5
$Q_p=$	1	6	$25\frac{1}{2}$	$103\frac{1}{3}$

By use of the expansion

$$\frac{B_{2p}}{(2p)!}=-2\frac{(-1)^p}{(2\pi)^{2p}}S_{2p},$$

$$S_{2p}=\sum_{n=1}^{\infty}\frac{1}{n^{2p}},$$

Q_p can also be expressed as

$$Q_p=\frac{1}{\pi^2}\cdot\frac{S_{2p}}{S_{2p-2}}\cdot(2^{2p}-1),$$

and it is easily seen from this formula that Q_p increases monotonously with p. Thus we have found that

$$R_1=Q_{p,\lambda}\cdot R_2=\lambda^{2p}(2^{2p}-1)\frac{B_{2p}}{4\pi^3(2p)!}[\psi^{(2p-1)}(t)]_0^T+o(\lambda^{2p}),$$

where $\lim_{\lambda\to 0}Q_{p,\lambda}=Q_p\geqslant 1$. In this sense R_1 is an approximation of R_2 on the "safe side".

We exemplify the use of these results in the following way.

Let $\varphi(t)$ be a characteristic function with continuous derivatives up to order 4 inclusively. Put

$$\varphi_1(t) = C\left(\frac{t}{T}\right)\varphi(t),$$

where $C(t)$ is the characteristic function deduced in chapter 1. Then φ_1 has continuous first and second derivatives, while the third derivative is discontinuous for t equal to $-T$, 0 and $+T$, except for special choices of φ. If we leave such special cases out of consideration we have

$$R_1 = \frac{\lambda}{4\pi^3}\sum(-1)^\nu \varphi_1(\lambda\nu) = 0(\lambda^4)$$

$$R_2 = -\frac{\lambda^2}{8\pi^2} T_\lambda(\varphi_1'') = 0(\lambda^4)$$

and $R_1 \sim R_2$ when $\lambda \to 0$.

This result is of special interest to us, since we have come to regard the multiplication by $C(t/T)$ as the "best" method to transform a given characteristic function into one, which is equal to zero outside the interval $(-T, T)$.

We summarize our results in the following approximation formula.

3.11. *Exact value:* $I = \frac{1}{2\pi}\int_{-T}^{T}\varphi(t)\,dt.$

Approximate value: $I_A = T_\lambda(\varphi) - R.$

Error bound: $|I - I_A| \leqslant R.$

$$T_\lambda(\varphi) = \frac{\lambda}{2\pi}\sum\varphi(\lambda\nu).$$

$$R = -\frac{\lambda^3}{16\pi^3}\sum\varphi''(\lambda\nu).$$

Approximate value of $R = \frac{\lambda}{4\pi^3}\sum(-1)^\nu\varphi(\lambda\nu).$

These formulas will be used in the numerical applications in chapter 5.

We will now conclude this chapter by comparing the approximation formulas (3.10) and (2.2). We write the integral I, which we want to approximate, in the following form

$$I = \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\varphi(t)\,dt = \frac{1}{2\pi}\int_{-T}^{T}\varphi_1(t)\,dt.$$

According to (2.2) we get the approximation

$$2\frac{1-\cos\lambda x}{\lambda^2 x^2} T_\lambda(\varphi_1).$$

We have seen that if φ_1 fulfills conditions $V1 - V5$ we have the following asymptotic expansion when $\lambda \to 0$:

$$T_\lambda(\varphi_1) = I + 0(\lambda^{2p}),$$

while
$$2\frac{1-\cos\lambda x}{\lambda^2 x^2} = 1 - \frac{\lambda^2 x^2}{12} + 0(\lambda^4).$$

For the approximation according to (2.2) we have thus the following asymptotic expansion

$$2\frac{1-\cos\lambda x}{\lambda^2 x^2} T_\lambda(\varphi_1) = I - \frac{\lambda^2 x^2}{12} I + 0(\lambda^4)$$

if $p \geq 2$.

According to (3.10) we get the approximation

$$T_\lambda(\varphi_1) + \frac{\lambda^2}{8\pi^2} T_\lambda(\varphi_1'').$$

As
$$T_\lambda(\varphi_1'') = 0(\lambda^{2p-2})$$

we have the following asymptotic expansion

$$T_\lambda(\varphi_1) + \frac{\lambda^2}{8\pi^2} T_\lambda(\varphi_1'') = I + 0(\lambda^{2p}).$$

Now suppose that $p \geq 2$. For x fixed and $\lambda \to 0$ we then conclude that (3.10) is a better approximation than (2.2) in the sense that the error term of (3.10) is $0(\lambda^{2p})$ while the error term of (2.2) is $0(\lambda^2)$.

Now for λ fixed we may consider the approximations as functions of x. Since

$$\varphi_1(t) = e^{-itx}\varphi(t)$$

it is easily seen that the approximation according to (3.10) may be written

$$I_A = \tau_0(x) + x \cdot \tau_1(x) + x^2 \cdot \tau_2(x),$$

where τ_0, τ_1 and τ_2 are certain trigonometric polynomials. Since a trigonometric polynomial is a periodic function and

$$\tau_2(x) = -\frac{\lambda^2}{8\pi^2} T_\lambda(\varphi_1) \leq 0$$

it follows that
$$\lim_{|x|\to\infty} \frac{I_A}{x^2}$$

exists and is < 0. Since the integral I itself tends to zero as $|x| \to \infty$ it is evident that I_A, according to (3.10), does not provide useful information about I when $|x| \to \infty$.

As for the approximation I_A according to (2.2) it is $0(x^{-2})$ when $|x| \to \infty$, since $T_\lambda(\varphi_1)$ is a periodic function. This does not say anything, however, about how fast the frequency function I tends to zero since the denominator x^2 in I_A is common to all characteristic functions to which (2.2) may be applied.

The somewhat critical aspects on (2.2) developped here apply mutatis mutandis to the other approximation formula studied in chapter 2, namely the one based on approximation of φ by parabolas.

Chapter 4

Approximation from the point of view of Fourier series

Let $f(x)$ be a frequency function and put

$$g(x) = \sum_k f\left(\frac{2k\pi + x}{\lambda}\right) \quad \text{for } |x| < \pi.$$

Then $g(x)$ is a frequency function and its n:th Fourier coefficient is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x)\,dx = \sum_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f\left(\frac{2k\pi + x}{\lambda}\right) dx$$

$$= \sum_k \frac{1}{2\pi}\int_{2k\pi-\pi}^{2k\pi+\pi} e^{-inx} f\left(\frac{x}{\lambda}\right) dx = \frac{\lambda}{2\pi}\varphi(-\lambda n),$$

where $\varphi(t)$ is the characteristic function of $f(x)$. Thus $g(x)$ has the Fourier series

$$g(x) \sim \frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty} e^{-inx}\varphi(n\lambda),$$

where the sign "$\sim$" indicates the purely formal relationship. We have not yet made any assumptions which assure us that the series will converge.

Let $P_n(x)$ be a non-negative trigonometric polynomial of order n:

$$P_n(x) = \sum_{-n}^{n} a_\nu e^{i\nu x}.$$

The convolution of $g(x)$ and $P_n(x)$ is denoted $g(x) * P_n(x)$:

$$g(x) * P_n(x) = \int_{-\pi}^{\pi} g(y) P_n(x-y)\,dy = \sum \int_{-\pi}^{\pi} g(y)\, a_\nu e^{i\nu(x-y)}\,dy = \lambda \sum_{-n}^{n} a_{-\nu}\varphi(\lambda\nu)\, e^{-i\nu x}.$$

We are in this chapter interested in the question to which extent this function is an approximation of $g(x)$. We first prove the following lemma.

4.1. *If* 1) $p_n(x) \geqslant 0$,

2) $p_n(x) \in L(-\pi, \pi)$,

3) $\int_{-\pi}^{\pi} p_n(x)\,dx \to 1$ *when* $n \to \infty$,

4) $\int_{-\pi}^{\pi} (1-\cos x)\,p_n(x)\,dx \to 0$ *when* $n \to \infty$,

then
$$\int_{-\pi}^{\pi} h(x-y)\,p_n(y)\,dy \to h(x),$$

if h *is a continuous function.*

Proof: $\int_{-\pi}^{\pi} (1-\cos y)\,p_n(y)\,dy \geqslant \int_{|y|>\varepsilon} (1-\cos y)\,p_n(y)\,dy \geqslant (1-\cos\varepsilon) \int_{|y|>\varepsilon} p_n(y)\,dy.$

Hence, according to assumptions 3 and 4

$$\left.\begin{aligned}\int_{|y|>\varepsilon} p_n(y)\,dy \to 0\\ \int_{|y|\leqslant\varepsilon} p_n(y)\,dy \to 1\end{aligned}\right\},$$

$$\int_{-\pi}^{\pi} h(x-y)\,p_n(y)\,dy = \int_{|y|>\varepsilon} h(x-y)\,p_n(y)\,dy + \int_{|y|\leqslant\varepsilon} h(x-y)\,p_n(y)\,dy.$$

Since $h(x-y)$ is continuous there is a constant M such that the first integral on the right hand side is less than

$$M \int_{|y|>\varepsilon} p_n\,dy \to 0.$$

Since $h(x-y)$ is continuous the second integral may be written

$$h(\xi) \int_{|y|\leqslant\varepsilon} p_n(y)\,dy.$$

If first $n \to \infty$ and then $\varepsilon \to 0$ this tends to $h(x)$, as was to be proved.

We turn now to the following choice of $P_n(x)$:

$$\begin{cases} P_n(x) = \dfrac{1}{2\pi} \sum_{-n}^{n} C\left(\dfrac{\nu}{n}\right) e^{i\nu x},\\ C\left(\dfrac{\nu}{n}\right) = \left(1 - \dfrac{|\nu|}{n}\right) \cos\dfrac{\pi\nu}{n} + \dfrac{1}{\pi} \sin\left|\dfrac{\pi\nu}{n}\right|. \end{cases}$$

That this is a non-negative trigonometric polynomial follows from (1.7) and the fact that

$$C(t) = (1 - |t|) \cos \pi t + \frac{1}{\pi} \sin |\pi t|$$

is a characteristic function.

Now
$$\int_{-\pi}^{\pi} P_n(x)\,dx = C(0) = 1.$$

Hence conditions 1, 2 and 3 of (4.1) are fulfilled. Furthermore

$$2 \int_{-\pi}^{\pi} (1 - \cos x)\, P_n(x)\, dx = \int_{-\pi}^{\pi} (2 - e^{ix} - e^{-ix})\, P_n(x)\, dx = 2 - 2\, C\left(\frac{1}{n}\right).$$

Hence
$$\frac{1 - \cos x}{1 - C\left(\frac{1}{n}\right)} P_n(x)$$

is a non-negative trigonometric polynomial and

$$\int_{-\pi}^{\pi} \frac{1 - \cos x}{1 - C\left(\frac{1}{n}\right)} P_n(x)\, dx = 1.$$

Furthermore

$$\int_{-\pi}^{\pi} (2 - 2 \cos x)^2 P_n(x)\, dx$$
$$= \int_{-\pi}^{\pi} (6 - 4\, e^{ix} - 4\, e^{-ix} + e^{2ix} + e^{-2ix})\, P_n(x)\, dx = 6 - 8\, C\left(\frac{1}{n}\right) + 2\, C\left(\frac{2}{n}\right).$$

But
$$C\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right) \cos \frac{\pi}{n} + \frac{1}{\pi} \sin \frac{\pi}{n} = 1 - \frac{\pi^2}{2\, n^2} + o\left(\frac{1}{n^2}\right)$$

and
$$C\left(\frac{2}{n}\right) = 1 - \frac{2\, \pi^2}{n^2} + o\left(\frac{1}{n^2}\right),$$

so that
$$6 - 8\, C\left(\frac{1}{n}\right) + 2\, C\left(\frac{2}{n}\right) = o\left(\frac{1}{n^2}\right),$$

$$\int_{-\pi}^{\pi} (1 - \cos x)\, \frac{1 - \cos x}{1 - C\left(\frac{1}{n}\right)} P_n(x)\, dx \to 0 \text{ when } n \to \infty.$$

We will now consider the convolution of $g(x)$ and $P_n(x)$. We are going to deduce an asymptotic formula, assuming that $g(x)$ is continuous as well as its first and second derivatives. Put

$$g_n(x) = g(x) * P_n(x).$$

Then

$$g_n(x) = \int_{-\pi}^{\pi} g(x-y) P_n(y)\, dy = \int_{-\pi}^{\pi} \left\{ g(x) - y \cdot g'(x) + \frac{y^2}{2} g''(x - \theta y) \right\} P_n(y)\, dy$$

$$= g(x) \int_{-\pi}^{\pi} P_n(y)\, dy - g'(x) \int_{-\pi}^{\pi} y P_n(y)\, dy + \tfrac{1}{2} \int_{-\pi}^{\pi} y^2 g''(x - \theta y) P_n(y)\, dy.$$

The first integral is equal to 1 and the second integral is equal to 0 since $P_n(y) = P_n(-y)$. The third integral is equal to

$$\left[1 - C\left(\frac{1}{n}\right)\right] \int_{-\pi}^{-\pi} g''(x - \theta y) \frac{y^2}{2(1 - \cos y)} \frac{1 - \cos y}{1 - C\left(\frac{1}{n}\right)} P_n(y)\, dy.$$

The function
$$g''(x - \theta y) \frac{y^2}{2(1 - \cos y)}$$

is continuous for $|y| \leqslant \pi$ and its value for $y = 0$ is $g''(x)$.

The functions
$$\frac{1 - \cos y}{1 - C\left(\frac{1}{n}\right)} P_n(y)$$

fulfill the conditions of (4.1). Hence the integral tends to $g''(x)$. Since

$$C\left(\frac{1}{n}\right) = 1 - \frac{\pi^2}{2n^2} + o\left(\frac{1}{n^2}\right),$$

we have proved that $g_n(x) = g(x) + \dfrac{\pi^2}{2n^2} g''(x) + o\left(\dfrac{1}{n^2}\right).$

Our next choice of $P_n(x)$ will be

$$P_n(x) = \frac{1}{2\pi} \sum_{-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{i\nu x}.$$

In this case we denote $g * P_n$ by σ_n.

$$\sigma_n(x) = \frac{\lambda}{2\pi} \sum_{-n}^{n} \left(1 - \frac{|\nu|}{n}\right) \varphi(\lambda\nu) e^{-i\nu x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{i\nu y} g(x-y)\, dy$$

$$= \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{1 - \cos ny}{1 - \cos y} g(x-y)\, dy = \frac{1}{2\pi n} \int_{0}^{\pi} \frac{1 - \cos ny}{1 - \cos y} \{g(x+y) + g(x-y)\}\, dy.$$

Since
$$\frac{1}{2\pi n}\int_0^\pi \frac{1-\cos n y}{1-\cos y} 2\,d y = 1,$$
we get
$$\sigma_n(x) - g(x) = \frac{1}{2\pi n}\int_0^\pi \frac{1-\cos n y}{1-\cos y}\{g(x+y)+g(x-y)-2g(x)\}\,d y.$$

If we assume that $g(x)$ is continuous as well as its first and second derivatives it follows that
$$h(x, y) = \frac{g(x+y)+g(x-y)-2g(x)}{1-\cos y}$$
is continuous for $0 \leqslant y \leqslant \pi$. Hence
$$\sigma_n(x) = g(x) + \frac{1}{2\pi n}\int_0^\pi h(x, y)\,d y - \frac{1}{2\pi n}\int_0^\pi \cos n y \cdot h(x, y)\,d y.$$

The last integral tends to zero as $n \to \infty$ according to the Riemann–Lebesque theorem. Hence
$$\sigma_n(x) = g(x) + \frac{2}{2\pi n}\int_0^\pi h(x, y)\,d y + o\left(\frac{1}{n}\right).$$

Summing up our results we may state the following theorem. Nothing is said about non-negativeness of $g(x)$, since we have not used this assumption in the proof.

4.2. *If* 1) $g(x)$ *is periodic with period* 2π,
2) $g(x)$, $g'(x)$ *and* $g''(x)$ *are continuous,*
3) *the n:th Fourier coefficient of* $g(x)$ *is* a_n, *i.e.*
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x)\,d x,$$
4) $g_n(x) = \sum_{-n}^{n} C\left(\frac{\nu}{n}\right) a_\nu e^{i\nu x}$,

5) $\sigma_n(x) = \sum_{-n}^{n}\left(1 - \frac{|\nu|}{n}\right) a_\nu e^{i\nu x}$,

then
$$g_n(x) = g(x) + \frac{\pi^2}{2n^2} g''(x) + o\left(\frac{1}{n^2}\right).$$

$$g_n(x) = g(x) + \frac{1}{2\pi n}\int_0^\pi h(x, y)\,d y + o\left(\frac{1}{n}\right),$$

where
$$C(t) = (1 - |t|) \cos \pi t + \frac{1}{\pi} \sin |\pi t|,$$

and
$$h(x, y) = \frac{g(x+y) + g(x-y) - 2g(x)}{1 - \cos y}.$$

Our intention is now to investigate how the term $o(1/n)$ in the expansion of $\sigma_n(x)$ depends on the analytical properties of $g(x)$. We prove the following theorem.

4.3. *If* 1) $g(x)$ *is periodic with period* 2π,
2) $g(x)$ *and its derivatives up to order* p *inclusive are continuous* $(p \geqslant 2)$,

then
$$\sigma_n(x) = g(x) + \frac{1}{2\pi n} \int_0^\pi h(x, y)\, dy + o(n^{1-p}).$$

Proof: From the proof of (4.2) we know that

$$\sigma_n(x) = g(x) + \frac{1}{2\pi n} \int_0^\pi h(x, y)\, dy - \frac{1}{2\pi n} \int_0^\pi \cos ny \cdot h(x, y)\, dy.$$

Now
$$g(x+y) = g(x) + y \cdot g'(x) + y^2 \int_0^1 (1-t)\, g''(x+ty)\, dt,$$

so that

$$h(x, y) = \frac{g(x+y) + g(x-y) - 2g(x)}{1 - \cos y} = \frac{y^2}{1 - \cos y} \int_0^1 (1-t)\, \{g''(x+ty) + g''(x-ty)\}\, dt.$$

From these formulas follows that $h(x, y)$ considered as a function of y for x fixed is periodic with period 2π. Also, since

$$\frac{y^2}{1 - \cos y}$$

is regular for $|y| < 2\pi$, it follows that $h(x, y)$ has continuous partial derivatives with respect to y up to order $(p-2)$ incl.

Now the integral

$$2 \int_0^\pi \cos ny \cdot h(x, y)\, dy = \int_{-\pi}^\pi e^{-iny}\, h(x, y)\, dy.$$

Integrating by parts and using the fact that $h(x, y)$ is periodic, this is easily seen to be equal to

$$\frac{1}{(in)^{p-2}} \int_{-\pi}^\pi e^{-iny} \frac{\partial^{p-2} h}{\partial y^{p-2}}\, dy,$$

and as $h(x, y)$ has a continuous partial derivative with respect to y of order $(p-2)$ this is $o(n^{2-p})$. Thus

$$\sigma_n(x) = g(x) + \frac{1}{2\pi n}\int_0^\pi h(x, y)\, dy + o(n^{1-p})$$

as was to be proved.

This formula explains why de la Vallée-Poussin's approximation

$$\tau_{2n}(x) = 2\sigma_{2n}(x) - \sigma_n(x)$$

is much better than $\sigma_{2n}(x)$ itself. For (4.3) gives the result

$$\tau_{2n}(x) = g(x) + o(n^{1-p}).$$

Actually it can be shown that

$$\tau_{2n}(x) = g(x) + o(n^{-p})$$

but this cannot be done with the methods developped here.

We are now going to compare the approximations obtained by (4.2) and (4.3) with the approximation obtained by straightforward summation of the $(n+1)$ first terms of the Fourier series.

We say that $T(x)$ is a trigonometric polynomial of degree n if

$$T(x) = \sum_{-n}^{n} a_\nu e^{i\nu x},$$

where $|a_n| + |a_{-n}| \neq 0$. We denote by H_n the class of trigonometric polynomials of degree $\leqslant n$.

Now let $g(x)$ be a continuous function for $|x| \leqslant \pi$. Put

$$E_n = \inf_{T \in H_n} \max_{|x| \leqslant \pi} |g(x) - T(x)|.$$

E_n is called the best approximation of $g(x)$ by polynomials from H_n. It can be shown that there is a unique $T(x)$ from H_n for which

$$E_n = \max_{|x| \leqslant \pi} |g(x) - T(x)|.$$

This $T(x)$ approximates $g(x)$ better than all other polynomials from H_n. These ideas go back to Chebyshev. Jackson has studied how E_n depends on the analytic properties of $g(x)$. From among his results we quote the following one.

4.4. *If* $g(x)$ *fulfills the conditions of* (4.3), *then*

$$E_n = o(n^{-p}).$$

As previously we denote the Fourier coefficients of $g(x)$ by a_n;

$$g(x) \sim \sum_{-\infty}^{+\infty} a_\nu e^{i\nu x}.$$

The sum of the $(n+1)$ first terms is denoted by $S_n(x)$:

$$S_n(x) = \sum_{-n}^{n} a_\nu e^{i\nu x}.$$

Lebesgue has shown that

$$|S_n(x) - g(x)| < (3 + \log n) E_n.$$

As $\log n$ increases slowly with n this means that $S_n(x)$ is "almost as good" an approximation of $g(x)$ as the best polynomial from H_n. This is a valuable result from the practical point of view.

If we combine Jackson's and Lebesgue's results we find

$$S_n(x) = g(x) + o\left(\frac{\log n}{n^p}\right).$$

Suppose now that $\varphi(t)$ is a characteristic function and that the integral

$$I = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \varphi(t)\, dt$$

exists. We want to perform a numerical integration of this integral and for this purpose we are going to use values of the integrand for $t = \nu\lambda$, where ν runs through all positive and negative integers.

We have seen that

$$\frac{\lambda}{2\pi} \varphi(-n\lambda)$$

may be considered as the n:th Fourier coefficient of $g(x)$, where

$$g(x) = \sum_k f\left(\frac{2k\pi + x}{\lambda}\right) \qquad |x| \leqslant \pi$$

that is

$$g(x) \sim \frac{\lambda}{2\pi} \sum_{-\infty}^{+\infty} \varphi(\nu\lambda) e^{-i\nu x},$$

or, if $g_1(x) = g(\lambda x)$, so that $g_1(x)$ is periodic with period $2\pi/\lambda$,

$$g_1(x) \sim \frac{\lambda}{2\pi} \sum_{-\infty}^{+\infty} \varphi(\nu\lambda) e^{-i\nu\lambda x}.$$

If certain conditions are fulfilled $g_1(x)$ will be determined by its Fourier coefficients. We quote just three theorems of this type.

If $g_1(x)$ is of bounded variation, then

$$\bar{g}_1(x) = \frac{\lambda}{2\pi} \sum_{-\infty}^{+\infty} \varphi(\lambda\nu) e^{-i\nu\lambda x}.$$

If $g_1(x)$ is continuous, then

$$\frac{\lambda}{2\pi}\sum_{-\infty}^{+\infty}\varphi(\lambda\nu)\,e^{-i\nu\lambda x}$$

is summable $(C, 1)$ to $g_1(x)$.

If $g_1(x)$ belongs to $L^2(-\pi/\lambda, \pi/\lambda)$, then

$$\frac{\lambda}{2\pi}\sum_{-n}^{n}\varphi(\lambda\nu)\,e^{-i\nu\lambda x}$$

converges in mean to $g_1(x)$.

The essential point is, that under certain conditions $g_1(x)$ may be determined from the discrete values $\varphi(\lambda\nu)$. Since $g_1(x)$ equals

$$g_1(x)=\sum_k f\left(\frac{2k\pi}{\lambda}+x\right),$$

it is evident that different functions $f(x)$ may correspond to the same function $g_1(x)$. This means that the set of values $\varphi(\lambda\nu)$ may determine $g_1(x)$ but cannot determine $f(x)$.

Among the frequency functions whose characteristic function takes on the values $\varphi(\lambda\nu)$ for $t=\lambda\nu$, we also find $g_1(x)$, for

$$\int_{-\pi/\lambda}^{\pi/\lambda} e^{i\nu\lambda x}\,g_1(x)\,dx=\int_{-\infty}^{+\infty} e^{i\nu\lambda x} f(x)\,dx=\varphi(\lambda\nu).$$

This means that if we wish to determine the frequency function starting from the discrete values $\varphi(\lambda\nu)$ of the characteristic function, we must make up our mind which frequency function to chose. The most natural choice seems to be the function

$$f_1(x)=\begin{cases} g_1(x)=\sum_k f\left(\dfrac{2k\pi}{\lambda}+x\right) & \text{for } |x|\leqslant\dfrac{\pi}{\lambda} \\ 0 & \text{for } |x|>\dfrac{\pi}{\lambda}. \end{cases}$$

Our problem may thus be formulated as follows. Given the discrete values $\varphi(\lambda\nu)$ of the characteristic function $\varphi(t)$ there is a corresponding frequency function which is zero outside the interval $(-\pi/\lambda, \pi/\lambda)$. An approximate value of this function in terms of $\varphi(\lambda\nu)$, $\nu=-n, -n+1, \ldots, n$, is to be found.

Consider then the integral

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\,\varphi(t)\,dt.$$

Truncate the integral outside $(-n\lambda, n\lambda)$ and apply a numerical integration formula with abscissas $\nu\lambda$ and weights $\lambda\gamma_{\nu,n}$. We obtain the following approximate value of the integral

$$\frac{\lambda}{2\pi}\sum_{-n}^{n}\gamma_{\nu,n}\,e^{-i\nu\lambda x}\,\varphi(\nu\lambda).$$

This is a trigonometric polynomial of degree $\leqslant n$. If we knew the explicit expression of $T(x)\in H_n$, for which the best approximation of $f_1(x)$ is obtained, then $\gamma_{\nu,n}$ could, of course, be choosen accordingly. Since we don't dispose of this explicit expression we have to be satisfied with something that is "next to the best". The foregoing discussions suggest the use of the $(n+1)$ first terms of the Fourier series, i.e. putting $\gamma_{\nu,n}=1$. The following formula is then obtained:

$$\frac{\lambda}{2\pi}\sum_{-n}^{n}e^{-i\nu\lambda x}\,\varphi(\lambda\nu).$$

This again may serve to support the impression that the trapezoidal rule should be preferred for the numerical integration of Fourier integrals where characteristic functions are involved.

Chapter 5

Numerical illustration

An insurance business will be briefly characterized as follows. The policyholders pay their premiums to the company. If a policyholder meets with an incident, which is covered by his policy, he makes a claim on the company. On each claim the company has to pay a certain amount, called the risk sum of the policy, to the policyholder.

We assume that the number of claims has a Poisson distribution. If we consider a period during which the expected number of claims is τ, the probability of getting exactly n claims will thus be

$$\frac{\tau^n}{n!}e^{-\tau}.$$

The risk sum has a known probability distribution. The probability that the risk sum of a policy, chosen at random, shall be less than or equal to u is $P(u)$, where $P(u)$ is a known distribution function. The characteristic function corresponding to $P(u)$ is denoted $\psi(t)$:

$$\psi(t)=\int_{-\infty}^{+\infty}e^{itu}\,dP(u).$$

The first two moments of $P(u)$ about the origin are denoted p_1 and p_2 respectively. Thus

$$p_1 = \int_{-\infty}^{+\infty} u\, d P(u),$$

$$p_2 = \int_{-\infty}^{+\infty} u^2\, d P(u).$$

Let $Y(\tau)$ denote the total amount of claims paid by the company during a period with an expected number of claims equal to τ. Then $Y(t)$ will have a compound Poisson distribution with characteristic function

$$\exp\{\tau \psi(t) - \tau\}.$$

From this formula follows that the mean and the variance of $Y(\tau)$ are $p_1 \tau$ and $p_2 \tau$ respectively. We will also consider the standardized variable $X(\tau)$, corresponding to $Y(\tau)$. I.e.

$$X(\tau) = \frac{Y(\tau) - p_1 \tau}{\sqrt{p_2 \tau}},$$

with characteristic function

$$\varphi(t) = \exp\left\{\tau \psi\left(\frac{t}{\sqrt{p_2 \tau}}\right) - \tau - i t \frac{p_1 \tau}{\sqrt{p_2 \tau}}\right\}.$$

Our intention is to calculate the distribution function corresponding to $X(\tau)$. As for the distribution function $P(u)$ of risk sums we make the following assumptions:

$$\begin{cases} \dfrac{d P(u)}{d u} = (1-p) \dfrac{\alpha}{e^{\alpha} - 1}\, e^{\alpha - \alpha u} & 0 \leqslant u < 1, \\ d P(1) = p. \end{cases}$$

The corresponding characteristic function is

$$\psi(t) = (1-p) \frac{\alpha}{\alpha - i t} \cdot \frac{e^{\alpha} - e^{it}}{e^{\alpha} - 1} + p\, e^{it}.$$

Then
$$p_1 = -i \psi'(0) = \frac{1-p}{\alpha} - \frac{1-p}{e^{\alpha} - 1} + p,$$

$$p_2 = -\psi''(0) = 2 \frac{1-p}{\alpha^2} - \frac{1-p}{e^{\alpha} - 1} - 2 \frac{1-p}{\alpha (e^{\alpha} - 1)} + p.$$

We will consider nine cases by letting α take on the values 1 and 5 and 10 while p takes on the values 0 and 0.01 and 0.02. The corresponding values of p_1 are shown in the following table:

$p \diagdown \alpha$	1	5	10
0	0.4180	0.1932	0.1000
0.01	0.4238	0.2013	0.1090
0.02	0.4297	0.2094	0.1180

The practical background to the probability mass p, situated at $u=1$, is the following. Every insurance company has a certain upper limit L of the risk sum of each individual policy. Should a policy have the risk sum S and $S>L$ then the company retains the sum L while the sum $(S-L)$ is covered by another company. This process is called reinsurance. The effect of this is that if $100\,p$ % of the policies sold by the company need reinsurance, then from the point of view of the company $100\,p$ % of its policies in force will have the risk sum L. From this it is evident that our choice of $P(u)$ implies that $P(u)$ is the probability of the risk sum being less than or equal to $u \cdot L$ and that the mean risk sum is $p_1 \cdot L$.

Apart from the nine distributions mentioned above we will also consider the following distribution obtained by letting $\alpha \to \infty$ and $p \to 0$. For $Y(\tau)$ we had the characteristic function

$$\exp\left\{\tau(1-p)\frac{\alpha}{\alpha-it}\cdot\frac{e^{\alpha}-e^{it}}{e^{\alpha}-1}+\tau p e^{it}-\tau\right\}.$$

Replacing t by αt gives

$$\exp\left\{\tau(1-p)\frac{1}{1-it}\cdot\frac{e^{\alpha}-e^{i\alpha t}}{e^{\alpha}-1}+\tau p e^{i\alpha t}-\tau\right\}.$$

Now let $\alpha \to \infty$ and $p \to 0$. We obtain

$$\exp\left\{\frac{\tau}{1-it}-\tau\right\}.$$

Replacing t by $t/\sqrt{2\tau}$ and multiplying by

$$\exp\left\{-\frac{it\tau}{\sqrt{2\tau}}\right\}$$

gives the characteristic function of the corresponding standardized variable

$$\varphi(t)=\exp\left\{\frac{\tau}{1-\frac{it}{\sqrt{2\tau}}}-\tau-\frac{it\tau}{\sqrt{2\tau}}\right\}.$$

We want to calculate the distribution function corresponding to $X(\tau)$ and we are going to use the integration formula (3.11). We truncate the characteristic function by adding to $X(\tau)$ a random variable Z with characteristic function

$$C\left(\frac{t}{32}\right)=\left(1-\frac{|t|}{32}\right)\cos\frac{\pi t}{32}+\frac{1}{\pi}\sin\frac{|\pi t|}{32}\quad\text{for } |t|\leqslant 32.$$

The characteristic function of $X(\tau)+Z$ is

$$\varphi(t)\cdot C\left(\frac{t}{32}\right),$$

which is zero for $|t|\geqslant 32$. The distribution function of $X(\tau)+Z$ is denoted $F_2(x)$ and is given by the formula

$$F_2(x+h)-F_2(x-h)=\frac{1}{2\pi}\int_{-32}^{32}2\frac{\sin ht}{t}e^{-itx}\varphi(t)\,C\left(\frac{t}{32}\right)dt.$$

The integrand is a characteristic function multiplied by $2h$. Since $2h>0$ we may apply (3.11), denoting the integrand by $g(t)$ for short.

Exact value: $I=\frac{1}{2\pi}\int_{-32}^{32}g(t)\,dt.$

Approximate value: $I_A=T_\lambda-R.$

Approximate error: $R.$

$$T_\lambda=\frac{\lambda}{2\pi}\sum g(\lambda\nu)$$

$$R=\frac{\lambda}{4\pi^3}\sum(-1)^\nu g(\lambda\nu).$$

The calculations started with $\lambda=32$ and were then repeated, each time with half the foregoing λ-value, until $R<0.00005$. This precision was as a rule attained with $\lambda=\frac{1}{2}$. The value I_A was then rounded of to 4 decimals. If we disregard the fact that R is itself an approximation we may then say that the value I_A arrived at has an error less than one unit in the fourth decimal.

Through the courtesy of The Swedish Board for Computing Machinery the computations were performed on one of their computers.

The variance of the additional variable Z is

$$\left(\frac{\pi}{32}\right)^2=0.0096$$

and its standard deviation = 0.1 approximately.

As the values in the table below refer to the distribution function $F_2(x)$ of $X(\tau)+Z$ we have to apply corrections according to chapter 1 to get information about the distribution function $F(x)$ of $X(\tau)$.

As an example let us find a lower bound of $F(2)-F(-2)$ corresponding to $\tau=250$, $\alpha=10$ and $p=0.01$. We make use of (1.1) and chose $\varepsilon=\pi/32$. That means that ε is equal to the standard deviation of Z. The table of the distribution function corresponding to $C(t/\pi)$ gives

$$F_1\left(\frac{\pi}{32}\right)=P\left\{Z\leqslant\frac{\pi}{32}\right\}=P\left\{\frac{32\,Z}{\pi}\leqslant 1\right\}=0.8492.$$

Interpolating in the table of $F_2(x)$ below we get

$$F_2\left(2-\frac{\pi}{32}\right)-F_2(0)\sim 0.4519.$$

$$F_2(0)-F_2\left(\frac{\pi}{32}-2\right)\sim 0.4885.$$

Then according to inequality (1.1)

$$F(2)-F(-2)\geqslant 1-\frac{1-0.4519-0.4885}{0.8492}=0.9298,$$

while

$$F_2(2)-F_2(-2)=0.9540.$$

The following heuristic reasoning will support the impression that our result is much better than these figures might indicate. If we compare the normal distribution with the distribution corresponding to $C(t)$ and the distribution F_2 in our table we find that the latter are both approximately normal. Now, if $X+Z$ and Z are normally distributed then X is normally distributed. As the variance of Z is approximately 0.01 and X is standardized the variance of $X+Z$ is 1.01. If $X+Z$ and Z were not only approximately but exactly normally distributed then it would be correct to say that F_2 is the distribution function of the variable $\sqrt{1.01}\cdot X$. Now, this it not so, but the impression subsists that the truth is somewhere in that direction.

τ	α	p	$10^4 [F_2(0) - F_2(x)]$											
			$x=$											
			−3.00	−2.75	−2.50	−2.25	−2.00	−1.75	−1.50	−1.25	−1.00	−0.75	−0.50	−0.25
25	∞	0	5279	5277	5269	5242	5172	5021	4745	4309	3694	2909	1993	1002
250	∞	0	5081	5069	5044	4991	4893	4722	4449	4045	3491	2782	1935	990
2500	∞	0	5016	5001	4970	4911	4807	4633	4362	3968	3430	2742	1916	985
25	10	0	5277	5276	5267	5241	5170	5018	4743	4306	3691	2908	1992	1001
250	10	0	5080	5069	5043	4991	4802	4721	4448	4045	3490	2781	1935	990
2500	10	0	5016	5001	4970	4911	4806	4632	4362	3968	3430	2742	1916	985
25	10	0.01	5416	5416	5414	5402	5358	5237	4979	4531	3872	3024	2046	1013
250	10	0.01	5124	5114	5093	5045	4951	4783	4509	4100	3534	2810	1949	993
2500	10	0.01	5030	5015	4986	4928	4825	4651	4380	3984	3443	2750	1920	986
25	10	0.02	5442	5441	5441	5434	5401	5291	5035	4572	3888	3019	2032	1004
250	10	0.02	5132	5123	5102	5056	4963	4795	4521	4110	3541	2814	1951	993
2500	10	0.02	5032	5018	4989	4931	4828	4655	4384	3987	3445	2752	1921	986
25	5	0	5249	5247	5237	5206	5131	4974	4697	4264	3657	2885	1980	998
250	5	0	5070	5059	5032	4979	4879	4708	4435	4033	3481	2775	1932	989
2500	5	0	5013	4997	4966	4907	4802	4628	4358	3964	3427	2740	1915	985
25	5	0.01	5264	5262	5253	5225	5152	4998	4720	4285	3673	2895	1985	999
250	5	0.01	5076	5064	5038	4985	4886	4715	4442	4039	3486	2778	1933	989
2500	5	0.01	5014	4999	4968	4909	4804	4630	4360	3966	3428	2741	1915	985
25	5	0.02	5271	5270	5262	5235	5163	5010	4733	4296	3681	2899	1986	999
250	5	0.02	5078	5066	5041	4988	4889	4718	4445	4042	3488	2780	1934	990
2500	5	0.02	5015	5000	4969	4910	4806	4632	4361	3967	3429	2741	1915	985
25	1	0	5183	5177	5161	5121	5035	4871	4594	4172	3587	2841	1961	994
250	1	0	5049	5036	5008	4952	4850	4678	4406	4007	3460	2762	1925	987
2500	1	0	5006	4990	4958	4899	4793	4619	4349	3956	3420	2735	1913	984
25	1	0.01	5183	5177	5162	5122	5035	4871	4594	4172	3587	2841	1961	994
250	1	0.01	5049	5036	5008	4952	4850	4678	4406	4007	3460	2762	1925	987
2500	1	0.01	5006	4990	4958	4899	4793	4619	4349	3956	3421	2735	1913	984
25	1	0.02	5183	5177	5162	5122	5035	4871	4594	4172	3587	2841	1961	994
250	1	0.02	5049	5036	5008	4952	4850	4678	4406	4007	3460	2762	1925	987
2500	1	0.02	5006	4990	4958	4899	4793	4619	4349	3956	3421	2735	1913	984

Explanation to tables

The tables give values of $10^4 | F_2(x) - F_2(0) |$, where $F_2(x)$ denotes the distribution function corresponding to the characteristic function

$$\varphi(t) \cdot C\left(\frac{t}{32}\right)$$

and where

$$C\left(\frac{t}{32}\right) = \begin{cases} \left(1 - \frac{|t|}{32}\right) \cos \frac{\pi t}{32} + \frac{1}{\pi} \sin \frac{|\pi t|}{32} & \text{for } |t| < 32 \\ 0 & \text{for } |t| \geqslant 32 \end{cases}$$

τ	α	p	$10^4\,[F_2(x) - F_2(0)]$, $x =$ 0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
25	∞	0	952	1807	2538	3135	3602	3954	4209	4388	4510	4590	4642	4674
250	∞	0	974	1876	2665	3316	3825	4202	4467	4645	4758	4827	4867	4889
2500	∞	0	980	1897	2705	3375	3898	4284	4552	4728	4837	4901	4937	4955
25	10	0	952	1807	2539	3136	3604	3955	4211	4390	4512	4592	4644	4676
250	10	0	974	1876	2665	3316	3825	4202	4468	4645	4759	4827	4867	4889
2500	10	0	980	1897	2705	3375	3898	4284	4552	4729	4838	4902	4937	4955
25	10	0.01	937	1761	2454	3016	3455	3788	4035	4212	4336	4422	4479	4517
250	10	0.01	970	1863	2639	3277	3777	4148	4411	4589	4704	4776	4819	4843
2500	10	0.01	979	1893	2696	3362	3882	4266	4534	4710	4820	4885	4922	4941
25	10	0.02	927	1741	2427	2984	3421	3754	4001	4179	4305	4392	4451	4490
250	10	0.02	969	1859	2632	3269	3767	4137	4400	4578	4694	4767	4810	4835
2500	10	0.02	979	1892	2695	3360	3879	4263	4530	4707	4817	4882	4919	4938
25	5	0	953	1815	2554	3159	3633	3989	4247	4427	4548	4627	4677	4708
250	5	0	975	1879	2671	3324	3836	4214	4480	4658	4770	4838	4878	4899
2500	5	0	980	1898	2707	3377	3902	4288	4556	4732	4841	4905	4940	4958
25	5	0.01	952	1810	2545	3146	3617	3971	4228	4408	4529	4609	4660	4692
250	5	0.01	974	1878	2668	3320	3830	4208	4474	4651	4764	4833	4872	4894
2500	5	0.01	980	1898	2706	3376	3900	4286	4554	4730	4839	4903	4938	4957
25	5	0.02	950	1806	2539	3139	3608	3961	4218	4398	4520	4600	4651	4683
250	5	0.02	974	1877	2666	3318	3827	4205	4471	4648	4761	4830	4870	4892
2500	5	0.02	980	1897	2705	3375	3899	4285	4553	4729	4838	4902	4938	4956
25	1	0	961	1838	2597	3220	3708	4073	4334	4513	4632	4707	4754	4781
250	1	0	977	1886	2684	3344	3860	4242	4509	4686	4797	4864	4901	4922
2500	1	0	981	1900	2711	3384	3910	4297	4566	4741	4850	4913	4948	4965
25	1	0.01	961	1838	2597	3220	3707	4072	4334	4513	4632	4707	4754	4781
250	1	0.01	977	1886	2684	3344	3860	4242	4509	4686	4797	4864	4901	4922
2500	1	0.01	981	1900	2711	3384	3910	4297	4566	4741	4850	4913	4948	4965
25	1	0.02	961	1838	2597	3220	3707	4072	4333	4513	4632	4707	4753	4781
250	1	0.02	977	1886	2684	3344	3860	4242	4509	4686	4797	4863	4901	4921
2500	1	0.02	981	1900	2711	3384	3910	4297	4566	4741	4850	4913	4948	4965

$$\varphi(t) = \exp\left\{\tau\,\psi\left(\frac{t}{\sqrt{p_2\,\tau}}\right) - \tau - i\,t\,\frac{p_1\,\tau}{\sqrt{p_2\,\tau}}\right\}$$

$$\psi(t) = (1-p)\,\frac{\alpha}{\alpha - i\,t}\cdot\frac{e^{\alpha} - e^{it}}{e^{\alpha} - 1} + p\,e^{it}$$

or

$$\psi(t) = \frac{1}{1 - i\,t}$$

the latter case being referred to in the tables as $\alpha = \infty$.

$$p_1 = -i\,\psi'(0)$$

$$p_2 = -\psi''(0).$$

APPENDIX

REFERENCES

Chapter 1.

Concerning Toeplitz forms see (1, pp. 16–19, 32–33). The essential part of the solution to the problem 1.4 is from (1, pp. 66–69). Cf. also (2, problems VII: 67–71). Theorem 1.5 follows from a theorem of L. Fejér (2, problem VI: 52).

Chapter 2.

The general idea of this approximation technique is due to Filon (3), who used the parabolic approximation.

Chapter 3.

Concerning Poisson's formula cf. (4), (5), (6), (7) and (8). The theorems 3.1 and 3.2 are from (8), apart from minor modifications. The theorem 3.3 is based on a theorem of Linfoot (5), who proves that

if 1) $g(t)$ is continuous,

2) $\sum g(\nu+t)$ is uniformly convergent

then
$$\sum g(\nu) = \sum_{(C,1)} \int_{-\infty}^{+\infty} e^{2\pi i \nu t} g(t)\, dt,$$

where $(C, 1)$ denotes summation by arithmetic means.

On the use of the trapezoidal rule on Fourier integrals cf. (9).

Chapter 4.

Concerning 4.2 cf. (10, pp. 140–149). As for the best approximation by trigonometric polynomials and its connection with the analytical properties of the function to be approximated see (10, pp. 59–82), Lebesgue's theorem concerning the rest term of Fourier series is deduced in (10, p. 135). The same question is dealt with in (11).

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