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On the existence of boundary values for harmonic functions in several variables

By Lennart Carleson

1. Let $u(z)$ be harmonic in $|z|<1$ and assume $u(z) \geqslant 0$. By the Poisson formula we have for $r<R<1$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \varphi)} u(Re^{i\varphi})\, d\varphi. \tag{1.1}$$

In particular,

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\varphi})\, d\varphi. \tag{1.2}$$

We can thus select a sequence $R_n \to 1$ so that $u(R_n e^{i\varphi})\, d\varphi$ converges weakly to some non-negative measure $d\mu$. We decompose $d\mu$ by Lebesgue's theorem:

$$d\mu = f(\varphi)\, d\varphi + ds(\varphi), \tag{1.3}$$

where $s(\varphi)$ is singular. Formula (1.1) becomes

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} P(r;\, \theta - \varphi) \left(f(\varphi) \frac{d\varphi}{2\pi} + \frac{1}{2\pi} ds(\varphi) \right), \tag{1.4}$$

where P is the Poisson kernel, i.e. the normal derivative of the Green's function.

The standard way to prove that

$$\lim u(z) \textit{ exists a.e., } z \to e^{i\theta} \textit{ non-tang.,} \tag{1.5}$$

is by means of a partial integration in (1.4) and Lebesgue's theorem on the existence of the derivative of an indefinite integral (see e.g. Nevanlinna [2], p. 190). This argument requires estimates of $\partial P/\partial \theta$, which makes generalizations difficult. However, using a slightly stronger version of Lebesgue's theorem, we obtain a proof not depending on partial integrations and therefore possible to generalize.

It is well known (see e.g. Zygmund [4], 65) that almost everywhere (θ)

$$\int_{-t}^{t}\{|f(\theta)-f(\varphi)|\,d\varphi+d s(\varphi)\}=o(t),\quad t\to 0. \tag{1.6}$$

We assume that (1.6) holds for $\theta=0$ and consider for simplicity only radial approach in (1.5). Choose $\delta>0$, fixed as $r\to 1$, and define N so that $2^N\eta\leqslant\delta<2^{N+1}\eta$, $\eta=1-r$. From (1.4) it follows

$$\begin{aligned}
|u(r)-f(0)| &\leqslant \left[\int_{-\eta}^{\eta}+\sum_{\nu=0}^{N}\int_{2^\nu\eta\leqslant|\varphi|<2^{\nu+1}\eta}\right]|f(\varphi)-f(0)|\,P\frac{d\varphi+ds}{2\pi}\\
&\quad +u(0)\operatorname*{Max}_{|\varphi|\geqslant\delta}P(r,\varphi)\\
&\leqslant o(\eta)\operatorname{Max}P+\sum_{\nu=0}^{N}o(2^\nu\eta)\operatorname*{Max}_{2^\nu\eta\leqslant|\varphi|}P+o(1)\\
&\leqslant o\left\{\sum_{\nu=0}^{N}\frac{2^\nu}{2^{2\nu}}\right\}=o(1).
\end{aligned} \tag{1.7}$$

2. We shall now use the above argument to prove a boundary value theorem for harmonic functions of several variables. It is closely related to previous works by Calderon and Stein [3]. The way of estimating the Green's function is taken from Calderon [1]. The lack of the method of conformal mapping introduces technical difficulties in proofs of rather evident results. This fact is clearly illustrated in section 4.

Before stating the theorem we introduce some notations. We consider points $P=(x_1, x_2, \dots, x_m; y)=(x; y)$ in $(m+1)$-dimensional Euclidean space. $|x|$ denotes distance on the m-dimensional subspace $X=\{P\,|\,y=0\}$, dx denotes the volume element in X. By $V_\alpha(x^0)$ we mean the cone

$$V_\alpha(x^0):\ |x-x^0|<\alpha y.$$

Theorem. *Let $u(P)$ be harmonic in $y>0$ and assume that for almost all $x\in X$, there is a cone $V_\beta(x)$ so that $u(P)$ is bounded from below in $V_\beta(x)$. Then*

$$\lim u(P),\quad P\to(x;\,0),\quad P\in V_\alpha(x), \tag{2.1}$$

exists a.e. on X for all α.

We consider only x's belonging to some bounded set, e.g. $|x|<1$. If we avoid an open subset O of measure $mO<\varepsilon$, we have for $y\leqslant y_0$ and a certain α independent of x, $u(P)\geqslant\text{Const}$, $P\in V_\alpha(x)$, $x\notin O$. We form the region

$$R=R(O)=\{\bigcup_{x\notin O}V_\alpha(x)\}\cap\{P\,|\,|x|<1,\ y<y_0\}.$$

If y_0 is large enough R is connected. We may assume that $u \geqslant 0$ in R. We observe that every boundary point P of R satisfies the Poincaré condition (some cone with vertex at P is contained in the complement of R). The Dirichlet problem can thus be solved for R. Let R_n be the part of R where $y > n^{-1}$ and let $G_n(P)$ be the Green's function for R_n with some fixed pole P_0. We need a uniform estimate of $G_n(P)$ (see Calderon [1]).

Let $\varphi(t)$ denote the distance from $t \in O$ to the complement O' of O and form

$$h(x;\, y) = y \int\limits_{O} \frac{\varphi(t)\, dt}{\{(t-x)^2 + y^2\}^{(m+1)/2}} = y\, h_1(x;\, y).$$

$h(x;\, y)$ is harmonic in $y > 0$. Observing that $\varphi(t) \geqslant \frac{1}{2}\varphi(x)$ if $|t-x| \leqslant \frac{1}{2}\varphi(x)$, we see that $h(x;\, C\varphi(x)) \geqslant \lambda_m C^{-m} \varphi(x)$, where λ_m only depends on m. (Points with $|x| = 1$ also have this property.) This implies that $h(x;\, z) \geqslant C'_\alpha \cdot z$, $z = y - n^{-1}$, for $(x;\, y)$ on the part of ∂R_n, where $n^{-1} < y < y_0$, $|x| < 1$. Let $G_n^*(P)$ be the Green's function for the cylinder $n^{-1} < y < y_0$, $|x| < 1$, with pole at P_0. Clearly, if $\delta > 0$ is given, there exist two constants c_1, $c_2 > 0$ so that

$$c_1 z \leqslant G_n^*(P) \leqslant c_2 z$$

if $|x| < 1 - \delta$ and $y < \delta$ say. The second relation holds for all $|x| < 1$. By the maximum principle

$$G_n(P) \geqslant G_n^*(P) - C_\alpha h(x;\, z) \quad \text{in} \quad R_n.$$

Hence for $c = c(\delta)$ independent of n and $|x| < 1 - \delta$, $y < \delta$,

$$G_n(P) \geqslant 2c(z - C_\alpha h(x;\, z)) = 2c \cdot z(1 - C_\alpha h_1).$$

We now need an estimate of $h_1(x;\, z) \leqslant h_1(x;\, 0)$. We have

$$\int\limits_{O'} h_1(x;\, 0)\, dx \leqslant \int\limits_{O} \varphi(t)\, dt \int\limits_{|x-t| \geqslant \varphi(t)} \frac{dx}{|x-t|^{m+1}}$$

$$\leqslant \lambda_m \int\limits_{O} dt = \lambda_m\, m\, O < \lambda_m\, \varepsilon.$$

Hence $h_1(x;\, z) \leqslant (2C_\alpha)^{-1}$ for all z, except when $x \in O_1$, $m\,O_1 < 2\lambda_m C_\alpha \varepsilon = \varepsilon_1$.

What will be needed of the above investigation of G_n is that

$$\frac{\partial G_n}{\partial n} \geqslant c \text{ for all } n, \quad P \in \partial R_n, \quad y = n^{-1},$$

except for x in a set S of measure $< \varepsilon + \varepsilon_1$.

We now consider the harmonic measure $\omega_n(e;\, P)$ of a certain subset e of ∂R_n at a point $P \in R_n$. If $P = P_0$, we delete the variable P. Harnack's inequality yields

$$M(P)^{-1} \leqslant \frac{\omega_n(e;\, P)}{\omega_n(e)} \leqslant M(P)$$

with $M(P)$ independent of n and e, $n > n(P)$. We can write $d\omega_n(\cdot\,;\, P) = K_n(\cdot\,;\, P)\, d\omega_n$. Here $K_n(P)$ is harmonic in P and satisfies the inequality above. Also $K_n(P_0) = 1$. We form $u_\varepsilon(P) = u(x;\, y+\varepsilon)$ and have

$$u_\varepsilon(P) = \int\limits_{\partial R_n} u_\varepsilon(Q)\, K_n(Q;\, P)\, d\omega_n(Q). \tag{2.2}$$

This formula corresponds to (1.1). Letting $n \to \infty$ we obtain with obvious notations

$$u_\varepsilon(P) = \int\limits_{\partial R} u_\varepsilon(Q)\, K(Q;\, P)\, d\omega(Q).$$

Letting $\varepsilon \to 0$ we get for a certain $f \in L^1(d\omega)$ and with s singular with respect to ω

$$u(P) = \int\limits_{\partial R} f(Q)\, K(Q;\, P)\, d\omega(Q) + \int\limits_{\partial R} K(Q;\, P)\, ds(Q).$$

(2.3) is the analogue of (1.4).

3. Let us consider a point $Q_0 = (x_0;\, 0) \in \partial R$ such that

(*a*) Q_0 is a point of density for the complement of $O_1 \cup O = S$;

(*b*) $\displaystyle\int\limits_{|x|<\varepsilon} |f(Q) - f(Q_0)|\, d\omega(Q) + \int\limits_{|x|<\varepsilon} ds(Q) = o(\varepsilon^m),\ \varepsilon \to 0,\ Q = (x + x_0;\, y).$

Since Lebesgue's theorem on symmetric derivatives holds for m dimensions, an inspection of the proof of (1.6) shows that (*b*), as well as (*a*), holds a.e. Namely, decompose $d\omega = \psi(Q)\, dQ + d\tau(Q)$ where $\psi \in L^1(dQ)$ and τ is singular with respect to Lebesgue measure. Then $f \in L^1(d\tau)$ and

$$\int\limits_{|x|<\varepsilon} |f(Q) - f(Q_0)|\, \psi(Q)\, dQ \leqslant \int\limits_{|x|<\varepsilon} |f(Q)\, \psi(Q) - f(Q_0)\, \psi(Q_0)|\, dQ$$

$$+ \int\limits_{|x|<\varepsilon} f(Q_0)\, |\psi(Q) - \psi(Q_0)|\, dQ = o(\varepsilon^m)$$

almost everywhere. Since τ is singular,

$$\int\limits_{|x|<\varepsilon} f(Q)\, d\tau(Q) = o(\varepsilon^m) \quad \text{a.e.}$$

We finally observe that $\partial G_n/\partial n \geqslant c$ for $(x; y) \in \partial R_n$, $x \notin S$. Since the surface element $d\sigma_n$ also satisfies an inequality $d\sigma_n \geqslant c\,dQ$, it follows that s is singular also with respect to Lebesgue measure.

Let us assume that $Q_0 = (0; 0)$ is a point, where (a) and (b) hold. We choose $A = (0; a)$, $a > 0$, and consider $u(A)$, as $a \to 0$. The general non-tangential approach is analogous. Define for a fixed $\delta > 0$

$$K_\nu = \{Q \mid Q \in \partial R,\ y < y_0,\ |x_i| < 2^\nu a\}$$

for $\nu = 0, 1, \dots, N$, $2^N a \leqslant \delta < 2^{N+1} a$, and

$$L_\nu = K_\nu - K_{\nu-1},\ \nu = 1, \dots, N,\ L_0 = K_0,$$

and

$$\Gamma = \partial R - K_N.$$

Formula (2.3) yields (cf (1.7))

$$|u(A) - f(Q_0)| \leqslant \left| \int (f(Q) - f(Q_0))\, K(Q; A)\, d\omega(Q) \right| + \int K(Q; A)\, ds(Q)$$

$$\leqslant \sum_{\nu=0}^{N} \sup_{Q \in L_\nu} K(Q; A)\, \varepsilon(\delta)\, 2^{m\nu} a^m + O(1) \sup_{Q \in \Gamma} K(Q; A).$$

We must study the harmonic functions $K(Q; A)$ for $Q = Q^{(\nu)} \in L_\nu$ and consider first the case $\nu = 0$.

Since $\partial G_n/\partial n \geqslant c$ for $(x; y) \in \partial R_n$, $x \notin O_1 \cup O$, it follows from condition (a) that the harmonic measure $v_0(P)$ of L_0 satisfies

$$v_0(P_0) \geqslant \gamma a^m, \tag{3.2}$$

where the constant γ is independent of a. We also observe that ∂R, $|x| < 1$, can be represented $y = \psi(x)$, where ψ satisfies a Lipschitz condition of order 1 and $\psi(x) = o(|x|)$, $|x| \to 0$.

We remove from R the set $|x_i| < 2a$, $y < ka$. The resulting domain is called R'. The harmonic measure of the part of $\partial R'$ with $|x_i| < 2a$ is called $v_0'(P)$. Since the harmonic measure of $\{P \mid P \in \partial R',\ |x_i| = 2a,\ y < ka\}$ with respect to R' is smaller than the harmonic measure of the same set with respect to $y > 0$, it follows that its value at $P_0 = O(k)\, a^m$. Hence $v_0'(P)$ also satisfies the inequality (3.2) if k is small enough.

We set $K(Q^{(0)}; A) = \mu_0$. From Harnack's inequality and the maximum principle it follows that

$$K(Q^{(0)}; P) \geqslant \text{Const. } \mu_0 v_0'(P).$$

Setting $P = P_0$ we find

$$\mu_0 \leqslant \text{Const. } a^{-m}. \tag{3.3}$$

We now choose $Q = Q^{(\nu)} = (x_\nu; y_\nu)$ and consider $B = (x_\nu; 2^\nu a)$, $\nu \leqslant N$. By (a), $\psi(x_\nu) = o(2^\nu a)$. We set $K(Q^{(\nu)}; B) = \mu_\nu$ and find as above

$$\mu_\nu \leqslant \text{Const. } a^{-m} 2^{-m\nu}.$$

On the other hand, $K(Q^{(\nu)}; P)/\mu_\nu$ is a positive harmonic function which vanishes on $\partial R - L_\nu$ and $=1$ for $P = B$. (In fact, one should first consider K_n; since all estimates are uniform, $n \to \infty$ causes no difficulty.) By the lemma in section 4 and the maximum principle

$$K(Q^{(\nu)}; P) \leqslant \text{Const. } \mu_\nu \cdot \int\limits_{(t;\, \psi(t) \in K_{\nu+1} - K_{\nu-2}} \frac{y\,dt}{\{(x-t)^2 + y^2\}^{(m+1)/2}}$$

in R. Inserting $P = A$ we find

$$K(Q^{(\nu)}; A) \leqslant \text{Const. } 2^{-\nu}\, 2^{-\nu m}\, a^{-m}. \tag{3.4}$$

Finally, if $Q \in \Gamma$, the argument giving (3.4) can be used for $\nu = N$ giving $\sup\limits_{Q \in \Gamma} K(Q; A) \to 0$, $a \to 0$. Inserting (3.3) and (3.4) we find $\lim\limits_{a \to 0} u(A) = f(Q_0)$ and the theorem is proved.

4. Lemma. *Let E be a subset of X in $|x| < 1$ and form for a fixed α*

$$R = \bigcup_{x \in E} V_\alpha(x) \cap \{P \mid |x| < 1,\ y < 1\}$$

and assume that the part Γ of ∂R with $|x| < 1$, $y < 1$ satisfies $y < \tfrac{1}{3}$. Let u be a positive harmonic function in R which vanishes continuously on ∂R except on the part of Γ which satisfies $|x| < \tfrac{1}{3}$. Then there exists a constant K, only depending on α, such that

$$u(x; y) < K \cdot u(0; \tfrac{1}{2}), \quad |x| = \tfrac{1}{2}. \tag{4.1}$$

By (2.2) it is sufficient to prove (4.1) when u is the harmonic measure of $\Gamma \cap \{P \mid |x - x_0| < \varrho\}$ for ϱ arbitrarily small and $|x_0| < \tfrac{1}{3}$. To simplify the notations we choose $x_0 = 0$. The proof shows that this is no restriction. We use the notation K_i for constants only depending on α.

Suppose that $(0; y_0) \in \Gamma$ and consider the sets D_ν:

$$D_\nu = R \cap \{P \mid |x| < 2^\nu \varrho,\ y \leqslant y_0 + K_1 2^\nu \varrho = \eta_\nu\},\quad \nu = 0, 1, \ldots .$$

If K_1 is large enough the boundary of D_ν consists of three parts: (1) a subset α_ν of Γ; (2) a subset β_ν of the cylinder $|x| = 2^\nu \varrho$; (3) a "disk" $\gamma_\nu : |x| < 2^\nu \varrho$, $y = \eta_\nu$. We use the notation $m_\nu = u(0; \eta_\nu)$. If K_1 is large enough it follows from Harnack's principle that

$$u(P) \leqslant K_2 m_\nu \text{ on } \gamma_\nu \tag{4.2}$$

and

$$m_{\nu-1} \leqslant K_2 m_\nu. \tag{4.3}$$

To be able to discuss $u(P)$ on β_ν we observe that R has the following property. If ξ is a given x-vector such that $|\xi| = 2^\nu \varrho$ and $\eta(\xi) < y < \eta_\nu$ is the corresponding subset of β_ν, then $\eta_\nu - \eta(\xi) < K_3 2^\nu \varrho$ and all points $(x; y)$ with $|x - \xi| < \alpha(y - \eta(\xi))$, $|x| < 1$, $y < 1$, belong to R. δ is a positive number to be determined later and we write $K_i(\delta)$ for functions of α and δ. (4.2) and the above mentioned property of R imply, again by Harnack's inequality, that

$$u(\xi;\, y) \leqslant K_4(\delta)\, m_\nu, \quad \eta(\xi) + \delta\, 2^\nu \varrho < y < \eta_\nu. \tag{4.4}$$

We shall now show by induction that, for ϱ small enough,

$$u(P) \leqslant K_5\, m_j, \quad P \in \beta_j \cup \gamma_j. \tag{4.5}$$

Let us first consider $j=0$. That (4.5) holds in this case is easily seen if we compare u with the harmonic measure of the bottom of a cylinder with radius ϱ and side $K_6\varrho$, evaluated at its center of gravity. We now assume that (4.5) holds for $j \leqslant \nu-1$. To prove (4.5) on $\beta_\nu \cup \gamma_\nu$ it is, by (4.2) and (4.4) only the part of β_ν with $\eta(\xi) < y \leqslant \eta(\xi) + \delta\, 2^\nu \varrho$ that has to be considered.

Let Σ be the following auxiliary domain

$$\Sigma = \{P \mid \alpha y > -|x|,\ |x-(-1, 0, \dots, 0)| > \tfrac{1}{2}\}$$

and let $\omega(P)$ be the harmonic measure of the part of $\partial\Sigma$ which is not the cone $\alpha y = -|x|$.

We now shrink Σ by a length factor $2^{\nu+1}\varrho$ and make a translation and rotation of the resulting domain to a domain with vertex of the cone at $(\xi;\, \eta(\xi))$ and axis of the cylinder along the y-axis. ω becomes ω_1 and it follows from the maximum principle, the induction assumption and (4.3) that

$$u(P) \leqslant K_5\, m_{\nu-1}\, \omega_1(P) \leqslant K_7\, m_\nu\, \omega_1(P)$$

in $D_\nu - D_{\nu-1}$. Since $\omega(0;\, y) \to 0$, $y \to 0$, it follows that $\omega_1(\xi;\, \eta(\xi) + s\, 2^\nu \varrho) < \varepsilon$ if $s < \delta(\varepsilon)$. Hence if $\delta = \delta(K_7^{-1} K_5)$, (4.5) is proved for $j = \nu$.

The induction can be continued as long as $2^\nu \varrho \leqslant \frac{1}{2}$. The maximum principle now shows that (4.1) holds.

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