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On linear dependence in closed sets

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1. Kronecker's theorem can be formulated as follows: A necessary and sufficient condition for $\lambda_2-\lambda_1, \lambda_3-\lambda_1, \ldots, \lambda_k-\lambda_1$ to be linearly independent (mod 2π) is

$$\inf_{a_\nu} \sup_{n\geqslant 0} \Big| \sum_1^k a_\nu e^{in\lambda_\nu} \Big| = \sum_1^k |a_\nu|.$$

Throughout the paper n represents integers and t arbitrary real numbers.

For an arbitrary closed set on $(0,2\pi)$ we define the following indices of linear independence:

$$(a)\quad P_C(E) = \inf_{\mu\in\Gamma^\circ} \sup_{n\geqslant 0} \Big| \int_E e^{inx}\, d\mu(x) \Big|$$

$$(b)\quad P_C^*(E) = \inf_{\mu\in\Gamma^\circ} \sup_{t\geqslant 0} \Big| \int_E e^{itx}\, d\mu(x) \Big|$$

$$(c)\quad P_H(E) = \inf_{\mu\in\Gamma^\circ} \sup_{n} \Big| \int_E e^{inx}\, d\mu(x) \Big|$$

$$(d)\quad P_H^*(E) = \inf_{\mu\in\Gamma^\circ} \sup_{t} \Big| \int_E e^{itx}\, d\mu(x) \Big|,$$

where Γ° is the class of functions μ which are constant outside E and satisfy

$$\int_E |d\mu| = 1.$$

An immediate consequence of the definitions is:

$$0 \leqslant P_C \leqslant \begin{cases} P_C^* \\ P_H \end{cases} \leqslant P_H^* \leqslant 1.$$

In Kronecker's theorem the condition $n\geqslant 0$ can be changed to $t\geqslant 0$, $-\infty < n < \infty$ or $-\infty < t < \infty$.

If E consists of a finite number of linearly independent points we thus have $P_C = P_C^* = P_H = P_H^* = 1$. We shall prove that this property holds for all sets E for which $P_H^*(E)=1$ (Theorem 1) and that $P_H^*(E)>0$ implies $P_C(E)>0$ (Theorem 2).

A set E is called a *Kronecker set* if $P_C(E)=1$ and a *weak Kronecker set* if $P_C(E)>0$.

An example of a perfect Kronecker set of the Cantor type was given in Rudin [4]. Theorem 2 provides a positive answer to the question, raised by Kahane-Salem in [3], concerning the equivalence of Carleson-sets and Helson-sets. Both notions are equivalent to weak Kronecker sets.

The following theorem by Carleson [1] is fundamental for the proof of Theorem 2:

Theorem A. *A necessary and sufficient condition that $P_H(E)>0$ is that for each $\delta>0$ every continuous function $\varphi(x)$ has a representation*

$$\varphi(x)=\sum_{-\infty}^{\infty} a_\nu e^{i\nu x},\ x\in E,$$

where

$$\sum_{-\infty}^{\infty}|a_\nu|\leqslant\frac{1}{P_H}+\delta$$

2. To prove Theorem 1 below we use

Lemma 1. *If $P_H^*(E)=1$ we have for all $\mu\in\Gamma^\circ$*

$$\overline{\lim_{|t|\to\infty}}\left|\int_E e^{itx}\,d\mu(x)\right|=1.$$

Proof. Suppose that the lemma is false. Then there exists a function $\mu\in\Gamma^\circ$ so that for $t=t_0$ we have

$$\int_E e^{it_0x}\,d\mu(x)=e^{i\varphi_0}.$$

μ cannot be a step function. In that case the lemma follows immediately from Kronecker's theorem. Without loss of generality we put $\varphi_0=0$ and we have $d\mu=e^{-it_0x}\,dp(x),\ dp(x)\geqslant 0$.

Choose $\xi\in E$, a point of continuity of μ, so that $e^{it_0\xi}\neq 1$ and form

$$d\mu_1=e^{-it_0x}\,dp(x)+d\delta(x-\xi).\ [\delta(x)=0 \text{ for } x<0,\ =1 \text{ for } x\geqslant 0].$$

There is no τ satisfying $\left|\int_E e^{i\tau x}\,d\mu_1\right|=2$.

Thus $\overline{\lim}_{|t|\to\infty}\left|\int_E e^{itx}\,d\mu_1(x)\right|=2$ and $\overline{\lim}_{|t|\to\infty}\left|\int_E e^{itx}\,d\mu(x)\right|=1$ which is a contradiction that proves Lemma 1.

Lemma 2. *If $P_H^*(E)=1$ we have for all $\mu\in\Gamma^\circ$*

$$\overline{\lim_{|n|\to\infty}}\left|\int e^{inx}\,d\mu(x)\right|=1.$$

Proof. The same as for Lemma 1 if t is changed to n.

Cor. 1. *If $P_H^*(E)=1$ we have for all $\mu\in\Gamma^\circ$*

$$\underline{\lim_{t\to\infty}}\int_E|e^{itx}-1|\,|d\mu|=0.$$

Proof. Choose an arbitrary $\varepsilon > 0$. By Lemma 1 there is a sequence $\{t_\nu\}_1^\infty$ so that

$$\left|\int_E e^{it_\nu x + i\varphi} d\mu - 1\right| < \frac{\varepsilon^2}{6}, \quad \nu \geqslant \nu_0.$$

Put
$$d\mu_2 = e^{it_{\nu_0} x + i\varphi} d\mu = d\mu_2' + i\, d\mu_2''.$$

Then we have
$$\int_E (|d\mu| - d\mu_2') < \frac{\varepsilon^2}{6} \text{ and } \int_E |d\mu_2''| < \frac{\varepsilon^2}{6}.$$

By the triangle inequality we obtain for $\nu \geqslant \nu_0$

$$\frac{\varepsilon^2}{6} > \left|1 - \int_E e^{it_\nu x + i\varphi} d\mu\right| = \left|1 - \int_E e^{i(t_\nu - t_{\nu_0})x} d\mu_2\right| \geqslant$$

$$\geqslant \left|\int_E 1 - e^{i(t_\nu - t_{\nu_0})x} |d\mu|\right| - \int_E |e^{i(t_\nu - t_{\nu_0})x}| (|d\mu| - d\mu_2') - \int_E |e^{i(t_\nu - t_{\nu_0})x} d\mu_2''|$$

whence
$$\int_E 1 - \cos\,(t_\nu - t_{\nu_0})\,x\, |d\mu| < \frac{\varepsilon^2}{6} + \frac{\varepsilon^2}{6} + \frac{\varepsilon^2}{6} = \frac{\varepsilon^2}{2}$$

and by Schwarz's inequality

$$\int_E |e^{i(t_\nu - t_{\nu_0})x} - 1|\,|d\mu| = 2\int_E \left|\sin \frac{(t_\nu - t_{\nu_0})\,x}{2}\right| |d\mu| \leqslant 2\sqrt{\int_E \sin^2 \frac{(t_\nu - t_{\nu_0})\,x}{2}\,|d\mu|}$$

$$\int |e^{i(t_\nu - t_{\nu_0})x} - 1|\,|d\mu| < \varepsilon \text{ for } \nu \geqslant \nu_0$$

and Cor. 1 is shown.

Cor. 2. *If* $P_H(E) = 1$ *we have for all* $\mu \in \Gamma^\circ$

$$\varliminf_{n \to \infty} \int_E |e^{inx} - 1|\,|d\mu| = 0.$$

Proof. The same as for Cor. 1 if t is changed to n and Lemma 2 is used instead of Lemma 1.

Cor. 3. $P_H(E) = 1$ *implies* $P_C(E) = 1$

Proof. Suppose that $P_C(E) < 1$. Then there exists a function $\mu \in \Gamma^\circ$ with $\left|\int_E e^{inx} d\mu(x)\right| < r < 1$, $n \geqslant 0$. According to Lemma 2 we choose n_0 so that

$$\left|\int_E e^{in_0 x} d\mu(x)\right| > \frac{1+r}{2}$$

and by Cor. 2 there exists n_1 with the properties: $n_1 + n_0 > 0$ and

$$\int_E |e^{in_1 x} - 1|\,|d\mu| < \frac{1-r}{2}.$$

The triangle inequality gives

$$r > \left|\int_E e^{i(n_0+n_1)x}\, d\mu(x)\right| \geq \left|\int_E e^{in_0x}\, d\mu(x)\right| - \int_E \left|e^{i(n_0+n_1)x} - e^{in_0x}\right| |d\mu(x)| > \frac{1+r}{2} - \frac{1-r}{2} = r.$$

This contradiction proves Cor. 3.

Lemma 3. *From a set of rational numbers with denominator m, it is sufficient to remove $m-1$ to get an integer as the sum of the others.*

Proof. Enumerate the numbers $\left\{\frac{P_\nu}{m}\right\}_1^N$ and from the sums

$$\sum_1^k \frac{P_\nu}{m} = n_k + \frac{\alpha_k}{m} \qquad k = 1,2, \ldots, m.$$

At least two of the α_k:s are identical, e.g. α_{k_1} and α_{k_2}. Thus

$$\sum_{k_2+1}^{k_1} \frac{P_\nu}{m} = n_1.$$

These P_ν are removed and the others are enumerated again from 1 to $N-(k_1-k_2)$. We obtain a similar sum:

$$\sum_{k'_2+1}^{k'_1} \frac{P'_\nu}{m} = n_2.$$

Analogously we can continue until the number of terms is less than $m-1$. The sum of the others is then an integer, which was to be proved.

Theorem 1. *If $P_H^*(E)=1$, then E is a Kronecker set.*

Proof: Suppose that $P_H^*(E)=1$ and $P_H(E)<1$. Then there exists a function $\mu \in \Gamma^\circ$ satisfying

$$\left|\int_E e^{inx}\, d\mu(x)\right| < P_1, \text{ all } n,\ P_1 < 1. \tag{1}$$

By Lemma 1 there is a sequence $\{t_j\}_1^\infty$ with the properties:

$$\left|\int_E e^{it_jx}\, d\mu(x)\right| \to 1$$

and $$t_j - [t_j] \to \alpha, \qquad [t_j] = n_j.$$

Put $$\delta = \frac{1-P_1}{8} \text{ and } m = \left[\frac{8\pi}{1-P_1}\right] + 1.$$

For $j \geq j_0$ we have

$$1 - \left|\int e^{in_jx} e^{i\alpha x}\, d\mu(x)\right| < \delta. \tag{2}$$

μ is continuous on E^C and discontinuous on E^S. $E = E^C \cup E^S$ and $\int_E |d\mu| = a$. $a \neq 0$ for otherwise the theorem follows directly from Kronecker's theorem.

Before proving the theorem we make the following statement: To each number β and each ψ there exist an integer N_β and a constant ψ_β with the properties

$$\int_{E^C} |e^{iN_\beta x + i\varphi_\beta} - e^{i\beta x + i\psi}| |d\mu| < 2\delta$$

and
$$\int_{E^S} |e^{iN_\beta x + i\varphi_\beta} - 1| |d\mu| < \delta.$$

To show this we divide E^C into disjoint subsets E_ν, $\cup E_\nu = E^C$, restricted by the condition

$$\frac{a\delta}{2m!} \leq \int_{E_\nu} |d\mu| \leq \frac{a\delta}{m!}.$$

We construct new functions μ_ν of bounded variation on E by choosing

$$\begin{cases} d\mu_\nu = e^{-i\beta x - i\psi} |d\mu(x)| \text{ on } E_\nu \\ d\mu_\nu = |d\mu(x)| \text{ otherwise.} \end{cases}$$

$\int_E |d\mu_\nu| = 1$ and thus by Lemma 1 there exists a sequence t_k^ν and a constant φ_ν so that

$$\int_E e^{it_k^\nu x + i\varphi_\nu} d\mu_\nu \to 1 \tag{3}$$

and
$$t_k^\nu - [t_k^\nu] \to \alpha_\nu \qquad [t_k^\nu] = n_k^\nu.$$

However, α_ν is a rational number with denominator less than m. For suppose that the contrary is true. Then there are two integers q_1 and q_2 satisfying

$$q_1 \alpha_\nu + \alpha - q_2 = h, \text{ where } |h| < \frac{1}{m}.$$

For $k \geq k_0$ we have according to (3) and the definition of μ_ν

$$\int_{E-E_\nu} |e^{i(n_k^\nu + \alpha_\nu)x + i\varphi_\nu} - 1| |d\mu| < \frac{\delta}{q_1}$$

and thus, if $q_1 n_k^\nu + q_2 = l_k^\nu$,

$$\delta > \int_{E-E_\nu} |e^{i(q_1 n_k^\nu + q_1\alpha_\nu)x + iq_1\varphi_\nu} - 1| |d\mu| = \int_{E-E_\nu} |e^{il_k^\nu x + iq_1\varphi_\nu} e^{ihx} - e^{i\alpha x}| |d\mu|$$

whence
$$\int_E e^{il_k^\nu x + iq_1\varphi_\nu} - e^{i\alpha x}| |d\mu| < \delta + 2\pi h + \frac{a\delta}{m!} < 4\delta.$$

This gives

$$\left|\int_E e^{in_jx} e^{i\alpha x} d\mu(x) - \int_E e^{i(l_k^\nu + n_j)x + iq_1\varphi_\nu} d\mu(x)\right| < 4\delta.$$

For $j \geq j_0$ we have by (2)

$$\left|\int_E e^{i(l_k^\nu + n_j)x} d\mu\right| > 1 - 5\delta > P_1$$

contrary to (1), and we have proved that all α_ν are rational numbers with denominator $m!$.

We put $\varepsilon_1 = \frac{a\delta^2}{4m!}$. If $k \geq k_1$ we have by (3) for all ν

$$\left|\left|\int_{E_\nu} e^{-i\beta x - i\psi} e^{i(n_k^\nu + \alpha_\nu)x + i\varphi_\nu} |d\mu|\right| + \int_{E-E_\nu} e^{i(n_k^\nu + \alpha_\nu)x + i\varphi_\nu} |d\mu| - 1\right| < \left(\frac{\varepsilon_1}{2^{\nu+1}}\right)^2.$$

Applying Schwarz's inequality as in the proof of Cor. 1 we get:

$$\int_{E_\nu} \left|e^{i(n_k^\nu + \alpha_\nu)x + i\varphi_\nu} - e^{i\beta x + i\psi}\right| |d\mu| < \frac{\varepsilon_1}{2^\nu}$$

and

$$\int_{E-E_\nu} \left|e^{i(n_k^\nu + \alpha_\nu)x + i\varphi_\nu} - 1\right| |d\mu| < \frac{\varepsilon_1}{2^\nu}.$$

Since all α_ν are rational numbers with denominator $m!$ we have by Lemma 3 $\sum \alpha_\nu = N$, where the summation runs over all ν except at most $m!$. The sets E_ν corresponding to these indices are removed and we get a new set $E^{C'}$.

$$\int_{E^C - E^{C'}} |d\mu| < \frac{m!\, a\delta}{m!} = a\delta.$$

By the triangle inequality

$$\int_{E^{C'}} \left|e^{i(N + \Sigma' n_k^\nu)x + \Sigma' i\varphi_\nu} - e^{i\beta x + i\psi}\right| |d\mu(x)| = \sum_\nu{}' \int_{E_\nu} \left|e^{i(\Sigma\alpha_\nu + \Sigma' n_k^\nu)x + \Sigma' i\varphi_\nu} - e^{i\beta x + i\psi}\right| |d\mu| \leq$$

$$\leq \frac{2m!}{a\delta} \cdot \sum \frac{\varepsilon_1}{2^\nu} \leq \delta \text{ for } k \geq k_1.$$

Put $\sum_\nu{}' n_{k_1}^\nu = N_{k_1}$ and $\sum{}' \varphi_\nu = \varphi_\beta$.

Now we conclude that

$$\int_{E^C} \left|e^{i(N + N_{k_1})x + i\varphi_\beta} - e^{i\beta x + i\psi}\right| |d\mu| \leq \delta + a\delta < 2\delta.$$

The triangle inequality also gives

$$\int_{E^S} |e^{i(N+N_{k_1})x+i\varphi_\beta}-1|\,|d\mu| \leqslant \sum \frac{\varepsilon_1}{2^\nu} < \delta.$$

Put $N+N_{k_1}=N_\beta$ and our statement is proved.

Returning to the proof of the theorem we assume that μ is continuous except at the points λ_S. $\lambda_S-\lambda_1$ are linearly independent since $P^*(E)=1$. There are by Kronecker's theorem an integer M and a constant ψ so that

$$\int_{E^S} |e^{-iMx+i\psi}-e^{i\alpha x}|\,|d\mu| < \delta.$$

According to the preceeding statement there exist an integer N_M and a constant φ_M with the properties

$$\int_{E^C} |e^{iN_Mx+i\varphi_M}-e^{iMx-i\psi}|\,|d\mu| < 2\delta$$

and

$$\int_{E^S} |e^{iN_Mx+i\varphi_M}-1|\,|d\mu| < \delta.$$

By the triangle inequality

$$\begin{cases} \int_{E^C} |e^{i(N_M-M)x+i\varphi_M+i\psi}-1|\,|d\mu| < 2\delta \\ \int_{E^S} |e^{i(N_M-M)x+i\varphi_M+i\psi}-e^{i\alpha x}|\,|d\mu| < 2\delta. \end{cases} \tag{4}$$

There also exist an integer N_α and a constant φ_α so that

$$\begin{cases} \int_{E^C} |e^{iN_\alpha x+i\varphi_\alpha}-e^{i\alpha x}|\,|d\mu| < 2\delta \\ \int_{E^S} |e^{iN_\alpha x+i\varphi_\alpha}-1|\,|d\mu| < \delta. \end{cases} \tag{5}$$

(4) and (5) imply

$$\int_E |e^{i(N_\alpha+N_M-M)x+i\varphi_\alpha+i\varphi_M+i\psi}-e^{i\alpha x}|\,|d\mu| < 7\delta.$$

Put $N_\alpha+N_M-M=N'$ and $\varphi_\alpha+\varphi_M+\psi=\varphi'$.

For $j \geqslant j_0$ we have by (1) and (2)

$$1-P_1-\delta < |\int_E e^{in_jx}e^{i\alpha x}\,d\mu(x) - \int_E e^{i(n_j+N')x+i\varphi'}\,d\mu < 7\delta$$

whence $8\delta > 1-P_1$, which is a contradiction.

Thus we have proved that $P_H^*(E)=1$ implies $P_H(E)=1$ but by Cor. 3 this implies $P_C(E)=1$ and E is a Kronecker set which was to be proved.

Theorem 2. *If $P_H^*(E)>0$, then E is a weak Kronecker set.*

Proof: Carleson has shown in [1] that $P_H^*(E) > 0$ implies $P_H(E) > 0$. It is therefore sufficient to prove that $P_H(E) > 0$ implies $P_C(E) > 0$. We give an indirect proof and assume that $P_C(E) = 0$ while $P_H(E) > 0$.

Choose $0 < \varepsilon < P_H$. Since $P_C = 0$ there exists a function $\mu \in \Gamma^\circ$ satisfying

$$\Big|\int_E e^{inx}\, d\mu(x)\Big| < \frac{\varepsilon P_H}{4}, \quad n \geqslant 0.$$

By the Radon-Nikodym theorem we obtain

$$\Big|\int_E e^{-inx}\, d\mu(x)\Big| = \Big|\int_E e^{inx}\, \overline{d\mu(x)}\Big| = \Big|\int_E e^{inx} f_0(x)\, d\mu(x)\Big|,$$

where $|f_0(x)| = 1$ and f_0 is measurable (μ).

Then we can approximate $f_0(x)$ with a continuous function $\varphi(x)$, $|\varphi(x)| \leqslant 1$ in the sense that

$$\int_E |\varphi(x) - f_0(x)|\,|d\mu(x)| < \frac{\varepsilon}{4}.$$

Since $P_H > 0$, $\varphi(x)$ can be represented

$$\varphi(x) = \sum_{-\infty}^{\infty} a_\nu e^{i\nu x}, \text{ where } \sum_{-\infty}^{\infty} |a_\nu| \leqslant \frac{1}{P_H} + \varepsilon.$$

Choose N so that $\sum_{|\nu| > N} |a_\nu| < \frac{\varepsilon}{4}$. By the triangle inequality we obtain for $n \geqslant N$

$$\Big|\int_E e^{-inx}\, d\mu(x)\Big| \leqslant \Big|\int_E e^{inx}(f_0(x) - \varphi(x))\, d\mu(x)\Big| + \Big|\int_E e^{inx}\varphi(x)\, d\mu(x)\Big| <$$

$$< \frac{\varepsilon}{4} + \Big|\int_E e^{inx} \sum_{-N}^{N} a_\nu e^{i\nu x}\, d\mu(x)\Big| + \sum_{|\nu| > N} |a_\nu| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$

i.e.

$$\Big|\int_E e^{inx}\, d\mu(x)\Big| < \varepsilon \qquad |n| \geqslant N. \tag{6}$$

$\Big|\int_E e^{inx}\, d\mu(x)\Big|$ assumes its greatest value, which is called P_0, $1 \geqslant P_0 \geqslant P_H$, (7)
when $n = n_0$, $|n_0| \leqslant N$.

Hence we have

$$\Big|\int_E e^{inx}(1 \pm e^{2iNx})\, d\mu(x)\Big| \quad \begin{cases} \leqslant P_0 + \varepsilon, & |n| < 3N \\ < 2\varepsilon, & |n| \geqslant 3N \end{cases} \tag{8}$$

We form the functions $\alpha_1(x)$ and $\alpha_2(x)$.

$$\alpha_1(x) = \int_0^x (1 + e^{2iNx})\, d\mu(x), \quad \alpha_2(x) = \int_0^x (1 - e^{2iNx})\, d\mu(x). \tag{9}$$

Choose $\delta = \frac{1}{5}(P_0 - \varepsilon)$ and suppose that

$$\int_E |d\alpha_1| = \int_E |1 + e^{2iNx}| \, |d\mu(x)| < 1 + \delta$$

and
$$\int_E |d\alpha_2| = \int_E |1 - e^{2iNx}| \, |d\mu(x)| < 1 + \delta$$

This gives

$$\int_E (|1 + e^{2iNx}| + |1 - e^{2iNx}| - 2) \, d\mu < 2\delta.$$

whence
$$\int_E \frac{2|1 + e^{2iNx}| \, |1 - e^{2iNx}| \, |d\mu(x)|}{|1 + e^{2iNx}| + |1 - e^{2iNx}| + 2} < 2\delta$$

and
$$\int_E |1 - e^{4iNx}| \, |d\mu(x)| < 2(1 + \sqrt{2})\delta < 5\delta.$$

But by (6)

$$\varepsilon > \left| \int_E e^{i(n_0 + 4N)x} d\mu(x) \right| \geqslant \left| \int_E e^{in_0 x} d\mu(x) \right| - \left| \int_E e^{in_0 x} (1 - e^{4iNx}) \, d\mu(x) \right| > P_0 - 5\delta.$$

This is a contradiction and thus

$$\max_{\nu = 1,2} \int_E |d\alpha_\nu| \geqslant 1 + \frac{P_0 - \varepsilon}{5}.$$

Suppose that the maximum is assumed by α_1.

The function
$$\mu_1(x) = \frac{\int_0^x d\alpha_1(x)}{\int_E |d\alpha_1(x)|}$$

then belongs to Γ° and by (8)

$$\left| \int_E e^{inx} d\mu_1(x) \right| < 2\varepsilon, \quad |n| \geqslant 3N \tag{6'}$$

$\left| \int_E e^{inx} d\mu_1(x) \right|$ assumes its greatest value, which is called P_1 for $n = n_1$, (7')

$$n_1 \leqslant 3N, \qquad P_1 \leqslant \frac{P_0 + \varepsilon}{1 + \frac{P_0 - \varepsilon}{5}}.$$

In the same way as (8) and (9) were formed from (6) and (7) we now form (8′) and (9′) and obtain a new function $\mu_2(x) \in \Gamma^\circ$ with the property

$$\left|\int_E e^{inx}\,d\mu_2(x)\right| \begin{cases} \leqslant P_2,\ |n|\leqslant 3^2\cdot N, = P_2 \text{ for } n=n_2 \\ \leqslant 2^2\cdot\varepsilon,\ n\geqslant 3^2\cdot N \end{cases}$$

where
$$P_2 \leqslant \frac{P_1+2\varepsilon}{1+\dfrac{P_1-2\varepsilon}{5}}.$$

From $\mu_2(x)$ we construct $\mu_3(x)$ etc.

We get functions $\mu_k(x)\in\Gamma^\circ$ satisfying

$$\left|\int_E e^{inx}\,d\mu_k(x)\right|\leqslant P_k, \text{ where } P_k\leqslant \frac{P_{k-1}+2^{k-1}\cdot\varepsilon}{1+\dfrac{P_{k-1}-2^{k-1}\cdot\varepsilon}{5}}.$$

However, the sequence given by $a_0=1$, $a_k=\dfrac{a_{k-1}}{1+\dfrac{a_{k-1}}{5}}$ tends to zero. Further we have $P_0\leqslant 1$ and $\dfrac{d}{dx}\left(\dfrac{x}{1+\dfrac{x}{5}}\right)>0$, $0\leqslant x\leqslant 1$, whence $P_k\leqslant a_k+r_k(\varepsilon)$, where $r_k(\varepsilon)\to 0$ when $\varepsilon\to 0$.

We choose k_0 such that $a_{k_0}<\dfrac{P_H}{3}$ and ε_0 such that $r_{k_0}(\varepsilon_0)<\dfrac{P_H}{3}$ and $2^{k_0}\varepsilon_0<\dfrac{P_H}{3}$. If we start with $\varepsilon=\varepsilon_0$ we obtain a function μ_{k_0} satisfying

$$\sup_n \left|\int_E e^{inx}\,d\mu_{k_0}(x)\right|<\frac{2P_H}{3}.$$

This contradicts our assumption and the theorem is proved.

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