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On the boundary behavior of the derivative of analytic functions

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1. Introduction

In 1929 R. Nevanlinna [5] introduced the class of functions $f(z)$, analytic and bounded in the unit circle $|z|<1$, for which the radial limits $\lim_{r\to 1-0} f(re^{ix})$ are of modulus 1 for almost all x in the interval $0\leqslant x\leqslant 2\pi$. This class will here be called class (N). The set of arguments x, such that $\lim_{r\to 1-0} f(re^{ix})$ does not exist or is not of modulus 1, will be called the exceptional set of the function $f(z)$.

Each function in class (N) admits a representation

$$f(z)=B(z)\,E(z), \tag{1.1}$$

where

$$B(z)=e^{i\gamma}z^m\prod_k \frac{\bar a_k(a_k-z)}{|a_k|(1-\bar a_k z)} \tag{1.2}$$

$$(\gamma \text{ real, } m \text{ integer } \geqslant 0,\ 0<|a_k|<1, \text{ and } \sum_k (1-|a_k|)<+\infty)$$

is the Blaschke product, finite or infinite, extended over the zeros of $f(z)$ ordered after increasing modulus, and where

$$E(z)=\exp\left\{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right\}, \tag{1.3}$$

with a non-decreasing function $\mu(t)$, defined in the interval $0\leqslant t\leqslant 2\pi$, and with the property that $\mu'(t)=0$ for all t in that interval except possibly for a set of measure zero. The first extensive description of the properties of the functions belonging to class (N) was given almost simultaneously by Frostman [2] and Seidel [7].

Except in the trivial case when $\mu(t)$ is identically constant, there is at least one argument x, such that the symmetric derivative

$$\lim_{h\to+0}\frac{\mu(x+h)-\mu(x-h)}{2h}=+\infty,$$

and for any such argument x we have

$$\lim_{r\to 1-0} f(re^{ix}) = 0.$$

The aim of this paper is to investigate the boundary behavior of the derivative of $f(z)$ for such a point e^{ix} and, in particular, some local conditions on the function $\mu(t)$ will be given to ensure that

$$\lim_{r\to 1-0} f'(re^{ix}) = 0.$$

In the following section we restrict ourselves to the case that $f(z) = E(z)$, i.e. $f(z)$ has no zeros in $|z| < 1$ and in section 3 we consider the general case. In section 4 we state some theorems on the Lebesgue function constructed on a set of the Cantor type, to be used in section 5, where we construct some examples of functions $f(z)$ belonging to (N). Each of these functions will have the radial limit of the derivative equal to zero for every point in the exceptional set, except in a set which, in a sense to be defined in section 5, is of measure zero.

2. The boundary behavior of $E'(z)$

Theorem 1. *Let $E(z)$ be a function given by (1.3) and let x be a point in the interval $0 < x < 2\pi$. Suppose that there exists a number $\eta > \frac{1}{2}$, such that*

$$\lim_{h\to +0} \left(\frac{\mu(x+h) - \mu(x-h)}{2h} + \eta \log h\right) = +\infty. \tag{2.1}$$

Then

$$\lim_{r\to 1-0} E(re^{ix}) = \lim_{r\to 1-0} E'(re^{ix}) = 0.$$

Writing $E(z) = \exp\{-w(z)\}$ and $w(z) = u(z) + iv(z)$ we have the following two lemmata.

Lemma 1. $u((1-h)e^{ix}) \geqslant \dfrac{\mu(x+h) - \mu(x-h)}{h}, \quad 0 < h \leqslant 1.$

Lemma 2. $|w'(re^{ix})| \leqslant \dfrac{2}{1-r^2} u(re^{ix}) \leqslant \dfrac{2}{1-r} u(re^{ix}), \quad 0 \leqslant r < 1.$

Lemma 1 is essentially due to Fatou [1], p. 340 (for the proof see Frostman [2], p. 107–109) and Lemma 2 is given by Zygmund [8], p. 72 (for the proof see Zygmund [9], I, p. 258).

Proof of Theorem 1. It follows immediately from (2.1) that

$$\lim_{h\to +0} \frac{\mu(x+h) - \mu(x-h)}{2h} = +\infty.$$

Hence, by Lemma 1,

$$\lim_{r\to 1-0} u(re^{ix}) = +\infty \quad \text{and} \quad \lim_{r\to 1-0} |E(re^{ix})| = \lim_{r\to 1-0} \exp\{-u(re^{ix})\} = 0.$$

From Lemma 2 we obtain

$$|E'(re^{ix})| = |w'(re^{ix})| \exp\{-u(re^{ix})\} \leqslant \frac{2}{1-r} u(re^{ix}) \exp\{-u(re^{ix})\}$$

$$= 2u(re^{ix}) \exp\left\{-\left(1-\frac{1}{2\eta}\right) u(re^{ix})\right\} (\exp\{-\tfrac{1}{2}u(re^{ix}) - \eta \log(1-r)\})^{1/\eta}.$$

Hence, since

$$\lim_{r\to 1-0} u(re^{ix}) \exp\left\{-\left(1-\frac{1}{2\eta}\right) u(re^{ix})\right\} = 0 \text{ for } \eta > \tfrac{1}{2},$$

and since, by Lemma 1, with $r = 1-h$

$$\exp\{-\tfrac{1}{2}u(re^{ix}) - \eta \log(1-r)\} \leqslant \exp\left\{-\frac{\mu(x+h)-\mu(x-h)}{2h} - \eta \log h\right\}$$

it follows from condition (2.1) that

$$\lim_{r\to 1-0} E'(re^{ix}) = 0.$$

This proves the theorem.

Incidentally we remark that for any argument x, where the symmetric derivative of $\mu(x)$ is infinite or zero

$$\lim_{z\to e^{ix}} |E'(z)|(1-|z|) = 0,$$

where the limit is uniform in every symmetric triangular neighbourhood of e^{ix}. This is a consequence of Lemma 2.

3. The boundary behavior of $f'(z)$

In this section we consider the general case, when there are zeros of $f(z)$ in $|z|<1$, i.e. when the Blaschke product does not reduce to a constant.

Theorem 2. *Let $f(z) = B(z)E(z)$ be a function in (N) and let x be a point in the interval $0<x<2\pi$. Suppose that there exists a number η, such that*

$$\lim_{h\to +0} \left(\frac{\mu(x+h)-\mu(x-h)}{2h} + \eta \log h\right) = +\infty, \tag{3.1}$$

and either (a) $\eta \geqslant 1$ or (b) $\tfrac{1}{2} < \eta < 1$, and

$$\sum_k \frac{1-|a_k|}{|e^{ix}-a_k|^{2(1-\eta)}} < +\infty. \tag{3.2}$$

Then

$$\lim_{r\to 1-0} f(re^{ix}) = \lim_{r\to 1-0} f'(re^{ix}) = 0.$$

Proof of the theorem. The proof of case (*a*), being similar to that of case (*b*), will be omitted.

Suppose we have x and η, $\frac{1}{2}<\eta<1$, such that (3.1) and (3.2) hold. Since $|B(z)|\leqslant 1$ it follows from Theorem 1 that $\lim_{r\to 1-0} f(re^{ix})=0$. Differentiating (1.1) we get

$$f'(re^{ix}) = B(re^{ix})\,E'(re^{ix}) + B'(re^{ix})\,E(re^{ix}),$$

where, by Theorem 1, the first term tends to zero when $z\to 1-0$. To prove that the second term tends to zero we differentiate (1.2) and obtain

$$B'(z) = \frac{m\,B(z)}{z} + B(z)\sum_k \frac{1-|a_k|^2}{(z-a_k)(1-\bar a_k z)} = \frac{m\,B(z)}{z} - \sum_k \frac{\bar a_k}{|a_k|} B_k(z)\frac{1-|a_k|^2}{(1-\bar a_k z)^2},$$

where $B_k(z)$ is the Blaschke product obtained from $B(z)$ by omitting the factor corresponding to a_k. Since this factor has a modulus $\leqslant 1$ and since

$$|1-\bar a_k z|^2 = |1-a_k\bar z|^{2\eta}\,|\bar z(\bar z^{-1}-a_k)|^{2-2\eta} \geqslant (1-r)^{2\eta} r^{2-2\eta}\,|e^{ix}-a_k|^{2-2\eta}$$

we obtain

$$|B'(re^{ix})\,E(re^{ix})| \leqslant \frac{m}{r}\,|B(re^{ix})\,E(re^{ix})| + \frac{2\,|E(re^{ix})|}{(1-r)^{2\eta}\,r^{2(1-\eta)}}\sum_k \frac{1-|a_k|}{|e^{ix}-a_k|^{2(1-\eta)}}. \tag{3.3}$$

Furthermore, by Lemma 1, we have (with $r=1-h$)

$$|E(re^{ix})|\,(1-r)^{-2\eta} \leqslant \exp\left\{-2\left(\frac{\mu(x+h)-\mu(x-h)}{2h} + \eta\log h\right)\right\}.$$

Hence, by (3.1)

$$\lim_{r\to 1-0} E(re^{ix})\,(1-r)^{-2\eta} = 0$$

and thus (3.3) together with (3.2) implies that

$$\lim_{r\to 1-0} B'(re^{ix})\,E(re^{ix}) = 0.$$

This completes the proof of the theorem.

For each argument x, where $\mu(x+0)-\mu(x-0)>0$, the conditions of Theorems 1 and 2 are fulfilled and thus $\lim_{r\to 1-0} f'(re^{ix})=0$. It is natural to ask if there can be other values of x for which $f'(re^{ix})\to 0$ when $r\to 1-0$. In section 5 we can answer this question affirmatively by stating examples.

4. Lebesgue functions

In this section we investigate a special type of functions $\mu(x)$, which will be used in stating the examples in section 5. We begin by constructing a class of perfect sets.

By a dissection of the type $[\xi]$, $0<\xi<\frac{1}{2}$, of an interval $I: a\leqslant x\leqslant b$, we mean the dissection

$$a\leqslant x\leqslant a+\xi(b-a),\ a+\xi(b-a)<x<b-\xi(b-a),\ b-\xi(b-a)\leqslant x\leqslant b$$

into two closed intervals, each of measure $\xi m(I)$, and an open interval of measure $(1-2\xi)m(I)$. Now let Ξ denote a sequence of real numbers ξ_k, $0<\xi_k<\frac{1}{2}$, $k=1, 2, \ldots$, such that $\lim_{p\to\infty} 2^p \prod_{k=1}^{p}\xi_k=0$. Starting with the interval I we first perform a dissection of the type $[\xi_1]$ and obtain two closed intervals, which we call $\delta_{1,1}$ and $\delta_{1,2}$. By putting $C_1=\delta_{1,1}\cup\delta_{1,2}$ we obtain a closed set of measure $2\xi_1 m(I)$. On each of the intervals $\delta_{1,1}$ and $\delta_{1,2}$ we then perform a dissection of the type $[\xi_2]$. We obtain four closed intervals $\delta_{2,1}$, $\delta_{2,2}$, $\delta_{2,3}$ and $\delta_{2,4}$, and a closed set $C_2=\bigcup_{k=1}^{4}\delta_{2,k}$ of measure $m(C_2)=4\xi_1\xi_2 m(I)$. By repeating this procedure we get a sequence of closed sets $C_p=\bigcup_{k=1}^{2^p}\delta_{p,k}$ of measure $m(C_p)=2^p\prod_{k=1}^{p}\xi_k m(I)$. Finally we put $C(I,\Xi)=\bigcap_{p=1}^{\infty}C_p$. This is the well-known Cantor set constructed on the interval I by the sequence Ξ. It is readily seen that $C(I,\Xi)$ is a perfect set of measure zero and that each point in $C(I,\Xi)$ admits a representation

$$x=a+\sum_{n=1}^{\infty}\varepsilon_n r_n,$$

where ε_n is 0 or 1 and $r_n=(1-\xi_n)\prod_{k=1}^{n-1}\xi_k m(I)$ $\left(\prod_{k=1}^{0}\xi_k=1\right)$. In the sequel we mainly consider Cantor sets constructed by sequences Ξ, such that $0<\xi_k\leqslant\xi<\frac{1}{2}$, $k=1, 2, \ldots$. To indicate this we write $\Xi\leqslant\xi<\frac{1}{2}$.

The following lemma, essentially originating from Hausdorff [3] (cf. Salem [6], p. 73), will be of frequent use.

Lemma 3. *Let* $x'=a+\sum_{n=1}^{\infty}\varepsilon_n' r_n$ *and* $x=a+\sum_{n=1}^{\infty}\varepsilon_n r_n$ *be two different points in* $C(I,\Xi)$, $\Xi\leqslant\xi<\frac{1}{2}$. *Suppose that* $x'>x$ *and let* p *be the natural number for which*

$$\varepsilon_n'=\varepsilon_n \quad \textit{if} \quad n\leqslant p-1, \quad \varepsilon_p'=1, \quad \textit{and} \quad \varepsilon_p=0.$$

Then

$$A\prod_{k=1}^{p-1}\xi_k\leqslant x'-x\leqslant B\prod_{k=1}^{p-1}\xi_k, \tag{4.1}$$

where $A=(1-2\xi)m(I)$ *and* $B=m(I)$.

Proof of Lemma 3. We have

$$x' - x = r_p + \sum_{n=p+1}^{\infty} (\varepsilon_n' - \varepsilon_n)\, r_n.$$

It follows that

$$r_p - \sum_{n=p+1}^{\infty} r_n \leqslant x' - x \leqslant r_p + \sum_{n=p+1}^{\infty} r_n$$

and since
$$\sum_{n=p+1}^{\infty} r_n = \sum_{n=p+1}^{\infty} (1-\xi_n) \prod_{k=1}^{n-1} \xi_k\, m(I) = \prod_{k=1}^{p} \xi_k\, m(I)$$

we obtain
$$A \prod_{k=1}^{p-1} \xi_k \leqslant (1-2\,\xi_p) \prod_{k=1}^{p-1} \xi_k\, m(I) \leqslant x' - x \leqslant B \prod_{k=1}^{p-1} \xi_k.$$

This proves the lemma. Incidentally we remark that the condition $\Xi \leqslant \xi < \frac{1}{2}$ is not necessary to prove the right-hand side of (4.1).

We now construct a non-decreasing, continuous function $\mu(x)$ increasing at every point of $C = C(I, \Xi)$ and constant in each interval contiguous to C. First we define $\mu(x)$ on C by putting

$$\mu(x) = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n} \quad \text{for} \quad x = a + \sum_{n=1}^{\infty} \varepsilon_n r_n.$$

We observe that the endpoints of the intervals $\delta_{p,k}$ are contained in C. Let x_1 be the right-hand endpoint of $\delta_{p,k}$ and x_2 the left-hand endpoint of $\delta_{p,k+1}$. These points admit the representations

$$x_1 = a + \sum_{n=1}^{s-1} \varepsilon_n r_n + \sum_{n=s+1}^{\infty} r_n \quad \text{and} \quad x_2 = a + \sum_{n=1}^{s-1} \varepsilon_n r_n + r_s$$

and hence it follows

$$\mu(x_1) = \sum_{n=1}^{s-1} \varepsilon_n 2^{-n} + \sum_{n=s+1}^{\infty} 2^{-n} = \sum_{n=1}^{s-1} \varepsilon_n 2^{-n} + 2^{-s} = \mu(x_2).$$

Thus, in order to get a non-decreasing function, we put, in each component of $I - C$, $\mu(x)$ equal to the well-defined value at the endpoints of the component. The function, obtained in this way, will be called the Lebesgue function constructed on the Cantor set C. It is easy to verify that $\mu(x)$ is a continuous, non-decreasing function, such that $\mu'(x) = 0$ almost everywhere in I.

For each $x = a + \sum_{n=1}^{\infty} \varepsilon_n r_n \in C$ we introduce two sets of integers

$$N_0(x) = \{n;\ \varepsilon_n = 0\} \quad \text{and} \quad N_1(x) = \{n;\ \varepsilon_n = 1\}.$$

Let n_j^{ν}, $j = 1, 2, \ldots$, be the elements of $N_\nu(x)$, $\nu = 0, 1$, ordered as an increasing sequence. If x is not the right-hand endpoint of any $\delta_{p,k}$, the set $N_0(x)$ is infinite, and if x is not the left-hand endpoint of any $\delta_{p,k}$, the set $X_1(x)$ is infinite.

We state two theorems concerning the right-hand and the left-hand derivatives of $\mu(x)$ at a point $x \in C$.

Theorem 3. *Let x be a point in the Cantor set $C(I, \Xi)$, $\Xi \leqslant \xi < \frac{1}{2}$, and suppose that x is not the right-hand endpoint of any $\delta_{p,k}$. Let $\mu(x)$ be the Lebesgue function constructed on $C(I, \Xi)$. Then*

$$\lim_{h \to +0} \frac{\mu(x+h)-\mu(x)}{h} = +\infty \tag{4.2}$$

if and only if

$$\lim_{j\to\infty} 2^{n_j^0-1} \prod_{k=1}^{n_j^0-1} \xi_k = 0. \tag{4.3}$$

Theorem 4. *Let x be a point in the Cantor set $C(I, \Xi)$, $\Xi \leqslant \xi < \frac{1}{2}$, and suppose that x is not the left-hand endpoint of any $\delta_{p,k}$. Let $\mu(x)$ be the Lebesgue function constructed on $C(I, \Xi)$. Then*

$$\lim_{h \to +0} \frac{\mu(x)-\mu(x-h)}{h} = +\infty$$

if and only if

$$\lim_{j\to\infty} 2^{n_j^1-1} \prod_{k=1}^{n_j^1-1} \xi_k = 0.$$

Proof of Theorem 3. Let x be a point in $C = C(I, \Xi)$, which is not the right-hand endpoint of any $\delta_{p,k}$. Since C is perfect, there is in every neighbourhood of x an $x' \in C$, where $x' > x$. Let us prove that the right-hand derivative of $\mu(x)$ is infinite at the point x if and only if

$$\lim_{\substack{x' \to x-0 \\ x' \in C}} \frac{\mu(x')-\mu(x)}{x'-x} = +\infty. \tag{4.4}$$

The necessity of (4.4) is obvious. To prove the sufficiency, we observe that if $x+h\ (h>0)$ is situated in an interval where $\mu(x)$ is constant, we have

$$\frac{\mu(x_1)-\mu(x)}{x_1-x} \leqslant \frac{\mu(x+h)-\mu(x)}{h} \leqslant \frac{\mu(x_2)-\mu(x)}{x_2-x},$$

where x_1 and x_2 are the right-hand and the left-hand endpoints of the interval. Hence the sufficiency of (4.4) follows. Let x and x' in C, $x' > x$, admit the representations

$$x = a + \sum_{n=1}^{\infty} \varepsilon_n r_n \quad \text{and} \quad x' = a + \sum_{n=1}^{\infty} \varepsilon_n' r_n.$$

By Lemma 3 there is an integer j, such that

$$\mu(x')-\mu(x)=2^{-n_j^0}+\sum_{n=n_j^0+1}^{\infty}(\varepsilon_n'-\varepsilon_n)\,2^{-n}=\sum_{n=n_j^0+1}^{\infty}(1+\varepsilon_n'-\varepsilon_n)\,2^{-n}$$

and such that

$$\left(B\prod_{k=1}^{n_j^0-1}\xi_k\right)^{-1}\prod_{n=n_j^0+1}^{\infty}(1-\varepsilon_n)\,2^{-n}\leqslant\left(B\prod_{k=1}^{n_j^0-1}\xi_k\right)^{-1}\sum_{n=n_j^0+1}^{\infty}(1+\varepsilon_n'-\varepsilon_n)\,2^{-n}\leqslant\frac{\mu(x')-\mu(x)}{x'-x} \tag{4.5}$$

and
$$\left(A\prod_{k=1}^{n_j^0-1}\xi_k\right)^{-1}\sum_{n=n_j^0+1}^{\infty}(1+\varepsilon_n'-\varepsilon_n)\,2^{-n}\geqslant\frac{\mu(x')-\mu(x)}{x'-x}. \tag{4.6}$$

We will now prove that (4.2) holds if and only if

$$\lim_{j\to\infty}\prod_{k=1}^{n_j^0-1}\xi_k^{-1}\sum_{n=n_j^0+1}^{\infty}(1-\varepsilon_n)\,2^{-n}=+\infty. \tag{4.7}$$

Since $x'\to x+0$ implies $j\to\infty$, the sufficiency of (4.7) follows immediately from (4.5) and (4.4). Suppose that (4.2) holds and consider the sequence of points

$$x_j=a+\sum_{n=1}^{\infty}\varepsilon_n^j\,r_n\in C,\ \ j=1,2,\ldots,$$

where
$$\varepsilon_n^j=\begin{cases}\varepsilon_n & \text{if } n<n_j^0\\ 1 & \text{if } n=n_j^0\\ 0 & \text{if } n>n_j^0\end{cases}\qquad j=1,2,\ldots.$$

Since $j\to\infty$ implies $x_j\to x+0$, the necessity of (4.7) follows from (4.6) applied to x and x_j. However, since

$$2^{-n_{j+1}^0}\leqslant\sum_{n=n_j^0+1}^{\infty}(1-\varepsilon_n)\,2^{-n}\leqslant 2^{-n_{j+1}^0+1}, \tag{4.8}$$

(4.7) holds if and only if (4.3) holds. This proves Theorem 3. The proof of Theorem 4, being similar to that of Theorem 3, will be omitted.

Let $x=a+\sum_{n=1}^{\infty}\varepsilon_n r_n$ be a point, which is not the right-hand endpoint of any $\delta_{p,k}$ and consider the sequence of points $x_j=a+\sum_{n=1}^{\infty}\varepsilon_n^j r_n\in C,\ j=1,2,\ldots,$

where
$$\varepsilon_n^j=\begin{cases}\varepsilon_n & \text{if } n\neq n_j^0\\ 1 & \text{if } n=n_j^0\end{cases}\qquad j=1,2,\ldots.$$

From the definition of Ξ we have

$$\lim_{j\to\infty} 2^{n_j^0-1} \prod_{k=1}^{n_j^0-1} \xi_k = 0,$$

and thus we conclude from (4.5) that $(\mu(x_j)-\mu(x))/(x_j-x)\to+\infty$ when $j\to\infty$. In the same way, if $y\in C$ is not the left-hand endpoint of any $\delta_{p,k}$, we can find a sequence of points $y_j<y$ such that $(\mu(y)-\mu(y_j))/(y-y_j)\to+\infty$ when $y_j\to y-0$. When proving this, we have not used the condition $\Xi\leqslant\xi<\frac{1}{2}$ and thus we have the following theorem.

Theorem 5. *Let $C(I,\Xi)$ be a Cantor set and $\mu(x)$ the corresponding Lebesgue function. Then, at each point $x\in C(I,\Xi)$,*

$$\limsup_{h\to+0} \frac{\mu(x+h)-\mu(x-h)}{2h} = +\infty.$$

5. Examples of functions in (N)

We make use of the Cantor sets $C(0\leqslant x\leqslant 2\pi,\Xi)$, $\Xi\leqslant\xi<\frac{1}{2}$, and their corresponding Lebesgue functions. We prove the following theorem.

Theorem 6. *Let $f(z)=B(z)E(z)$ be composed of a Blaschke product $B(z)$ and a function*

$$E(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right\},$$

where $\mu(t)$ is the Lebesgue function constructed on the Cantor set

$$C = C(0\leqslant t\leqslant 2\pi,\Xi),\ \Xi\leqslant\xi<\tfrac{1}{2}.$$

Let x be a point in this set, such that at least one of the conditions

$$\lim_{j\to\infty} n_{j+1}^0\, 2^{n_{j+1}^0} \prod_{k=1}^{n_j^0-1} \xi_k = 0, \tag{5.1}$$

and

$$\lim_{j\to\infty} n_{j+1}^1\, 2^{n_{j+1}^1} \prod_{k=1}^{n_j^1-1} \xi_k = 0 \tag{5.2}$$

holds. Then

$$\lim_{r\to1-0} f(re^{ix}) = \lim_{r\to1-0} f'(re^{ix}) = 0.$$

Proof. We carry through the proof only for a point x, such that (5.1) holds. By Theorem 3, condition (5.1) implies that (4.2) holds. Hence, if $x+h\ (h>0)$ is situated in an interval where $\mu(x)$ is constant, and if h is sufficiently small, we have

$$\frac{\mu(x+h)-\mu(x-h)}{2h}+\log h \geq \frac{\mu(x+h)-\mu(x)}{2h}+\log h \geq \frac{\mu(x_1)-\mu(x)}{2(x_1-x)}+\log(x_1-x).$$

where x_1 is the right-hand endpoint of the interval of constancy. Thus, to prove the theorem it is enough to prove, by Theorem 2,

$$\lim_{\substack{x'\to x+0\\ x'\in C}}\left(\frac{\mu(x')-\mu(x)}{2(x'-x)}+\log(x'-x)\right)=+\infty. \tag{5.3}$$

However, by (4.5), (4.8), and Lemma 3, there is for every $x'\in C$ with $x'>x$ an integer j, such that

$$\frac{\mu(x')-\mu(x)}{2(x'-x)}+\log(x'-x)\geq\left(2B\,2^{n_j^*+1}\prod_{k=1}^{n_j^*-1}\xi_k\right)^{-1}+\log\left(A\prod_{k=1}^{n_j^*-1}\xi_k\right)$$

$$=\left(2^{n_j^*+1}\prod_{k=1}^{n_j^*-1}\xi_k\right)^{-1}\left(\frac{1}{2B}+\left(2^{n_j^*+1}\prod_{k=1}^{n_j^*-1}\xi_k\right)\log\left(A\,2^{n_j^*+1}\prod_{k=1}^{n_j^*-1}\xi_k\right)-\right.$$

$$\left.-\left(n_{j+1}^0 2^{n_j^*+1}\prod_{k=1}^{n_j^*-1}\xi_k\right)\log 2\right).$$

Hence, since $x'\to x+0$ implies $j\to\infty$, and since (5.1) holds, (5.3) follows, and we have proved the theorem.

Let S_0 be the set of points x, for which (5.1) holds and S_1 the set of points, for which (5.2) holds. In order to investigate to what extent the points in C belong to S_0 and S_1, we introduce for a set $S\subset C$ the measure

$$\mu(S)=\int_0^{2\pi}\chi_S(t)\,d\mu(t), \tag{5.4}$$

where $\chi_S(t)$ is the characteristic function of the set S and $\mu(t)$ is the Lebesgue function constructed on C. We say that S is measurable (μ) if the integral in (5.4) exists. Denote by S' the image of S under the transformation

$$\mu(x)\colon C\to\{x;\ 0\leqslant x\leqslant 1\}.$$

By the theorem of Lebesgue [4], p. 87, S is measurable (μ) if and only if S' is measurable in the ordinary sense, and then

$$\mu(S)=\int_0^{2\pi}\chi_S(t)\,d\mu(t)=\int_0^1\chi_{S'}(x)\,dx=m(S').$$

Theorem 7. *S_0 and S_1 are measurable (μ) and $\mu(S_0) = \mu(S_1) = 1$.*

Proof. For symmetric reason we restrict ourselves to S_0. Since $\mu(C) = 1$ it is enough to show that, for any ε, $0 < \varepsilon < 1$, there is a set $S_\varepsilon \subset S_0 \subset C$, such that $\mu(S_\varepsilon) > 1 - \varepsilon$ or, what is the same, $\mu(C - S_\varepsilon) < \varepsilon$.

Let ε be a number, such that $0 < \varepsilon < 1$, and let a be a number such that $a > 1$ and $a^2 < (-\log \xi)/\log 2$, where $\Xi \leq \xi < \frac{1}{2}$. We choose an integer K, such that

$$\sum_{k=K}^{\infty} 2^{-(a^{k+1} - a^k - 1)} < \varepsilon. \tag{5.5}$$

Next we devide the set of natural numbers into subsets Z_r, $r = 0, 1, 2, \ldots$, defined by

$$Z_0 = \{n: 1 \leq n < a^K\},\quad Z_r = \{n: a^{K+r-1} \leq n < a^{K+r}\},\quad r = 1, 2, \ldots.$$

Since $\varepsilon < 1$, (5.5) implies that $a^{K+r} - a^{K+r-1} > 1$, $r = 1, 2, \ldots$, and thus $Z_r \neq O$ ($=$ the empty set). Finally we put

$$S_\varepsilon = \{x: x \in C,\ Z_r \cap X_0(x) \neq O \text{ for each } r = 1, 2, \ldots\}.$$

Let $x \in S_\varepsilon$ and let $n_i^0 \in Z_r$ $(r \geq 1)$. Then n_{i+1}^0 belongs to Z_r or Z_{r+1} and hence

$$\frac{n_{i+1}^0}{n_i^0} < \frac{a^{K+r+1}}{a^{K+r-1}} = a^2. \tag{5.6}$$

Since $a^2 < (-\log \xi)/\log 2$, a simple calculation shows that (5.6) implies (5.1) and thus $S_\varepsilon \subset S_0$. In order to estimate $\mu(C - S_\varepsilon)$ we use

$$C - S_\varepsilon = \bigcup_{r=1}^{\infty} T_r, \text{ where } T_r = \{x;\ x \in C,\ Z_r \subset X_1(x)\},\ r = 1, 2, \ldots. \tag{5.7}$$

It is readily seen that T_r is measurable (μ) and that

$$\mu(T_r) = m(T_r') < 2^{-(a^{K+r} - a^{K+r-1} - 1)}$$

and thus, by (5.7), S_ε is measurable (μ) and

$$\mu(C - S_\varepsilon) \leq \sum_{r=1}^{\infty} 2^{-(a^{K+r} - a^{K+r-1} - 1)} < \varepsilon.$$

This completes the proof.

From Theorem 7 we see that S_0 and S_1 have the power of continuum. In fact, since $\mu(S_0) = 1$ we have $m(S_0') = 1$. Hence S_0' has the power of continuum and therefore also S_0.

In view of Theorem 7, each function $f(z)$ of the type defined in Theorem 6 has the property that $\lim_{r \to 1-0} f'(re^{ix}) = 0$ almost everywhere (μ) in the exceptional set, although the associated function $\mu(x)$ is continuous. Thus we have answered the question raised in section 3.

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