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A property of bounded normal operators in Hilbert space

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Let H be a complex Hilbert space (i.e. a complete inner product space over the complex numbers) with scalar product $(\, . \, , \, . \,)$, norm $\| \, . \, \|$, and identity operator I. For any bounded linear operator B in H we denote the spectrum by $\operatorname{sp}(B)$ and the point-spectrum by $\operatorname{psp}(B)$. The smallest circular disc containing $\operatorname{sp}(B)$ is denoted by d_B, the boundary of d_B by c_B, and its center and radius by z_B and R_B, respectively. For a plane set S we denote the convex hull by $\operatorname{conv}(S)$. For u and v in H we define

$$\sin(u, v) = \left(1 - \frac{|(u, v)|^2}{\|u\|^2 \|v\|^2}\right)^{\frac{1}{2}}.$$

(If u or v is zero, we define $\sin(u, v) = 0$.)

The purpose of this note is to prove and discuss the following

Theorem 1. *Let N be a bounded normal operator in H. Then*

$$\sup_{u \neq 0} \left(\frac{\|Nu\|^2}{\|u\|^2} - \frac{|(Nu, u)|^2}{\|u\|^4} \right) = R_N^2, \tag{1}$$

or equivalently,

$$\sup_{u \neq 0} \frac{\|Nu\|}{\|u\|} \sin(u, Nu) = R_N. \tag{2}$$

Proof. We may evidently suppose that $R_N > 0$. For any complex λ and any $u \in H$ with $\|u\| = 1$,

$$\|Nu\|^2 - |(Nu, u)|^2 = \|Nu - \lambda u\|^2 - |(Nu, u) - \lambda|^2. \tag{3}$$

Hence, taking $\lambda = z_N$,

$$\sup_{u \neq 0} \left(\frac{\|Nu\|^2}{\|u\|^2} - \frac{|(Nu, u)|^2}{\|u\|^4} \right) = \sup_{\|u\|=1} \left(\|Nu - z_N u\|^2 - |(Nu, u) - z_N|^2\right)$$

$$\leqslant \sup_{\|u\|=1} \|Nu - z_N u\|^2 = \|N - z_N I\|^2 = R_N^2, \tag{4}$$

since the norm and the spectral radius of the normal operator $N - z_N I$ are equal.

We will now exhibit a sequence $u_n \in H$ such that

$$\left.\begin{aligned} &\|u_n\| \to 1, \\ &\|Nu_n - z_N u_n\|^2 \to R_N^2, \\ \text{and} \quad &(Nu_n, u_n) \to z_N \end{aligned}\right\} \quad \text{when} \quad n \to \infty. \tag{5}$$

The theorem will then be proved.

We first establish that $z_N \in \operatorname{conv}(\operatorname{sp}(N) \cap c_N)$. Otherwise, we could find a closed half-circle $h \subset c_N$ having no point in common with sp (N). Since h and sp (N) are compact, they would have a positive distance, and it would follow that sp (N) is contained in a smaller circular disc than d_N.

Thus there exist two or three distinct complex numbers $\lambda_1, \ldots, \lambda_r \in \operatorname{sp}(N)$ such that ($r = 2$ or 3)

$$\left.\begin{aligned} &|\lambda_i - z_N|^2 = R_N^2, \\ &z_N = \sum_{i=1}^{r} m_i \lambda_i \quad \text{with} \quad \sum_{i=1}^{r} m_i = 1 \quad \text{and} \quad m_i > 0. \end{aligned}\right\} \tag{6}$$

By the properties of spectrum there exist sequences $\{u_n^{(i)}\}_{n=1}^{\infty}$ ($i = 1, \ldots, r$) in H with

$$\|u_n^{(i)}\| = 1, \tag{7}$$

$$\|Nu_n^{(i)} - \lambda_i u_n^{(i)}\| \to 0 \quad \text{when} \quad n \to \infty. \tag{8}$$

Since N is normal and the λ_i are distinct, (8) implies

$$(u_n^{(i)}, u_n^{(j)}) \to 0 \quad \text{for} \quad i \neq j \quad \text{when} \quad n \to \infty. \tag{9}$$

Finally we define $u_n = \sum_{i=1}^{r} m_i^{\frac{1}{2}} u_n^{(i)}$. The verification of (5) is immediate, using (6), (7), (8) and (9). This completes the proof.

The proof can, of course, also be carried through by means of the spectral representation of N (for notation, cf. [2]),

$$N = \int z\, E(dx\, dy), \quad z = x + iy.$$

In this case the problem reduces to finding $\sup_{d\alpha \in \mathfrak{M}} I(d\alpha)$, where

$$I(d\alpha) = \int |z - z_N|^2 \, d\alpha(z) - \left| \int z \, d\alpha(z) - z_N \right|^2$$

and $\mathfrak{M}$ is the set of positive measures on sp (N) of the form $d\alpha = (E(dx\,dy)\,u, u)$ with $\|u\| = 1$. We may also write

$$2I(d\alpha) = \iint |z - \zeta|^2 \, d\alpha(z)\, d\alpha(\zeta).$$

One of the authors ([1], Theorems 5 and 6) has studied the problem of maximizing this integral for all positive measures on sp (N) with total mass 1, using essentially the above argument.

It follows from the proof that the sup in (1) and (2) is attained if and only if we can find $u \in H, \|u\|=1$, such that

$$\|Nu - z_N u\|^2 = \int |z - z_N|^2 (E(dx\,dy)\,u, u) = R_N^2,$$

$$(Nu, u) = \int z(E(dx\,dy)\,u, u) = z_N.$$

These equations express that $(E(dx\,dy)\,u, u)$ has its support on c_N and its center of gravity in z_N. In particular, the sup is attained if the points λ_i in (6) and (8) can be chosen as eigenvalues (and the sequences $u_n^{(i)}$ as constant sequences of eigenvectors), i.e. if $z_N \in \operatorname{conv}(\operatorname{psp}(N) \cap c_N)$. This situation occurs for instance if $\operatorname{sp}(N) = \operatorname{psp}(N)$, which is, of course, always the case if H is finite-dimensional.

If N is self-adjoint, so that sp (N) is a subset of the real line, then R_N is half the diameter of the spectrum, and the sup in (1) and (2) can be attained only in the case discussed above.

It is easy to construct an example of a (necessarily non self-adjoint) normal operator N in a (necessarily infinite-dimensional) Hilbert space H such that psp (N) is empty and still the sup in (1) and (2) is attained. (One can, for example, arrange that $\operatorname{sp}(N) = c_N$.)

We will apply Theorem 1 and its proof to the following problem: Given $u, f \in H$, what can be said about the spectrum of any bounded normal operator with $Nu = f$? The following result follows at once from (4).

Corollary 1. *Let $u, f \in H$, $\|u\| = 1$, and let $r = \|f\| \sin(f, u)$. Let N be a bounded normal operator such that $Nu = f$. Then*

$$r^2 \leqslant R_N^2 - |(f, u) - z_N|^2. \tag{10}$$

In particular $R_N \geqslant r$.

Corollary 1 has the following consequence for self-adjoint operators:

Corollary 2. *Let $u, f \in H$, $\|u\| = 1$, $r = \|f\| \sin(f, u)$, and let (f, u) be real. Let A be a self-adjoint operator such that $Au = f$. Then* conv(sp(A)) *contains some closed interval $[(f, u) - \alpha,\ (f, u) + \beta]$ with $\alpha\beta = r^2$.*

Corollary 2 is immediately obtained from Corollary 1. In fact, conv(sp(A)) $= [z_A - R_A, z_A + R_A]$ and we have to prove that

$$(z_A + R_A - (f, u))\ \ ((f, u) - z_A + R_A) \geqslant r^2,$$

which in this case is equivalent to (10).

As a trivial consequence of Corollary 2 we get:

Corollary 3. *Let $u, f \in H$, $\|u\| = 1$, $r = \|f\| \sin(f, u)$, and let $(f, u) > 0$. Let A be a non-negative, self-adjoint operator with $Au = f$. Then*

$$\sup_{\lambda \in \mathrm{sp}(A)} \lambda \geqslant (f,u) + r^2/(f,u) = \|f\|^2/(f,u).$$

This estimate, which can also be easily obtained directly, is stronger than the obvious ones, (f,u) and $\|f\|$. In fact, we have by the Cauchy inequality,

$$(f,u) \leqslant \|f\| \leqslant \|f\|^2/(f,u).$$

We conclude this paper by discussing the following problem: Given $u, f \in H$ with $\|u\| = 1$, can we always find a normal operator N with $Nu = f$ such that we get equality in (10)? The answer is contained in the following

Theorem 2. *Let $u, f \in H$, $\|u\| = 1$, and let $r = \|f\| \sin (f,u) > 0$. Assume that the complex number z and the positive number R satisfy*

$$r^2 = R^2 - |(f,u) - z|^2. \tag{11}$$

Then there exists a normal operator N with $Nu = f$ such that $R_N = R$ and $z_N = z$. In particular, for given $u, f \in H$, $\|u\| = 1$, it is always possible to find a normal operator N such that $Nu = f$ and $R_N = r$. In this case, necessarily $z_N = z = (f,u)$.

Proof. We first prove the theorem for the case $z = 0$ and (f,u) real. Consider the subspace G in H, spanned by u and f. We will prove that we can find two orthogonal vectors ψ_1 and ψ_2 such that

$$\left.\begin{aligned} u &= \psi_1 + \psi_2, \\ f &= R\psi_1 - R\psi_2. \end{aligned}\right\} \tag{12}$$

In fact, (12) is solved by

$$\psi_1 = \frac{1}{2}\left(u + \frac{f}{R}\right),$$

$$\psi_2 = \frac{1}{2}\left(u - \frac{f}{R}\right).$$

Using the definition of r, we get from (11)

$$(\psi_1, \psi_2) = \frac{1}{4}\left(1 - \frac{\|f\|^2}{R^2}\right) = \frac{R^2 - r^2 - (f,u)^2}{4R^2} = 0.$$

The linear operator A defined by

$$A(x_1\psi_1 + x_2\psi_2 + v) = x_1 R\psi_1 - x_2 R\psi_2, \quad v \perp G,$$

then satisfies the conditions in Theorem 2. The operator A is even self-adjoint.

The general case can now be reduced to the case considered above. Put

$$f' = e^{i\theta}(f - zu),$$

where the real number θ is chosen such that

$$(f', u) = e^{i\theta}((f, u) - z)$$

is real. We easily find (cf. (3)) that

$$r' = \|f'\| \sin(u, f') = \|f\| \sin(u, f) = r,$$

and therefore $$r'^2 = R^2 - (f', u)^2.$$

We can thus find a self-adjoint operator A with conv(sp(A)) equal to the closed interval $[-R, R]$ and such that $Au = f'$. The normal operator $N = e^{-i\theta} A + zI$ then satisfies the conditions in Theorem 2.

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