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A generalization of two-norm spaces

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Introduction

In some concrete Banach spaces Fichtenholz [8] introduced a kind of convergence weaker than that generated by the given norm. These considerations were generalized by Alexiewicz [1, 2], who introduced the concept of two-norm convergence or γ-convergence. Roughly speaking, a two norm space is a linear space provided with two norms $\| \ \|$ and $\| \ \|^*$, the second dominated by the first. A sequence $(x_n)_1^\infty$ in a two-norm space is said to be γ-convergent to x_0 if $\sup_n \|x_n\| < \infty$ and $\|x_n - x_0\|^* \to 0$, $n \to \infty$. The two-norm spaces, and in particular their linear functionals continuous with respect to the γ-convergence, were examined in great detail in a series of papers [3, 4, 5] of Alexiewicz and Semadeni. The Saks spaces, which have many properties in common with the two-norm spaces, have been studied by Orlicz [11, 12] and Orlicz and Pták [13]. It is natural to ask whether on a given two-norm space E there exists a topology, which generates the two-norm convergence, and thus makes it possible to apply the theory of linear topological spaces. The first to construct such a topology was Wiweger [15, 16], and he also proved its uniqueness under certain additional requirements. In this paper we intend to give an extension of the theory of two norm spaces to the more general situation of a linear space E provided with two locally convex topologies μ and τ, such that every τ-bounded set is μ-bounded. Thus we construct in a natural way a third topology μ^τ on E, called the mixed topology, which is uniquely determined and coincides with Wiweger's topology in the special case of a two-norm space. In this way the theory takes a form which very clearly shows its connection with well-known results in the theory of locally convex linear spaces, in particular with Grothendieck's construction [9] of the completion of the conjugate space E' of a locally convex linear space E. Although Wiweger's topology is well defined in the general case too, it does not seem to give an adequate generalization of the two-norm convergence. Using the theory of locally convex linear spaces we then give a detailed study of the space E endowed with the topology μ^τ and of its conjugate space, thus generalizing and sharpening known results for two-norm spaces. As an application of the theory we obtain in section 2 a characterization of (semi-) reflexive bornological spaces. For Banach spaces a similar criterion was given by Alexiewicz and Semadeni [5].

1. The mixed topology

Throughout the paper E shall denote a real or complex linear space, and whenever we speak of a topology τ on E we shall suppose that τ is locally convex and separated.

A subset M of E considered as a topological space in the topology induced by τ shall be denoted by $M[\tau]$. If τ_1 and τ_2 are two given topologies on E, we write $\tau_1 \leqslant \tau_2$ when τ_1 is weaker (coarser) than τ_2 or, equivalently, τ_2 is stronger (finer) than τ_1.

Definition 1.1. *A triplet $\{E, \mu, \tau\}$ of a linear space E and two topologies μ and τ on E such that every τ-bounded subset of E is μ-bounded is called a bitopological space. The finest locally convex topology on E which is identical with μ on the τ-bounded subsets of E is called the mixed topology on E and is denoted by μ^τ.*

Hence, by definition, μ^τ is the finest locally convex topology on E which has the property that the canonical injections

$$\varphi_B\colon B[\mu] \to E$$

are continuous for each τ-bounded subset B. It is obvious that one may here restrict oneself to all sets B in a fundamental system for the bounded subsets of $E[\tau]$, hence in particular to all absolutely convex τ-bounded subsets. Therefore a fundamental system of neighborhoods of 0 in $E[\mu^\tau]$ consists of all absolutely convex subsets W of E such that $\varphi_B^{-1}(W) = W \cap B$ is a neighborhood of 0 in $B[\mu]$ for each absolutely convex τ-bounded subset B of E.

It follows immediately from the definition of the mixed topology that

$$\mu \leqslant \mu^\tau, \tag{1.1}$$

and that $\mu^{\tau_1} = \mu^{\tau_2}$ for any two topologies τ_1 and τ_2 on E which have the same bounded subsets. In particular, if β denotes the bornological structure on E associated with the topology τ (see [6, Ch. III, § 2, ex. 13]) we have

$$\mu^\tau = \mu^\beta. \tag{1.2}$$

We also notice that

$$\mu \leqslant \beta; \tag{1.3}$$

for the identical mapping $I\colon E[\beta] \to E[\mu]$ of the bornological space $E[\beta]$ onto $E[\mu]$ maps bounded subsets onto bounded subsets, and thus is continuous. Another consequence of Definition 1.1 is that, if μ_1 and μ_2 induce the same topology on τ-bounded subsets, then $\mu_1^\tau = \mu_2^\tau$. In particular

$$\mu^\tau = (\mu^\tau)^\tau. \tag{1.4}$$

Proposition 1.1. *If $\{E, \mu, \tau\}$ is a bitopological space, then every τ-bounded subset B of E is μ^τ-bounded.*

Proof. Let $(x_n)_1^\infty$ be a sequence of elements in the τ-bounded set B and $(\lambda_n)_1^\infty$ a sequence of positive numbers such that $\lambda_n \to 0$ for $n \to \infty$. Because B is also bounded in the associated bornological space $E[\beta]$, it follows that $\lambda_n x_n \to 0$ in $E[\beta]$ and hence, on account of (1.3), in $E[\mu]$. But the sequence $(\lambda_n x_n)_1^\infty$ is clearly τ-bounded, and since μ^τ is identical with μ on τ-bounded subsets, we deduce that $\lambda_n x_n \to 0$ in $E[\mu^\tau]$. This shows that B is bounded in $E[\mu^\tau]$, and thus the proposition is proved.

Corollary 1.1. *If $\{E, \mu, \tau\}$ is a bitopological space and β the bornological structure on E associated with τ, then $\mu \leqslant \mu^\tau \leqslant \beta$.*

Proof. The first inequality is (1.1), and the second is proved exactly as (1.3).

Proposition 1.2. *Let $\{E, \mu, \tau\}$ be a bitopological space. Then a linear mapping u of $E[\mu^\tau]$ into a locally convex linear space F is continuous if and only if, for each τ-bounded subset B, its restriction to $B[\mu]$ (or, which is the same, to $B[\mu^\tau]$) is continuous. Moreover, among the topologies on E which are identical with μ on τ-bounded subsets, μ^τ is the only one that has this property.*

Proof. The first part of the proposition is an immediate consequence of the definition of μ^τ. On the other hand, let σ be a topology on E which satisfies the first statement of the proposition and which is identical with μ on τ-bounded subsets. Then it is obvious that the restriction to every τ-bounded subset B of the identical mapping I: $E[\sigma]\to E[\mu^\tau]$ is continuous. Hence I is continuous, that is $\mu^\tau \leqslant \sigma$. Since the inverse inequality is valid by definition, we conclude that $\mu^\tau = \sigma$. The proof is complete.

Among other things this result shows that the mixed topology is uniquely determined by the two conditions: (1) μ^τ is identical with μ on τ-bounded subsets; (2) a linear mapping u of $E[\mu^\tau]$ into a locally convex space F is continuous if and only if its restriction to $B[\mu]$ is continuous for every τ-bounded subset B of E. It follows in particular that, in the case of two-norm spaces, μ^τ coincides with the topology introduced by Wiveger ([16] Theorem 2.6.1).

Another immediate conclusion is the following:

Corollary 1.2. *Let $\{E,\mu,\tau\}$ be a bitopological space and F a complete locally convex linear space. Then the space $\mathcal{L}(E[\mu^\tau],F)$ of all linear continuous mappings of $E[\mu]$ into F is complete in the topology of uniform convergence on τ-bounded subsets of E and in the topology of uniform convergence on μ^τ-bounded subsets of E.*

Proof. We give the proof of the first case only. The second case follows from the proof of the first and Proposition 1.1. If ϕ is a Cauchy filter on $\mathcal{L}(E[\mu^\tau],F)$, the projected Cauchy filter $\phi(x)$ on F converges for each fixed $x\in E$, since F is complete. The limit $u(x)$ is a linear mapping of $E[\mu^\tau]$ into F, whose restriction to every τ-bounded subset B is continuous, for by assumption ϕ converges uniformly to $u(x)$ on B. Hence the desired result follows from Proposition 1.2.

In the theory of two-norm spaces the normal spaces play a fundamental role (see [3, 4, 16]). This notion has several possible generalizations to the general case. We shall adopt the following

Definition 1.2. *A bitopological space $\{E,\mu,\tau\}$ is called a-, b- and c-normal, respectively, if it satisfies the following requirements:* (a) *$E[\tau]$ has a fundamental system $\mathcal{B}$ for its bounded subsets such that every $B\in\mathcal{B}$ is absolutely convex and μ-closed;* (b) *$E[\tau]$ satisfies* (a) *and $\mathcal{B}$ is in addition countable;* (c) *$E[\tau]$ has a fundamental system $\mathcal{N}$ for its neighborhoods of* 0 *such that every $V\in\mathcal{N}$ is absolutely convex and μ-closed.*

Clearly every b- or c-normal bitopological space is a-normal. In the special case of a two-norm space all three kinds of normality coincide. It is also immediate that, if $\{E,\mu,\tau\}$ is a- or b-normal, the space $\{E,\mu,\beta\}$ is a- or b-normal, respectively.

Theorem 1.1. *If $\{E,\mu,\tau\}$ is b- or c-normal, then $E[\tau]$ and $E[\mu^\tau]$ have the same bounded subsets.*

Proof. Suppose first that $\{E,\mu,\tau\}$ is c-normal. In view of Proposition 1.1, it suffices to show that every μ^τ-bounded subset B is τ-bounded. This will in turn follow if we can prove that every sequence $(x_n)_1^\infty$ such that $x_n\to 0$ in $E[\mu^\tau]$ is bounded in $E[\tau]$. In fact, let B be μ^τ-bounded and assume that this condition is already proved. Then, if $(x_n)_1^\infty$ is a sequence in B and $(\lambda_n)_1^\infty$ a sequence of positive numbers with $\lim_{n\to\infty}\lambda_n=0$, it follows that $\sqrt{\lambda_n}x_n\to 0$ in $E[\mu^\tau]$, that is, $(\sqrt{\lambda_n}\,x_n)_1^\infty$ is bounded in $E[\tau]$. Hence $\lambda_n x_n \to 0$ in $E[\tau]$, which proves the desired fact that B is τ-bounded.

To conclude the proof, let $(x_n)_1^\infty$ be a sequence such that $x_n\to 0$ in $E[\mu^\tau]$ and suppose that $(x_n)_1^\infty$ is not τ-bounded. Since $\{E,\mu,\tau\}$ is c-normal, there exists an absolutely convex μ-closed neighborhood V of 0 in $E[\tau]$ and a subsequence $(x_{k_n})_1^\infty$ of $(x_n)_1^\infty$ such that

$$x_{k_n}\notin nV \quad (n=1,2,\ldots).$$

But all nV are closed in $E[\mu]$, so that we can find a sequence $(U_n)_1^\infty$ of neighborhoods of 0 in $E[\mu]$ with the property that

$$x_{k_n}\notin(nV+U_n) \quad (n=1,2,\ldots).$$

Consequently the absolutely convex set

$$W=\bigcap_{n=1}^{\infty}(nV+U_n)$$

does not contain any of the elements x_{k_n}, $n=1,2,\ldots$. Hence we are through, if we can prove that W is a neighborhood of 0 in $E[\mu^\tau]$; for we have then obtained a contradiction to the fact that $x_n\to 0$ in $E[\mu^\tau]$. But if B is any τ-bounded absolutely convex subset, then $B\subset nV$ for all sufficiently large n, say $n>n_0$. Therefore

$$W\cap B=\bigcap_{n=1}^{\infty}(nV+U_n)\cap B\supset\bigcap_{n=1}^{n_0}(U_n\cap B),$$

which proves that $W\cap B$ is a neighborhood of 0 in $B[\mu]$ for each τ-bounded absolutely convex subset B, that is, W is a neighborhood of 0 in $E[\mu^\tau]$. Hence the proof is complete in the case $\{E,\mu,\tau\}$ is c-normal. The other case is proved analogously.

In order to clarify the connection between the mixed topology and the γ-convergence introduced by Alexiewicz [2], we state the following consequence of Theorem 1.1.

Corollary 1.3. *Let $\{E,\mu,\tau\}$ be a b- or c-normal bitopological space. Then $x_n\to x_0$ in $E[\mu^\tau]$ if and only if $(x_n)_1^\infty$ is bounded in $E[\tau]$ and $x_n\to x_0$ in $E[\mu]$.*

Proof. If $x_n\to x_0$ in $E[\mu^\tau]$, then $(x_n)_1^\infty$ is bounded in $E[\mu^\tau]$, hence in $E[\tau]$, and $x_n\to x_0$ in $E[\mu]$ on account of (1.1). The converse is a direct consequence of the definition of μ^τ.

We shall now prove a result which shows, roughly speaking, that in the most interesting cases the mixed topology is in certain respects no "good" topology.

Proposition 1.3. *Let $\{E,\mu,\tau\}$ be a c-normal bitopological space such that $\tau=\beta$ is bornological and τ is not identical with μ on τ-bounded subsets. Then $E[\mu^\tau]$ is not quasi-tonnelé.*[1]

Proof. Let V be an absolutely convex and μ-closed neighborhood of 0 in $E[\tau]$. Since $\mu\leqslant\mu^\tau$, V is closed in $E[\mu^\tau]$, and since by Theorem 1.1 $E[\mu^\tau]$ and $E[\tau]$ have the same

[1] I borrow the French term.

bounded subsets, V absorbs every μ^τ-bounded subset of E. Therefore, if $E[\mu^\tau]$ is quasi-tonnelé, V is a neighborhood of 0 in $E[\mu^\tau]$, which proves that $\tau \leqslant \mu^\tau$. Hence, according to Corollary 1.1, $\tau = \mu^\tau$. This, however, contradicts the assumptions of the proposition, so that the proof is complete.

In particular, under the conditions of Proposition 1.3, the mixed topology is neither tonnelé, nor bornological. If in addition $E[\tau]$ has a countable fundamental system for its bounded subsets, which consists of sets that are metrizable in the topology μ, then $E[\mu^\tau]$ is not a (DF)-space. For a (DF)-space with this property is quasi-tonnelé (see [10, p. 71]). We conclude this section by noticing the following result, which is an immediate consequence of Definition 1.1. and a theorem of Raikov [14].

Preposition 1.4. *If* $\{E,\mu,\tau\}$ *has a countable fundamental system for its bounded sets consisting of subsets which are complete (and hence closed) in the topology induced by* μ, *then* $E[\mu^\tau]$ *is complete.*

2. Duality in bitopological spaces

The dual space of a locally convex linear space $E[\mu]$ will be denoted by $E[\mu]'$. Provided that nothing else is indicated, $E[\mu]'$ will be considered in the strong topology, that is, uniform convergence on μ-bounded subsets of E. For brevity we shall often use the notation $\sigma\mu$ for the weak topology $\sigma(E, E[\mu]')$ on E belonging to the duality between E and $E[\mu]'$.

An important property of a two-norm space $\{E,\mu,\tau\}$ is that $E[\mu^\tau]'$ is a closed subspace of $E[\tau]'$. This was shown by Orlicz and Pták [13]. In our general case one obtains:

Theorem 2.1. *Let* $\{E,\mu,\tau\}$ *be a bitopological space and, as before,* β *the bornological structure on* E *associated with* τ. *Then* $E[\mu^\tau]'$ *is complete and we have the (topological) inclusions*

$$E[\mu]' \subset E[\mu^\tau]' \subset E[\beta]'. \tag{2.1}$$

Moreover, $E[\mu^\tau]'$ *is a complete and hence closed subspace of* $E[\beta]'$.

Proof. The inclusions (2.1) follow at once from Corollary 1.1, and the rest of the theorem is a consequence of Corollary 1.2.

The theorem which we are now going to prove was stated for two-norm spaces by Alexiewicz and Semadeni in [3]. In view of Proposition 1.2 and Theorem 2.1 it is in fact a simple consequence of a more general result of Grothendieck [9]. However, for the convenience of the reader we give here a short proof extending the arguments in [3].

Theorem 2.2. *If* $\{E,\mu,\tau\}$ *is an a-normal bitopological space, the closure of* $E[\mu]'$, *considered as a subspace of* $E[\beta]'$, *is identical with* $E[\mu^\tau]'$.

Proof. In virtue of Theorem 2.1 it is sufficient to show that $E[\mu]'$ is dense in $E[\mu^\tau]'$ in the topology induced by $E[\beta]'$, i.e. in the topology of uniform convergence on τ-bounded subsets. Therefore, let η_0 be a given element in $E[\mu^\tau]'$ and B an arbitrary absolutely convex τ-bounded subset of E. Let

$$V(B,\eta_0)=\{\eta: |<x,\eta-\eta_0>| \leqslant 1,\, x\in B,\, \eta\in E[\mu^\tau]'\}$$

be the corresponding neighborhood of η_0. We shall prove that $V(B,\eta_0)$ contains some element $\xi\in E[\mu]'$. It is no restriction to assume that $\eta_0 \neq 0$. Let x_0 be a point in E for which $<x_0,\eta_0> = 1$, and denote by H the closed hyperplane in $E[\mu^\tau]$ on which η_0 equals zero. Then every $x\in E$ can be written uniquely in the form

$$x=h(x)+t(x)\cdot x_0,$$

where $h(x)\in H$ and $t(x)$ is a complex number. Since H is also closed in $E[\beta]$, the linear form $x\to t(x)$ on $E[\beta]$ is continuous, so that

$$\sup_{x\in B} |t(x)| < \infty.$$

Hence $h(x)$ runs through a τ-bounded subset $h(B)$ of H when x runs through B, for $h(B)\subset H \cap (B+t(B)\cdot x_0)$. Let B' be an absolutely convex μ-closed τ-bounded subset of E which contains $h(B)$. Then the closed absolutely convex hull B_0 of $h(B)$ in $E[\mu]$ is identical with the closed absolutely convex hull of $h(B)$ in $B'[\mu^\tau]$, and hence $B_0\subset H$. But $x_0 \notin B_0$, for $x_0\notin H$, so that according to the Hahn–Banach theorem there exists an element $\xi\in E[\mu]'$ such that $<x_0,\xi> = 1$ and $|<x,\xi>| \leqslant 1$ for $x\in B_0$.
Hence we obtain

$$|<x,\xi-\eta_0>| = |<h(x)+t(x)\cdot x_0,\xi-\eta_0>| = |<h(x),\xi>| \leqslant 1$$

for each $x\in B$, that is, $\xi\in V(B,\eta_0)$. The proof is complete.

Corollary 2.1. *If $\{E,\mu,\tau\}$ is an a-normal bitopological space, the weak topologies $\sigma\mu$ and $\sigma\mu^\tau$ induce the same topology on τ-bounded subsets of E.*

Corollary 2.2. *Let $\{E,\mu_1,\tau\}$ and $\{E,\mu_2,\tau\}$ be two a-normal bitopological spaces. If $E[\mu_1]'=E[\mu_2]'$, then $E[\mu_1^\tau]'=E[\mu_2^\tau]'$.*

Corollary 2.3. *If $\{E,\mu,\tau\}$ is a-normal, then the bitopological spaces $\{E,\sigma\mu,\tau\}$ and $\{E,\sigma\mu^\tau,\tau\}$ are a-normal, and we have the following identities:*

$$(\sigma\mu)^\tau=(\sigma\mu^\tau)^\tau, \tag{2.2}$$

$$\sigma\mu^\tau=\sigma(\sigma\mu)^\tau=\sigma(\sigma\mu^\tau)^\tau. \tag{2.3}$$

Proof. As the closure of any absolutely convex subset of E is the same in both of the toplogies μ and $\sigma\mu$, we conclude that $\{E,\sigma\mu,\tau\}$ is a-normal. Since $\sigma\mu\leqslant\sigma\mu^\tau$, this implies the a-normality of $\{E,\sigma\mu^\tau,\tau\}$, too. The equality (2.2) is an immediate consequence of Corollary 2.1 and the definition of the mixed topology. The second equality in (2.3) follows from (2.2), and the first is a consequence of Corollary 2.2, since $E[\mu]'=E[\sigma\mu]'$.

Let us shortly examine the two extreme cases that might occur in Theorem 2.1, that is, the cases $E[\mu^\tau]'=E[\beta]'$ and $E[\mu]'=E[\mu^\tau]'$.

Proposition 2.1. *For an a-normal bitopological space $\{E,\mu,\tau\}$ we have $E[\mu^\tau]'=E[\beta]'$ if and only if $\sigma\mu$ and $\sigma\beta$ are identical on τ-bounded subsets of E.*

Proof. The necessity of the condition follows at once from Corollary 2.1. Suppose conversely that the condition of the proposition is satisfied and let B be a τ-bounded set. The restriction to B of every $\eta \in E[\beta]'$ is continous in the topology $\sigma\beta$ and thus in $\sigma\mu$. Since $\sigma\mu \leqslant \mu$, we deduce that η is continuous on $B[\mu]$ for every τ-bounded set B and therefore, by Proposition 1.2, $\eta \in E[\mu^\tau]'$. Hence $E[\beta]' \subset E[\mu^\tau]'$, which proves the proposition, since the inverse inclusion is always valid.

For two-norm spaces the equality $E[\mu]' = E[\mu^\tau]'$ can only occur in the trivial case $\mu = \tau$ (see [3]). Since by (1.4) we have $E[\mu^\tau]' = E[(\mu^\tau)^\tau]'$, this proposition does not hold true in the general situation without certain restrictions on $\{E,\mu,\tau\}$. We have the following result.

Proposition 2.2. *Let $\{E,\mu,\tau\}$ be a b-normal bitopological space such that $E[\mu]$ is bornological and $E[\tau]$ and $E[\mu]$ have a countable fundamental system for its bounded sets. Then $E[\mu]' = E[\mu^\tau]'$ if and only if $\mu = \beta$. In particular $\mu = \tau$ if $E[\tau]$ is bornological.*

Proof. Since the dual space of a bornological space is complete, $E[\mu]'$ is a Fréchet space. The same conclusion can be drawn for $E[\mu^\tau]'$. In fact, $E[\mu^\tau]'$ is complete by Theorem 2.1 and the b-normality implies that $E[\mu^\tau]$ has a countable fundamental system for its bounded sets, i.e., $E[\mu^\tau]'$ is metrizable. In virtue of (2.1) the identical mapping I: $E[\mu]' \to E[\mu^\tau]'$ is continuous and hence, by a well-known theorem of Banach, $E[\mu]'$ and $E[\mu^\tau]'$ are isomorphical. This implies, however, that $E[\mu]$ and $E[\mu^\tau]$, and hence $E[\mu]$ and $E[\beta]$ (Theorem 1.1.), have the same bounded subsets. Since μ and β are bornological, we infer from this that $\mu = \beta$. The proof is finished.

A locally convex linear space $E[\mu]$ is said to be reflexive if the dual space $E[\mu]''$ of $E[\mu]'$ is identical with E (semi-reflexive in the sense of Bourbaki [6]), and it is said to be completely reflexive if it is reflexive and the strong topology on $E[\mu]'' = E$ is identical with the initial topology μ (reflexive in the sense of Bourbaki). We recall that $E[\mu]$ is completely reflexive if and only if it is reflexive and tonnelé. Hence it follows from Proposition 1.3 that we cannot expect $E[\mu^\tau]$ to be completely reflexive. We shall say that the bitopological space $\{E,\mu,\tau\}$ is μ-, μ^τ- and β-reflexive, when $E[\mu]$, $E[\mu^\tau]$ and $E[\beta]$, respectively, are reflexive.

Theorem 2.3. *Let $\{E,\mu,\tau\}$ be b- or c-normal. Then $E[\mu^\tau]$ is reflexive if and only if every τ-bounded subset B of E is relatively compact in the topology $\sigma\mu$.*

Proof. It follows from Theorem 1.1 that $E[\mu^\tau]$ and $E[\tau]$ have the same bounded sets. Moreover, the normality condition of the theorem implies that $\sigma\mu$ and $\sigma\mu^\tau$ induce the same topology on each τ-bounded subset B of E (Corollary 2.1), and that B has the same closure in $E[\mu^\tau]$ as in $E[\mu]$. Therefore the result is a consequence of the general criterion on reflexivity [6, Ch. IV, § 3].

Corollary 2.4. *If $\{E,\mu,\tau\}$ is b- or c-normal and μ-reflexive, then $E[\mu^\tau]$ is reflexive.*

Corollary 2.5. *If $\{E,\mu,\tau\}$ is b- or c-normal, $\mu \leqslant \tau$ and $E[\tau]$ reflexive, then $E[\mu^\tau]$ is reflexive.*

Proof. Let B be a τ-bounded subset of E and $\bar{B}$ its closure in $E[\sigma\mu]$. Then $\bar{B}$ is τ-bounded, and hence, in view of the reflexivity of $E[\tau]$, compact in $\sigma\tau$. But $\sigma\mu \leqslant \sigma\tau$, since $\mu \leqslant \tau$, and therefore $\bar{B}$ is compact in $\sigma\mu$, too. This proves the corollary.

As an application of the theory we shall now give a criterion on reflexivity of bornological spaces. This criterion is the extension to bornological spaces of a somewhat sharper result for Banach spaces, which was proved by Alexiewicz and Semadeni [5]. We also wish to point out that, according to the remark before Theorem 2.2, the proof of the necessity of the theorem is essentially a corollary to the theorem of Grothendieck [9].

Theorem 2.4. *Necessary and sufficient for a bornological space $E[\beta]$ to be reflexive is that $E[\mu]'$ is dense in $E[\beta]'$ for every locally convex topology μ on E weaker than β.*

Proof. Suppose that $E[\beta]$ is reflexive and let μ be a given topology on E weaker than β. Then every β-bounded subset B of E is compact in the topology $\sigma\beta$, and since $\sigma\mu \leqslant \sigma\beta$, we conclude that $\sigma\mu$ induces the same topology as $\sigma\beta$ on B. This implies in particular the a-normality of $\{E,\mu,\beta\}$, but also, in virtue of Proposition 2.1, that $E[\mu^\beta]' = E[\beta]'$. Hence, by Theorem 2.2, $E[\mu]'$ is dense in $E[\beta]'$.

Let conversely the condition of the theorem be fulfilled and suppose that $E[\beta]$ is not reflexive. Then there exists an element $x'' \neq 0$ of $E[\beta]''$ such that $x'' \notin E$, i.e. there is a closed hyperplane H of $E[\beta]'$, which is dense in the topology $\sigma(E[\beta]', E)$. The linear spaces E and H are therefore in duality with respect to the bilinear from $<x, x'>$. The associated weak topology $\sigma(E,H)$ on E is weaker than $\sigma\beta$, and hence also weaker than β. But the dual space of E in the topology $\sigma(E,H)$ is identical with H. Hence we have reached a contradiction, for by construction H is not dense in $E[\beta]'$. This completes the proof of the theorem.

3. Relativization of the mixed topology

If F is a subspace of the locally convex linear spaces $E[\mu]$, we write μ_F for the relativization of the topology μ to F. Every linear subspace F of a bitopological space $\{E,\mu,\tau\}$ determines in a natural way a bitopological space, namely $\{F,\mu_F,\tau_F\}$. It is easily seen that, if $\{E,\mu,\tau\}$ is a-, b- or c-normal, then $\{F,\mu_F,\tau_F\}$ is a-, b- or c-normal, respectively. One always has

$$(\mu^\tau)_F \leqslant \mu_F^{\tau_F}. \tag{3.1}$$

For $(\mu^\tau)_F$ induces the same topology as μ_F on bounded subsets of $F[\tau]$, and thus (3.1) is a consequence of the definition of $\mu_F^{\tau_F}$. The inverse inequality is in general false. Hence the problem arises of determining sufficient conditions on $\{E,\mu,\tau\}$ and F for the equality sign to be valid in (3.1). As will be seen below, this problem is also connected with the question, whether μ^τ is the *finest* and not only the *finest locally convex* topology on E which is identical with μ on τ-bounded subsets, in which case a subset H of $E[\mu^\tau]$ is closed if and only if $H \cap B$ is closed in $B[\mu]$ for every τ-bounded subset B of E.

Theorem 3.1. *Let $\{E,\mu,\tau\}$ be a bitopological space such that μ^τ is the finest topology on E which induces the same topology as μ on τ-bounded subsets. Then $(\mu^\tau)_F = \mu_F^{\tau_F}$ for every μ^τ-closed subspace F of E.*

Proof. On account of (3.1), it is enough to prove that every $\mu_F^{\tau_F}$-closed subset H of F is closed in the topology $(\mu^\tau)_F$. Let B_F be a τ_F-bounded subset of F, that is, $B_F = B \cap F$, where B is bounded in $E[\tau]$. Then $H \cap B_F = H \cap B$ is closed in $B_F[\mu]$. Since F is μ^τ-closed, B_F is closed in $B[\mu]$. Hence $H \cap B$ is closed in $B[\mu]$ for each τ-bounded subset B of E, i.e., H is closed in $E[\mu^\tau]$ and therefore in $F[\mu^\tau]$. This proves the theorem.

We shall now define an important type of bitopological spaces that satisfy the hypothesis of Theorem 3.1.

Definition 3.1. *A bitopological space* $\{E,\mu,\tau\}$ *is said to be* μ^τ*-compact if every* τ*-bounded subset of* E *is relatively compact in the topology induced by* μ.

In particular, if $\{E,\mu,\tau\}$ is *b*- or *c*-normal and μ^τ-compact, $\sigma\mu$ and μ induce the same topology on τ-bounded subsets, and $E[\mu^\tau]$ is reflexive (Theorem 2.3).

Proposition 3.1. *If* $\{E,\mu,\tau\}$ *is b-normal and* μ^τ*-compact, then* μ^τ *is the finest topology on* E *which induces the same topology as* μ *on* τ*-bounded subsets.*

Proof. The hypotheses imply that $E[\mu^\tau]'$ is a Fréchet space. Moreover, from the above remark it follows that E can be identified with the dual space of $E[\mu^\tau]'$ and that μ^τ is the finest locally convex topology which coincides with $\sigma\mu$ on τ-bounded subsets. However, a subset B of E is τ-bounded if and only if it is equi-continuous, regarded as a subset of the dual space of $E[\mu^\tau]'$. Hence the proposition is a consequence of a result of Dieudonné and Schwartz [7, p. 84].

Corollary 3.1. *If* $\{E,\mu,\tau\}$ *is b-normal and* μ^τ*-compact, then* $(\mu^\tau)_F=\mu_F{}^{\tau_F}$ *for every* μ^τ*-closed subspace* F *of* E.

It is an immediate consequence of the general criterion on reflexivity that, if $E[\mu^\tau]$ is reflexive, then $F[\mu^\tau]$ is reflexive for any μ^τ-closed subspace F of E. A little less trivial is the following:

Proposition 3.2. *Let* $\{E,\mu,\tau\}$ *be a b- or c-normal bitopological space and* F *a subspace of* E. *Then* $F[\mu^\tau]$ *is reflexive if and only if* $F[\mu_F{}^{\tau_F}]$ *is reflexive. In particular, if* $\{E,\mu,\tau\}$ *is* μ^τ*-reflexive and* F *is* μ^τ*-closed then* $\{F,\mu_F,\tau_F\}$ *is* $\mu_F{}^{\tau_F}$*-reflexive.*

Proof. Since in view of (3.1) we have

$$F\subset F[\mu_F{}^{\tau_F}]''\subset F[\mu^\tau]'',$$

$F[\mu_F{}^{\tau_F}]$ is reflexive if $F[\mu^\tau]$ is reflexive. The normality condition on $\{E,\mu,\tau\}$ implies that $F[\mu^\tau]$ and $F[\mu_F{}^{\tau_F}]$ have the same bounded sets as $F[\tau]$. Therefore, if $F[\mu_F{}^{\tau_F}]$ is reflexive and B is a bounded subset of $F[\mu^\tau]$, B is relatively compact in $\sigma(\mu_F{}^{\tau_F})$ and hence, by Corollary 2.1, in $\sigma\mu_F$. But for any locally convex topology μ and any subspace F of E we have $\sigma\mu_F=(\sigma\mu)_F$, so that, using Corollary 2.1 again, B is relatively compact in $(\sigma\mu^\tau)_F$. Hence $F[\mu^\tau]$ is reflexive, and the proof is complete.

In Corollary 3.1 we gave a sufficient condition for the equality $(\mu^\tau)_F=\mu_F{}^{\tau_F}$ to hold. However, it is also interesting to know when

$$\sigma(\mu^\tau)_F=\sigma\mu_F{}^{\tau_F}. \tag{3.2}$$

For it is easily seen that (3.2) is a necessary and sufficient condition for each linear functional on F, continuous with respect to the topology $\mu_F{}^{\tau_F}$, to have an extension to the whole of E, which is continuous on $E[\mu^\tau]$. We shall say that a subspace F satisfying (3.2) has the *extension property*. Corollary 3.1, or more general Theorem 3.1, gives, of course, a sufficient condition for F to have the extension property. However the following result, which is a generalization of a theorem of Wiweger [17], is more general.

Theorem 3.2. *If $\{E,\mu,\tau\}$ is b- or c-normal and μ^τ-reflexive, then every μ^τ-closed subspace F of E has the extension property.*

Proof. It is easily verified that $\{E,\sigma\mu,\tau\}$ satisfies the same normality conditions as $\{E,\mu,\tau\}$. Since $\sigma\mu \leqslant \sigma\mu^\tau$, the same proposition holds for $\{E,\sigma\mu^\tau,\tau\}$. The reflexivity of $E[\mu^\tau]$ implies that $\{E,\sigma\mu,\tau\}$ is $(\sigma\mu)^\tau$-compact. Since $\sigma\mu$ and $\sigma\mu^\tau$ induce the same topology on τ-bounded subsets of E (Corollary 2.1), it follows that $\{E,\sigma\mu^\tau,\tau\}$ is $(\sigma\mu^\tau)$-compact. Hence, by Corollary 3.1,

$$((\sigma\mu^\tau)^\tau)_F = (\sigma\mu^\tau)_F{}^{\tau F} \tag{3.3}$$

for each subspace F, which is closed in μ^τ, or, which in virtue of (2.3) is the same, in $(\sigma\mu^\tau)^\tau$.

By repeated use of (2.3) and the fact that, for any locally convex topology τ on E and any subspace $F, \sigma\tau_F = (\sigma\tau)_F$, we deduce from (3.3) that

$$\sigma\mu_F{}^{\tau F} = \sigma(\sigma\mu_F)^{\tau F} = \sigma(\sigma\mu)_F{}^{\tau F} = \sigma(\sigma\mu^\tau)_F{}^{\tau F} = \sigma((\sigma\mu^\tau)^\tau)_F = (\sigma(\sigma\mu^\tau)^\tau)_F = (\sigma\mu^\tau)_F = \sigma(\mu^\tau)_F.$$

Hence (3.2) is valid and the theorem is proved.

Corollary 3.2. *If $\{E,\mu,\tau\}$ is b- or c-normal and $E[\beta]$ reflexive, then F has the extension property for every β-closed subset F of E.*

Proof. In virtue of Corollary 2.5, $\{E,\mu,\beta\}$ is μ^β-reflexive, or, which is the same, $\{E,\mu,\tau\}$ is μ^τ-reflexive. Moreover, it follows from Theorem 2.4 that $\sigma\mu^\tau = \sigma\mu^\beta = \sigma\beta$. Hence every β-closed subspace is μ^τ-closed, so that the result is a consequence of Theorem 3.2.

BIBLIOGRAPHY

1. ALEXIEWICZ, A., On sequences of operations (II). Studia Math. *11*, 200–236 (1950).
2. —— On the two-norm convergence. Studia Math. *14*, 49–56 (1954).
3. ALEXIEWICZ, A., and SEMADENI, Z., Linear functionals on two-norm spaces. Studia Math. *17*, 121–140 (1958).
4. —— The two-norm spaces and their conjugate spaces. Studia Math. *18*, 275–239 (1959).
5. —— Some properties of two-norm spaces and a characterization of reflexivity in Banach spaces. Studia Math. *19*, 115–132 (1960).
6. BOURBAKI, N., Espaces vectoriels topologiques. Ch. I–V (Act. Sci. Ind. 1189, 1229), Paris, 1955).
7. DIEUDONNÉ, J., and SCHWARTZ, L., La dualité dans les espaces (F) et (LF). Ann. Inst. Fourier *1*, 61–101 (1949).
8. FICHTENHOLZ, G., Sur les fonctionnelles linéaires, continues au sens généralisé. Mat. Sb. *4*, 193–214 (1938).
9. GROTHENDIECK, A., Sur la complétion du dual d'un espace vectoriel localement convexe. C. R. Acad. Sci. *230*, 605–606 (1950).
10. GROTHENDIECK, A., Sur les espaces (F) et (DF). Summa Bras. Math. *3*, 57–122 (1954).
11. ORLICZ, W., Linear operations in Saks spaces (I). Studia Math. *11*, 237–272 (1950).
12. —— Linear operations in Saks spaces (II). Studia Math. *15*, 1–25 (1956).
13. ORLICZ, W., and PTÁK, V., Some remarks on Saks spaces. Studia Math. *16*, 56–68 (1957).
14. RAIKOV, D., On completeness in locally convex spaces. Uspehi Mat. Nauk *14*, 1 (85) 223–229 (1959).
15. WIWEGER, A., A topologisation of Saks spaces. Bull. Pol. Acad. Sci. *5*, 773–777 (1957).
16. —— Linear spaces with mixed topologies. Studia Math. *20*, 47–68 (1961).
17. —— Some applications of the mixed topology to the theory of two-norm spaces. Bull. Pol. Acad. Sci. *9*, 571–574 (1961).

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