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On sign-independent and almost sign-independent convergence in normed linear spaces

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1. We shall say that a series of vectors in a normed vector space

$$X_1 + X_2 + X_3 + \cdots \tag{1.1}$$

is *sign-independently convergent (divergent)* if the series

$$\varepsilon_1 X_1 + \varepsilon_2 X_2 + \varepsilon_3 X_3 + \cdots \tag{1.2}$$

is convergent (divergent) for every sequence of signs $\{\varepsilon_k = \pm 1\}$.

It is an easy consequence of a result of Orlicz that in a Banach space the series (1.1) is sign-independently convergent if and only if it is unconditionally convergent, i.e. if and only if every series obtained from it by permutation of its terms is convergent. Therefore sign-independent convergence is equivalent to absolute convergence in finite-dimensional spaces. This is no longer the case in infinite-dimensional spaces. The well-known theorem of Dvoretzky and Rogers proves that if B is infinite-dimensional and $\{a_k\}$ a sequence of numbers such that $\sum_1^\infty a_k^2 < \infty$, then there is a sequence of unit vectors $\{x_k\}$ in B such that $\sum_1^\infty a_k x_k$ is sign-independently convergent.

Concerning sign-independent divergence of (1.1) in finite-dimensional spaces the necessary and sufficient condition that $\|X_k\| \not\to 0$ has been proved by Dvoretzky and Hanani. Here the infinite-dimensional spaces present a different situation, too. In 2° we shall show that if B is infinite-dimensional and $\sum_1^\infty a_k^2 = \infty$ then there is a sequence of unit vectors $\{x_k\}$ in B such that $\sum_1^\infty a_k x_k$ is sign-independently divergent.

We shall say that the series (1.1) is *almost sign-independently convergent (divergent)* if the series (1.2) is convergent (divergent) for almost every sequence of signs $\{\varepsilon_k = \pm 1\}$. This means that the series

$$\sum_1^\infty X_k \varphi_k(t),$$

where $\{\varphi_k(t)\}$ is the Rademacher function system, i.e.

$$\varphi_k(t) = \text{sign} \sin 2^k \pi t,$$

is convergent (divergent) almost everywhere in $0 \leqslant t \leqslant 1$.

A result of Rademacher states that if $X_k = a_k$ are real numbers and $\sum_1^\infty a_k^2 < \infty$ then the series (1.1) is almost sign-independently convergent. A counterpart of this is the theorem of Kolmogoroff, that $\sum_1^\infty a_k^2 = \infty$ implies almost sign-independent divergence of (1.1).

In 3°–5° we shall give generalizations of the theorems of Rademacher and Kolmogoroff, when the X_k:s are vectors in Hilbert space and certain Banach spaces.

I wish to express my gratitude to Professor Lennart Carleson for suggesting the problem and for invaluable advice.

2. Theorem 2.1. *Suppose B is an infinite-dimensional real normed vector space and $\{a_k\}$ is a sequence of real numbers such that*

$$\sum_1^\infty a_k^2 = \infty.$$

Then there is a sequence of unit vectors $\{x_k\}$ in B such that the series

$$\sum_1^\infty a_k x_k$$

sign-independently divergent.

The idea of the proof is the same as in Dvoretzky-Rogers's proof of their theorem. First we prove a geometrical lemma about n-dimensional Euclidean spaces and symmetric convex bodies there.

Lemma 2.1. *Let B be an n-dimensional real normed space; then, if $m(m-1) < n$, there exist points $x_1, \ldots, x_m$ of norm one in B, such that*

$$\left\| \sum_{i \leqslant m} t_i x_i \right\| \geqslant [1 - (m(m-1)/n)^{\frac{1}{2}}] \left(\sum_{i \leqslant m} t_i^2\right)^{\frac{1}{2}}$$

for all real $t_1, \ldots, t_m$.

Proof. Circumscribe to C, the unit ball of B, the ellipsoid E of minimum volume. By a linear transformation of B we may turn E into the Euclidean ball whose coordinates satisfy $\sum_{i \leqslant n} t_i^2 \leqslant 1$. Using subscripts to distinguish the norm with unit ball E from that with unit ball C, we want now to search for x_i with $\|x_i\|_E = \|x_i\|_C = 1$ and the x_i approximately orthogonal. By induction on i we shall find part of an orthonormal basis u_i, $i \leqslant m$, in the Euclidean space determined by E and points x_i, $i \leqslant m$, with $\|x_i\|_E = \|x_i\|_C = 1$, such that

(a) $x_i = \sum_{j \leqslant i} b_{ij} u_j$, and all $b_{ij} \geqslant 0$, and

(b) $b_{i1}^2 + \cdots + b_{i\,i-1}^2 = 1 - b_{ii}^2 \leqslant (i-1)/n$.

To begin the proof take $u_1 = x_1$ to be any point of contact of the surfaces C and E, and, for the moment let $u_2, \ldots, u_n$ be any vectors completing an orthonormal basis in E.

Suppose that $x_1, \ldots, x_i$ and $u_1, \ldots, u_i$, $1 \leqslant i < m$, have been found to satisfy (a) and (b) for all $j \leqslant i$; fill out an orthonormal basis with any suitable $u_{i+1}, \ldots, u_n$ and consider for $\varepsilon > 0$ the "spheroid" E_ε of points whose coordinates in this basis satisfy

$$(1+\varepsilon)^{-(n-i)}(\beta_1^2 + \cdots + \beta_i^2) + (1+\varepsilon+\varepsilon^2)^i(\beta_{i+1}^2 + \cdots + \beta_n^2) \leqslant 1.$$

The volume of E_ε is easily calculated to be smaller than that of E, so there is a point p_ε in C and, thus, in E, but outside E_ε. Thus, if $\beta_1, \ldots, \beta_n$ are the coordinates of p_ε,

$$\beta_1^2 + \cdots + \beta_n^2 \leqslant 1,$$

$$(1+\varepsilon)^{-(n-i)}(\beta_1^2 + \cdots + \beta_i^2) + (1+\varepsilon+\varepsilon^2)^i(\beta_{i+1}^2 + \cdots + \beta_n^2) > 1.$$

Subtracting the first of these inequalities from the second gives

$$[(1+\varepsilon)^{-(n-i)} - 1](\beta_1^2 + \cdots + \beta_i^2) + [(1+\varepsilon+\varepsilon^2)^i - 1](\beta_{i+1}^2 + \cdots + \beta_n^2) > 0.$$

By compactness, there is a subsequence of ε's tending to 0 such that the corresponding p_ε converge to some point x_{i+1} common to the surfaces C and E. Dividing the last inequality by ε and taking the limit gives, if x_{i+1} has the coordinates $b_1, \ldots, b_n$,

$$-(n-i)(b_1^2 + \cdots + b_i^2) + i(b_{i+1}^2 + \cdots + b_n^2) \geqslant 0.$$

Choosing u_{i+1} orthogonal to $u_1, \ldots, u_i$ in the space spanned by these and x_{i+1} and completing this sequence to a new orthonormal basis gives a representation for x_{i+1} which can now be seen to satisfy the conditions (a) and (b) for $i+1$. This induction process defines x_i and u_i for all $i \leqslant m$.

(b) implies that

$$\| x_i - u_i \|_E^2 = (1-b_{ii})^2 + \sum_{j<i} b_{ij}^2 \leqslant 2(i-1)/n.$$

Since C is inside E, $\|x\|_C \geqslant \|x\|_E$ for every x, so

$$\Big\| \sum_{i \leqslant m} t_i x_i \Big\|_C \geqslant \Big\| \sum_{i \leqslant m} t_i x_i \Big\|_E \geqslant \Big\| \sum_{i \leqslant m} t_i u_i \Big\|_E - \Big\| \sum_{i \leqslant m} t_i (x_i - u_i) \Big\|_E$$

$$\geqslant \Big\| \sum_{i \leqslant m} t_i u_i \Big\|_E - \sum_{i \leqslant m} |t_i| \, (2(i-1)/n)^{\frac{1}{2}}.$$

Use of Schwarz's inequality shows that this is

$$\geqslant \Big(\sum_{i \leqslant m} t_i^2\Big)^{\frac{1}{2}} - \Big(\sum_{i \leqslant m} t_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i \leqslant m} 2(i-1)/n\Big)^{\frac{1}{2}} = [1 - (m(m-1)/n))^{\frac{1}{2}}] \Big(\sum_{i \leqslant m} t_i^2\Big)^{\frac{1}{2}},$$

and the lemma is proved.

The following lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2. *If $4m(m-1)<n$ then in every n-dimensional real normed space B there exist m unit vectors $x_1, \ldots, x_m$, such that*

$$\|\sum_{i\leqslant m} t_i x_i\| \geqslant \tfrac{1}{2}(\sum_{i\leqslant m} t_i^2)^{\frac{1}{2}}.$$

Proof of Theorem 2.1. Take $\{a_k\}$ such that $\sum_1^\infty a_k^2=\infty$. Then group the terms in this sum in blocks

$$\{k: N_j < k \leqslant N_{j+1}\}$$

so that

$$\sum_{k=N_j+1}^{N_{j+1}} a_k^2 \geqslant 1.$$

In B find a sequence of finite-dimensional subspaces B_j each of such high dimension that in B_j vectors $x_{N_j+1}, \ldots, x_{N_{j+1}}$ satisfying Lemma 2.2 can be constructed. The series $\sum_1^\infty a_k x_k$ is obviously sign-independently divergent, as is seen by Cauchy's principle of convergence.

3. Theorem 3.1. *Let $\{x_k\}$ be a sequence of unit vectors in a complex Hilbert space and $\{a_k\}$ a sequence of complex numbers such that*

$$\sum_1^\infty |a_k|^2 < \infty.$$

Then the series

$$\sum_1^\infty a_k x_k$$

is almost sign-independently convergent.

Before we proceed to the proof we remark that Rademacher's theorem is a special case of Theorem 3.1 (all vectors x_k equally directed). Another special case of the theorem is an immediate consequence of Riesz-Fischer's theorem ($\{x_k\}$ an orthonormal system).

We shall need the following lemma, due to Hardy and Littlewood.

Lemma 3.1. *Suppose $f(t)$ is defined on $(0, 1)$ and put*

$$\theta(s; f) = \sup_\sigma \,(s-\sigma)^{-1}\int_\sigma^s f(t)\,dt, \qquad 0 \leqslant \sigma \leqslant 1.$$

If f belongs to $L^p(0,1)$, $p>1$, then $\theta(s;|f|)$ also belongs to $L^p(0,1)$ and

$$\int_0^1 \theta^p(s;|f|)\,ds \leqslant A_p \int_0^1 |f|^p\,dt,$$

where A_p depends only on p.

Proof of Theorem 3.1. The space spanned by the vectors x_k is mapped into $L^2(0,1)$ in the well-known manner. The vectors x_k are thereby mapped onto functions $f_k(x)$, and

$$\int_0^1 |f_k(x)|^2 dx = 1, \qquad k = 1, 2, \dots$$

The problem is then to show that the sequence $\{S_n(x,t)\}$, where

$$S_n(x,t) = \sum_1^n a_k f_k(x) \varphi_k(t),$$

converges in mean (in x) for almost every t in $(0, 1)$, as $n \to \infty$.

By the theorem of Beppo Levi we have

$$\int_0^1 \left(\sum_1^\infty |a_k f_k(x)|^2\right) dx = \sum_1^\infty \int_0^1 |a_k f_k(x)|^2 dx = \sum_1^\infty |a_k|^2 < \infty.$$

Thus $\sum_1^\infty |a_k f_k(x)|^2 < \infty$ for almost every x in $(0, 1)$. It follows by Riesz–Fischer's theorem, noting $\{\varphi_k(t)\}$ is an ON-system and regarding $a_k f_k(x)$ as coefficient of the function $\varphi_k(t)$, that for almost every x in $(0, 1)$ there is a function $F(x,t) = F_x(t) \in L^2(0,1)$, such that

$$\lim_{n=\infty} \int_0^1 |F(x,t) - S_n(x,t)|^2 dt = \lim_{n=\infty} \sum_{n+1}^\infty |a_k f_k(x)|^2 = 0. \tag{3.1}$$

Another integration gives

$$\lim_{n=\infty} \|F - S_n\|_U^2 = \lim_{n=\infty} \int_0^1 \int_0^1 |F(x,t) - S_n(x,t)|^2 dx\, dt = \lim_{n=\infty} \sum_{n+1}^\infty |a_k|^2 = 0. \tag{3.2}$$

(U stands for the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$, and $\|\dots\|_U$ for the norm in the Hilbert space of two-place functions $L^2(U)$.)

By Fubini's theorem

$$G_n(t) = \int_0^1 |F - S_n|^2 dx$$

exists for almost every t in $(0, 1)$. We want to show that $G_n(t) \to 0$ a.e. as $n \to \infty$. To this end we introduce the functions

$$H_N(t) = \sup_{n \geqslant N} G_n(t).$$

$\{H_N(t)\}$ is a monotonic decreasing sequence of positive functions. If we can show

$$\lim_{N=\infty} \int_0^1 H_N(t)\, dt = 0,$$

this implies $H_N(t) \to 0$ a.e., and a fortiori $G_n(t) \to 0$ a.e.

Now we have

$$\int_0^1 H_N(t)\,dt=\int_0^1 (\sup_{n\geqslant N}\int_0^1 |F-S_n|^2\,dx)\,dt\leqslant \int_0^1\int_0^1 \sup_{n\geqslant N} |F-S_n|^2\,dx\,dt=\|\sup_{n\geqslant N}|F-S_n|\,\|_U^2,$$

and Minkowski's inequality gives

$$\|\sup_{n\geqslant N}|F-S_n|\,\|_U\leqslant \|F-S_N\|_U+\|\sup_{n\geqslant N}|S_n-S_N|\,\|_U.$$

The first term in the right member tends to zero as $N\to\infty$ by (3.2). We want to show that the second term is majorized by the first term, multiplied by a constant.

Suppose t is a point of (0, 1) and I_n the interval of the type $(j\cdot 2^{-n}, (j+1)\cdot 2^{-n})$ which contains the point t. For $k\leqslant n$ the Rademacher functions $\varphi_k(t)$ are constant in I_n. Thus

$$S_n(x,t)=\frac{1}{|I_n|}\int_{I_n} S_n(x,t)\,dt, \tag{3.3}$$

where $|I_n|$ denotes the length of I_n. For $k>n$ the parts of I_n where $\varphi_k(t)$ is positive resp. negative have the same measure, thus

$$\frac{1}{|I_n|}\int_{I_n} S_n(x,t)\,dt=\frac{1}{|I_n|}\int_{I_n} S_m(x,t)\,dt, \qquad m\geqslant n. \tag{3.4}$$

Formula (3.1) gives (by Schwarz's inequality) that

$$\frac{1}{|I_n|}\int_{I_n} (F(x,t)-S_m(x,t))\,dt\to 0, \text{ as } m\to\infty. \tag{3.5}$$

For $n\geqslant N$ we conclude from (3.3), (3.4) and (3.5)

$$S_n(x,t)-S_N(x,t)=\frac{1}{|I_n|}\int_{I_n} (F(x,t)-S_N(x,t))\,dt,$$

and from this

$$\sup_{n\geqslant N}|S_n-S_N|\leqslant \sup_{n\geqslant N}\frac{1}{|I_n|}\int_{I_n}|F-S_N|\,dt\leqslant \theta(t;|F-S_N|),$$

where θ is the function of Lemma 3.1. Hence

$$\int_0^1 \sup_{n\geqslant N}|S_n-S_N|^2\,dt\leqslant A_2\int_0^1 |F-S_N|^2\,dt$$

by Lemma 3.1, and another integration gives

$$\| \sup_{n \geq N} |S_n - S_N| \|_U \leqslant \sqrt{A_2} \| F - S_N \|_U.$$

This completes the proof.

Theorem 3.2. *Let $\{f_k(x)\}$ be a sequence of functions in $L^p(0, 1)$, $1 < p < \infty$, such that*

$$\|f_k\| = \left(\int_0^1 |f_k(x)|^p \, dx \right)^{1/p} = 1, \qquad k = 1, 2, \ldots$$

Let $\{a_k\}$ be a sequence of complex numbers such that

a) $\sum_1^\infty |a_k|^p < \infty$ *in the case* $1 < p < 2$

b) $\sum_1^\infty |a_k|^2 < \infty$ *in the case* $2 \leqslant p < \infty$.

Then

$$\sum_1^\infty a_k f_k(x)$$

is almost sign-independently convergent in the metric of the space.

Lemma 3.2. *If $\{\varphi_k(t)\}$ is the Rademacher function system and*

$$s_n(t) = \sum_1^n a_k \varphi_k(t),$$

then for $p > 0$

$$\int_0^1 |s_n - s_m|^p \, dt \leqslant B_p \left(\sum_{m+1}^n |a_k|^2 \right)^{p/2},$$

where B_p depends only on p.

Lemma 3.2 is well known under the name of *Khintchine's inequality.*

Proof of Theorem 3.2. Minkowski's inequality and Hardy–Littlewood's lemma are still valid in the L^p-spaces. The reference made to Schwarz's inequality can be altered to Hölder–Riesz's inequality. Consequently, the theorem may be proved in the same manner as Th. 3.1 if only the existence of a function $F(x, t)$, satisfying (3.1) and (3.2) (with the exponent 2 replaced by p) can be motivated by other arguments.

Lemma 3.2 gives

$$\int_0^1 |S_n - S_m|^p \, dt \leqslant B_p \left(\sum_{m+1}^n |a_k f_k(x)|^2 \right)^{p/2}.$$

When $1 < p < 2$ this is

$$\leqslant B_p \sum_{m+1}^n |a_k|^p |f_k(x)|^p.$$

When $2 \leqslant p < \infty$ it is

$$\leqslant B_p \left(\sum_{m+1}^{n} |a_k|^2 \right)^{p/2-1} \cdot \sum_{m+1}^{n} |a_k|^2 \, |f_k(x)|^p$$

by Hölder's inequality. Another integration gives

$$\int_0^1 \int_0^1 |S_n - S_m|^p \, dx \, dt \leqslant \begin{cases} B_p \sum_{m+1}^{n} |a_k|^p & \text{for} \quad 1 < p < 2 \\ B_p \left(\sum_{m+1}^{n} |a_k|^2\right)^{p/2} & \text{for} \quad 2 \leqslant p < \infty. \end{cases}$$

From this follows the strong convergence of $S_n(x, t)$ to a function $F(x, t)$ in $L^p(U)$, i.e. the formula corresponding to (3.2) with the exponent 2 replaced by p. Use of B. Levi's theorem on the last integration (with the sum taken from $m+1$ to ∞) also gives the formula corresponding to (3.1).

4. The following examples show that the condition on the sequence $\{a_k\}$ in Theorem 3.2 cannot be weakened.

a) $1 < p < 2$.

Choose a sequence of intervals $\{I_k\}$ such that $I_k \subset (0, 1)$ for every k and $I_j \cap I_k = \emptyset$ when $j \neq k$. Put

$$f_k(x) = \begin{cases} |I_k|^{-\frac{1}{p}}, & x \in I_k \\ 0 & \text{elsewhere,} \end{cases}$$

where $|I_k|$ as before is the length of I_k. Then

$$\int_0^1 |f_k(x)|^p \, dx = 1$$

for every k, and, if we put $s_n(x) = \sum_1^n \varepsilon_k a_k f_k(x)$, where $\{\varepsilon_k\}$ is an arbitrary sequence of signs, we get

$$\int_0^1 |s_n(x)|^p \, dx = \sum_1^n |a_k|^p.$$

Hence, if $\sum_1^\infty |a_k|^p = \infty$, the series $\sum_1^\infty a_k f_k(x)$ is sign-independently divergent.

b) $2 \leqslant p < \infty$.

Let $\{f_k(x)\}$ be an ON-system (in L^2-sense) of functions in $L^p(0, 1)$ (take, for instance, a system of bounded functions). Then

$$\int_0^1 |s_n(x)|^p \, dx \geqslant \left(\int_0^1 |s_n(x)|^2 \, dx \right)^{p/2} = \left(\sum_1^n |a_k|^2 \right)^{p/2}.$$

Hence, if $\sum_1^\infty |a_k|^2 = \infty$, the series $\sum_1^\infty a_k f_k(x)$ is sign-independently divergent.

5. Theorem 5.1. *If* $\{x_k\}$ *is a sequence of unit vectors in a complex Hilbert space and* $\{a_k\}$ *a sequence of complex numbers, such that*

$$\sum_1^\infty |a_k|^2 = \infty,$$

then the series

$$\sum_1^\infty a_k x_k$$

is almost sign-independently divergent.

Proof: As in Theorem 3.1 we prove the theorem in the space $L^2(0,1)$. Proceeding indirectly, we suppose that the series $\sum a_k \varphi_k(t) f_k(x)$ is convergent in mean for $t \in E$, $mE > 0$. By Egoroff's theorem we can choose E in such a way that the convergence will be uniform in E. Then there is a constant M, such that

$$\|S_n - S_m\|_x = \left(\int_0^1 |S_n(x,t) - S_m(x,t)|^2 dx\right)^{\frac{1}{2}} < M$$

for $t \in E$. For

$$\|S_n - S_m\|_x \leqslant \|S_n - S_{N_0}\|_x + \|S_m - S_{N_0}\|_x$$

and we choose N_0 so that

$$\|S_n - S_{N_0}\|_x < \varepsilon \quad \text{for} \quad n > N_0, \; t \in E.$$

For $n < N_0$, $t \in E$, we obtain

$$\|S_n - S_{N_0}\|_x \leqslant \sum_1^{N_0} |a_k| = A,$$

and we may take $M = 2(A + \varepsilon)$.

Consequently

$$M^2 \cdot mE > \int_E \|S_n - S_m\|_x^2 dt = \int_0^1 \int_E \left|\sum_{m+1}^n a_k \varphi_k f_k\right|^2 dx\, dt =$$

$$= \int_0^1 (mE \cdot \sum_{m+1}^n |a_k f_k|^2 + 2\, Re\{\sum_{m+1 \leqslant i < k \leqslant n} a_i \bar{a}_k f_i \bar{f}_k \int_E \varphi_i \varphi_k\, dt\, dx\}. \tag{4.1}$$

The system of functions $\{\varphi_i \varphi_k\}_{1 \leqslant i < k,\, 1 < k < \infty}$ is an ON-system and the inequality of Bessel gives

$$\sum_{m+1 \leqslant i < k \leqslant n} (\int_E \varphi_i \varphi_k\, dt)^2 \leqslant \frac{(mE)^2}{9} \quad \text{for} \quad m, n > N(E),$$

since the left member is the sum of the squares of the coefficients in the development of the characteristic function of E.

An application of Cauchy's inequality gives

$$\left| Re \left\{ \sum_{m+1\leqslant i<k\leqslant n} a_i \bar{a}_k f_i \bar{f}_k \int_E \varphi_i \varphi_k \, dt \right\} \right| \leqslant \frac{mE}{3} \cdot \left(\sum_{m+1\leqslant i<k\leqslant n} |a_i a_k f_i f_k|^2 \right)^{\frac{1}{2}} \leqslant \frac{mE}{3} \cdot \sum_{m+1}^{n} |a_k f_k|^2.$$

From (4.1) we obtain

$$M^2 \cdot mE > \int_0^1 \left(mE \cdot \sum_{m+1}^{n} |a_k f_k|^2 - \frac{2mE}{3} \sum_{m+1}^{n} |a_k f_k|^2 \right) dx =$$

$$= \frac{mE}{3} \cdot \sum_{m+1}^{n} |a_k|^2 \int_0^1 |f_k|^2 \, dx = \frac{mE}{3} \cdot \sum_{m+1}^{n} |a_k|^2.$$

Thus $$\sum_{m+1}^{n} |a_k|^2 < 3M^2 \quad \text{for} \quad m, n > N(E).$$

This implies $\sum_1^\infty |a_k|^2 < \infty$ against the assumption, and theorem 5.1 is proved.

Added in proof: In a letter Prof. Dvoretzky has called my attention to his article "A theorem on convex bodies and applications to Banach spaces", Proc. Nat. Acad. Sci. (USA) 45, 223–226 (1959). With the help of his Theorem 1 it is easy to prove the following improvement of Theorem 2.1 of this note: Let B be an infinite-dimensional real normed space, $\{a_k\}$ and $\{b_k\}$ sequences of real numbers such that

$$\sum a_k^2 < \infty, \quad \sum b_k^2 = \infty.$$

Then there exists a sequence of unit vectors $\{x_k\}$ in B such that

$$\sum a_k x_k$$

is sign-independently convergent, while

$$\sum b_k x_k$$

is sign-independently divergent.

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