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Bilateral Tauberian theorems of Keldyš type

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Introduction

Let T be the class of increasing (i.e. non-decreasing) functions defined for $x \geqslant 0$ and identically zero in neighbourhoods of the origin. Let $\varphi \in T(\alpha,\beta,a)$ or shorter $\varphi \in T(\alpha,\beta)$ if $\varphi \in T$, is differentiable and satisfies the basic inequality

$$\alpha\varphi(x) \leqslant x\varphi'(x) \leqslant \beta\varphi(x) \tag{1}$$

for $x > a$, $a = \text{constant} > 0$. Here α and β are constants for which always $0 \leqslant \alpha < \beta < \alpha + 1$. Avoiding the case $\varphi \equiv 0$ we can and shall also always assume $\varphi(a) > 0$.

We write $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to +\infty$, later on also when the independent variable tends to $-\infty$ or tends to infinity in the complex plane in certain ways. The integral part of a number β is denoted $[\beta]$.

Keldyš [1] has proved

Theorem A. *Let $\varphi \in T(\alpha,\beta)$ and suppose in case $\alpha = 0$ that $\varphi(x) \to \infty$ as $x \to \infty$. Write $[\beta] = m$. If for $\psi \in T$*

$$\int_0^\infty (u+x)^{-m-1} d\varphi(u) \sim \int_0^\infty (u+x)^{-m-1} d\psi(u)$$

then $\varphi \sim \psi$.

In his proof Keldyš first deduces the following theorem which, however, is not explicitly formulated in his paper.

Theorem B. *Let φ and ψ satisfy the same conditions as in Theorem A. Let k be a constant such that $\beta < k < \alpha + 1$. It then follows from*

$$\int_0^\infty (u+x)^{-k} d\varphi(u) \sim \int_0^\infty (u+x)^{-k} d\psi(u)$$

that $\varphi \sim \psi$.

In this paper we deduce theorems of similar type (with $\alpha > 0$) in the bilateral case in which integrals are considered over $(-\infty, +\infty)$ instead of over $(0, \infty)$. As preparation we study in section 1 unilateral Tauberian and Abelian theorems with the kernel $u^\delta(u+x)^{-1}$, where δ is a constant with $0 \leqslant \delta < 1$. Section 2 deals with the bilateral case for the kernel $(u+x)^{-1}$. In section 3 a unilateral Tauberian theorem is deduced for the kernel $u^{-h}(u+x)^{-m-1}$. By the help of these results a bilateral theorem with kernel $(u+x)^{-m-1}$ is obtained in the last section.

Å. Pleijel [2] has investigated the bilateral case with the kernel $u^{-h}(u+x)^{-1}$ for special explicitly given functions φ. I thank him for directing my interest to the problem of generalizing the results by Keldyš.

1. The unilateral case with the kernel $u^{\delta}(u+x)^{-1}$

In Keldyš's paper [1] we find

Lemma 1. *If $\varphi \in T(\alpha,\beta,a)$ then $x^{-\alpha}\varphi(x)$ increases and $x^{-\beta}\varphi(x)$ decreases for $x \geqslant a$.*

For our estimates we also need

Lemma 2. *Let $\varphi \in T(\alpha,\beta,a)$ with $0<\alpha<\beta<1$ and let δ be a constant such that $0 \leqslant \delta < 1-\beta$. Then there are positive constants A_1, A_2 so that*

$$A_1 x^{\delta-1}\varphi(x) \leqslant \int_0^{\infty} u^{\delta}(u+x)^{-1}d\varphi(u) \leqslant A_2 x^{\delta-1}\varphi(x) \tag{2}$$

is valid for all $x>a$.

Proof. The integral is diminished if taken only over $(x,2x)$. Since φ is increasing and since according to (1) $d\varphi(u) \geqslant \alpha u^{-1}\varphi(u)du$ for $u>a$, it is clear that $d\varphi(u) \geqslant \alpha\varphi(x)u^{-1}du$ in $(x,2x)$ provided $x>a$. This proves the left hand side of (2). To show the right-hand side the domain of integration is divided into $(0,a)$, (a,x) and $(x,+\infty)$. The integral over $(0,a)$ is evidently $O(x^{-1})$ which can be replaced by $O(x^{\delta-1}\varphi(x))$ since $x^{\delta}\varphi(x)$ has a positive lower bound when $x>a$. According to Lemma 1, $u^{-\alpha}\varphi(u) \leqslant x^{-\alpha}\varphi(x)$ for $a \leqslant u \leqslant x$ and $u^{-\beta}\varphi(u) \leqslant x^{-\beta}\varphi(x)$ for $x \leqslant u < \infty$. These estimates added to $d\varphi(u) \leqslant \beta u^{-1}\varphi(u)du$ give the appropriate estimates for the integrals over (a,x) and $(x,+\infty)$. The convergence of the occurring integrals is clear since $\alpha>0$, $\delta \geqslant 0$ and $\beta+\delta<1$.

Theorem 1. *Let $\varphi \in T(\alpha,\beta,a)$ with $0<\alpha<\beta<1$ and let $\psi \in T$. Let δ be a constant such that $0 \leqslant \delta < 1-\beta$ and write*

$$f(x) = \int_0^{\infty} u^{\delta}(u+x)^{-1}d\varphi(u),$$

$$g(x) = \int_0^{\infty} u^{\delta}(u+x)^{-1}d\psi(u).$$

Then $f \sim g$ if and only if $\varphi \sim \psi$.

Proof. Assume first $f \sim g$. If $\delta=0$ theorem A shows that $\varphi \sim \psi$. Only the case $\delta>0$ remains to be considered. Write

$$F(x) = \int_0^{x} t^{-\delta}(x-t)^{\delta-1}f(t)dt,$$

$$G(x) = \int_0^{x} t^{-\delta}(x-t)^{\delta-1}g(t)dt.$$

If the expressions for f and g as integrals are introduced and the order of integration inverted one finds

$$F(x)=\pi(\sin \pi\delta)^{-1}\int_0^\infty (u+x)^{\delta-1}d\varphi(u), \tag{3}$$

$$G(x)=\pi(\sin \pi\delta)^{-1}\int_0^\infty (u+x)^{\delta-1}d\psi(u).$$

If we can prove that $f\sim g$ implies $F\sim G$ the result $\varphi\sim\psi$ follows from Theorem B.

To every $\varepsilon>0$ we can choose $c>a$ so large that if $g(t)=f(t)+\eta(t)f(t)$ the function $\eta(t)$ satisfies $|\eta(t)|<\varepsilon$ for $t\geqslant c$. Take $x>2c$ and write.

$$F(x)=F_1(x)+F_2(x)=\int_0^c t^{-\delta}(x-t)^{\delta-1}f(t)\,dt+\int_c^x t^{-\delta}(x-t)^{\delta-1}f(t)\,dt,$$

$$G(x)=G_1(x)+G_2(x)=\int_0^c t^{-\delta}(x-t)^{\delta-1}g(t)\,dt+\int_c^x t^{-\delta}(x-t)^{\delta-1}g(t)\,dt.$$

We evidently have

$$F_1(x)=O(x^{\delta-1}), \tag{4}$$

$$G_1(x)=O(x^{\delta-1}). \tag{4'}$$

We want to get a lower bound for $F_2(x)$. A partial integration of (3) shows that

$$F(x)=\pi(1-\delta)[\sin\pi\delta]^{-1}\int_0^\infty (u+x)^{\delta-2}\varphi(u)\,du.$$

The last integral is diminished if taken only over $(x, 2x)$ and since $\varphi(u\geqslant\varphi(x)$ in this interval we get a constant $A>0$ such that

$$F(x)\geqslant 2A\varphi(x)x^{\delta-1}.$$

Since $F_2(x)=F(x)-F_1(x)$ and $\varphi(x)\to\infty$ when $x\to\infty$ we conclude by (4) that

$$F_2(x)>A\varphi(x)x^{\delta-1} \tag{5}$$

for large values of x. Because of the choice of c

$$f(t)(1-\varepsilon)<g(t)<f(t)(1+\varepsilon)$$

when $t>c$. It follows that

$$F_2(x)(1-\varepsilon)<G_2(x)<F_2(x)(1+\varepsilon). \tag{6}$$

Writing

$$\frac{F}{G}=\frac{F_1/F_2+1}{G_1/F_2+G_2/F_2}$$

and observing that $\varphi(x)\to\infty$ when $x\to\infty$ we see from (4), (4′), (5) and (6) that $F\sim G$. As was already mentioned this implies that $\varphi\sim\psi$ is a consequence of $f\sim g$.

We proceed to prove that $f \sim g$ is a consequence of $\varphi \sim \psi$. The cases $\delta = 0$ and $\delta > 0$ are now considered simultaneously. If

$$\psi(u) = \varphi(u) + \eta_1(u)\varphi(u) \tag{7}$$

our assumption means that $\eta_1(u) \to 0$ as $u \to \infty$. Relation (7) certainly holds for $u \geqslant a$. The proof is however easily adapted also to cover the case when ψ cannot be written in this way for $u < a$. Partial integrations of the formulas for f and g lead to the relations

$$f(x) = \int_0^\infty K(x, u)\varphi(u)\,du,$$

$$g(x) = \int_0^\infty K(x, u)\psi(u)\,du,$$

where $K(x, u) = u^\delta(u + x)^{-2} - \delta u^{\delta - 1}(u + x)^{-1}$. From (7) it follows that

$$g(x) = f(x) + \int_0^\infty K(x, u)\eta_1(u)\varphi(u)\,du.$$

For $u > C = C(\varepsilon)$ we have $|\eta_1(u)| < \varepsilon$. The integral from C to $+\infty$ is less than

$$\varepsilon \int_C^\infty (u^\delta(u + x)^{-2} + \delta u^{\delta - 1}(u + x)^{-1})\varphi(u)\,du.$$

Assume $x > C > a$. Then, according to Lemma 1, $u^{-\alpha}\varphi(u) \leqslant x^{-\alpha}\varphi(x)$ in (C, x) and $u^{-\beta}\varphi(u) \leqslant x^{-\beta}\varphi(x)$ in (x, ∞). It follows that

$$\left| \int_C^\infty K(x, u)\eta_1(u)\varphi(u)\,du \right| < \varepsilon C' x^{\delta - 1}\varphi(x)$$

where C' is independent of $C = C(\varepsilon)$. By a rough estimate of $K(x, u)$ it is easily seen that

$$x \int_0^C K(x, u)\eta_1(u)\varphi(u)\,du$$

is bounded. By the help of (2) it then follows that $g \sim f$.

2. The bilateral case with the kernel $(u + x)^{-1}$

Lemma 3. *If φ, $\varphi \not\equiv$ constant, and ψ are monotone in the same way for $x > 0$ and for $x < 0$ and if*

$$\int_{-\infty}^{+\infty} (u - z)^{-1} d\varphi(u) \sim \int_{-\infty}^{+\infty} (u - z)^{-1} d\psi(u) \tag{8}$$

as z tends to infinity along one non-real half-ray from the origin, then the same relation holds for all such half-rays.

It is evidently no restriction to assume $0<c<\pi$ for the argument of the given half-ray. Denote the left- and right-hand sides of (8) by $f(z)$ and $g(z)$. If $z=x+iy$ then

$$f(z) = I_1 - xI_2 + iyI_2,$$

where
$$I_1 = \int_{-\infty}^{+\infty} \frac{u\,d\varphi(u)}{(u-x)^2+y^2}, \quad I_2 = \int_{-\infty}^{+\infty} \frac{d\varphi(u)}{(u-x)^2+y^2}.$$

We first treat the case $d\varphi(u)\geqslant 0$ for all u. Since $\varphi \not\equiv$ constant we have $I_2>0$ and we can assume $0<\arg f(z)<\pi$ in the upper half-plane. Similar considerations allow us to assume $0<\arg g(z)<\pi$. Hence we get $-\pi<\arg f(z)-\arg g(z)<\pi$ and a consequence is that every branch of log $(f(z)/g(z))$ is univalent and analytic in the upper half-plane. Thus $h(z)=\exp\{i\log[f(z)/g(z)]\}$ is analytic if $y>0$ and we get $\exp(-\pi)<|h(z)|<\exp\pi$. Our assumption $f\sim g$ when z tends to infinity along $\arg z=c$ yields $h(z)\to 1$ when $|z|\to\infty$ along the same half-ray. It follows from Montel's theorem that $h(z)\to 1$ uniformly in $\delta\leqslant \arg z\leqslant \pi-\delta$ for every $\delta>0$. In consequence $f\sim g$ when $|z|\to\infty$ along any half-ray in the upper half-plane.

When $d\varphi(u)\geqslant 0$ for $u>0$ and $d\varphi(u)\leqslant 0$ for $u<0$ we consider

$$zf(z) = xI_1 - (x^2+y^2)I_2 + iyI_1.$$

We observe that $I_1>0$ and repeating the discussion above with $zf(z)$ and $zg(z)$ instead of $f(z)$ and $g(z)$ we obtain the desired result.

Now we can prove the following Tauberian theorem.

Theorem 2. *Let for $x\geqslant 0$, $\varphi(x)\in T(\alpha,\beta,a)$ with $0<\alpha<\beta<1$ and $\psi(x)\in T$. Suppose also that either $\varphi(-x)\in T(\alpha,\beta,a)$, $\psi(-x)\in T$ or $-\psi(-x)\in T(\alpha,\beta,a)$, $-\psi(-x)\in T$ when $x\geqslant 0$. Assume*

$$A_1 \leqslant |\varphi(-x)/\varphi(x)| \leqslant A_2 \tag{9}$$

for $x>a$ where A_1, A_2 are positive constants. If under these assumptions

$$\int_{-\infty}^{\infty} (u-z)^{-1} d\varphi(u) \sim \int_{-\infty}^{\infty} (u-z)^{-1} d\psi(u)$$

when $|z|\to\infty$ along a non-real half-ray from the origin then $\varphi(x)\sim\psi(x)$ as $x\to+\infty$ and as $x\to-\infty$.

According to Lemma 3 we may assume the half-ray to be $z=it$, $t>0$. The assumption of the theorem then reads

$$\int_{-\infty}^{\infty} (u-it)^{-1} d\psi(u) = (1+\eta(t)) \int_{-\infty}^{\infty} (u-it)^{-1} d\varphi(u),$$

where $\eta(t)=\eta_1(t)+i\eta_2(t)$ tends to zero when $t\to+\infty$. Splitting real and imaginary parts we obtain after simple transformations of the occurring integrals the formulas

$$\int_0^{\infty} \sqrt{u}(u+t^2)^{-1} d\psi_1(u) = (1+\eta_1(t)) \int_0^{\infty} \sqrt{u}(u+t^2)^{-1} d\varphi_1(u) - t\eta_2(t) \int_0^{\infty} (u+t^2)^{-1} d\varphi_2(u), \tag{10}$$

$$\int_0^\infty (u+t^2)^{-1}d\psi_2(u) = (1+\eta_1(t))\int_0^\infty (u+t^2)^{-1}d\varphi_2(u) + t^{-1}\eta_2(t)\int_0^\infty \sqrt{u}(u+t^2)^{-1}d\varphi_1(u), \tag{11}$$

where $\psi_1(u)=\psi(\sqrt{u})+\psi(-\sqrt{u})$, $\psi_2(u)=\psi(\sqrt{u})-\psi(-\sqrt{u})$, $\varphi_1(u)=\varphi(\sqrt{u})+\varphi(-\sqrt{u})$, $\varphi_2(u)=\varphi(\sqrt{u})-\varphi(-\sqrt{u})$.

Let us first consider the case when $d\varphi(u)\geqslant 0$ for $u>0$ and $d\varphi(u)\leqslant 0$ for $u<0$. On account of the conditions for φ and ψ it is easily seen that $\varphi_1(u)\in T(\alpha/2,\beta/2)$ and $\psi_1(u)\in T$. It is also clear that $|\varphi_2(u)|\leqslant\varphi_1(u)$. According to the last remark

$$\left|\int_0^\infty (u+t^2)^{-1}d\varphi_2(u)\right| \leqslant \int_0^\infty (u+t^2)^{-1}d\varphi_1(u)$$

and by Lemma 2 the right-hand side is uniformly $O(t^{-2}\varphi_1(t^2))$. This lemma also shows that

$$\int_0^\infty \sqrt{u}(u+t^2)^{-1}d\varphi_1(u) > Bt\varphi_1(t^2)$$

with B positive for large values of t. Since $\eta_1(t)$ and $\eta_2(t)$ tends to zero as $t\to\infty$ it so follows from (10) that

$$\int_0^\infty \sqrt{u}(u+t^2)^{-1}d\psi_1(u) \sim \int_0^\infty \sqrt{u}(u+t^2)^{-1}d\varphi_1(u).$$

Here $\psi_1\in T$ and $\varphi_1\in T(\alpha/2,\beta/2)$ where according to the conditions on α and β the inequalities $\beta/2<\frac{1}{2}$ and $\alpha/2>0$ are valid. Theorem 1 gives $\psi_1(x)\sim\varphi_1(x)$ when $x\to+\infty$, and from the Abelian part of the same theorem it then follows that

$$\int_0^\infty (u+t^2)^{-1}d\psi_1(u) = (1+\eta_3(t))\int_0^\infty (u+t^2)^{-1}d\varphi_1(u), \tag{12}$$

where $\eta_3(t)\to 0$ when $t\to+\infty$. The result of adding (11) and (12) is

$$2\int_0^\infty (u+t^2)^{-1}d\psi(\sqrt{u}) = 2\int_0^\infty (u+t^2)^{-1}d\varphi(\sqrt{u}) + \eta_1(t)\int_0^\infty (u+t^2)^{-1}d\varphi_2(u)$$
$$+\eta_2(t)t^{-1}\int_0^\infty \sqrt{u}(u+t^2)^{-1}d\varphi_1(u) + \eta_3(t)\int_0^\infty (u+t^2)^{-1}d\varphi_1(u). \tag{13}$$

The factor of η_1 is $O(t^{-2}\varphi_1(t^2))$ and the same holds for the factors of η_2 and η_3. An application of Lemma 2 shows that for large values of t

$$\int_0^\infty (u+t^2)^{-1}d\varphi(\sqrt{u}) > B't^{-2}\varphi(t),$$

where B' is a positive constant. Since $\varphi(-t)/\varphi(t)$ is bounded it then follows from (13) that

$$\int_0^\infty (u+t^2)^{-1} d\psi(\sqrt{u}) \sim \int_0^\infty (u+t^2)^{-1} d\varphi(\sqrt{u}).$$

Theorem 1 then shows that $\varphi(x) \sim \beta(x)$ as $x \to +\infty$. On the other hand, if we subtract (11) and (12) we obtain by similar considerations as above that

$$\int_0^\infty (u+t^2)^{-1} d\psi(-\sqrt{u}) \sim \int_0^\infty (u+t^2)^{-1} d\varphi(-\sqrt{u})$$

and we conclude by theorem 1 that $\varphi(x) \sim \psi(x)$ as $x \to -\infty$. Then we have proved the theorem under the assumption $d\varphi(u) \leqslant 0$ for $u<0$.

Suppose $d\varphi(u) \geqslant 0$ for $u<0$. Then we first consider (10) and by the help of Lemma 2 and the Tauberian part of theorem 1 we get $\psi_2 \sim \varphi_2$. The next step is to apply the Abelian part of theorem 1 with $\delta = \frac{1}{2}$ to the functions φ_2 and ψ_2 and combining this result and (10) we get in the same way as above the desired result.

Remark. Without (9) our proof only shows that $\varphi_1 \sim \psi_1$ if $d\varphi(u) \leqslant 0$ for $u<0$ and $\varphi_2 \sim \psi_2$ in case $d\varphi(u) \geqslant 0$ for $u<0$. If we are interested in obtaining $\varphi \sim \psi$ as $x \to +\infty$ we only use the right part of (9) and if we want $\varphi(x) \sim \psi(x)$ as $x \to -\infty$ we use the left part of (9).

3. The unilateral case with the kernel $u^{-h}(u+x)^{-m-1}$

In this section we shall prove a Tauberian theorem for integrals of the form

$$\int_0^\infty u^{-h}(u+x)^{-m-1} d\varphi(u), \tag{14}$$

where h is a positive constant. Writing $\varphi_1(x) = \int_0^x u^{-h} d\varphi(u)$ we get (14) equal to

$$\int_0^\infty (u+x)^{-m-1} d\varphi_1(u)$$

and the integral is of the form already treated in Theorem A. Hence we start proving some properties of integrals like φ_1.

Lemma 4. *Let $\varphi \in T(\alpha,\beta,a)$. Consider $\varphi_1(x) = \int_0^x t^{-h} d\varphi(t)$ where $h>0$ is a constant and assume $\varphi_1(x) \to +\infty$ when $x \to +\infty$. Then $x^{-\alpha'}\varphi_1(x)$ increases for every $\alpha' < \alpha - h$ and $x^{-\beta'}\varphi_1(x)$ decreases for every $\beta' > \beta - h$ when x is sufficiently large. If $\alpha > h$ the condition $\varphi_1(x) \to +\infty$ is fulfilled by itself and $\varphi_1 \in T(\alpha',\beta')$ provided $\alpha' < \alpha - h$ and $\beta' > \beta - h$ are chosen so that $0 < \beta' < \alpha' + 1$.*

It is easily seen that

$$x\varphi_1'(x) - (\alpha - h)\varphi_1(x) = x^{-h}(x\varphi'(x) - \alpha\varphi(x)) + h\int_0^x t^{-h-1}(t\varphi'(t) - \alpha\varphi(t))\,dt.$$

In fact the terms $x\varphi_1'(x)$ and $x^{-h+1}\varphi'(x)$ as well as the other terms not containing α cancel. The rest of the identity is the result of a partial integration in the integral defining $\varphi_1(x)$. Since $\varphi \in T(\alpha,\beta,a)$ relation (1) shows that $x\varphi'(x)-\alpha\varphi(x)\geqslant 0$ for $x>a$. Thus for these x

$$x\varphi_1'(x)-(\alpha-h)\varphi_1(x)\geqslant C,$$

where C is a constant. If $\varphi_1(x)\to+\infty$ this implies that for a given $\varepsilon>0$

$$x\varphi_1'(x)\geqslant(\alpha-h-\varepsilon)\varphi_1(x)$$

when x is greater than a certain constant $a'=a'(\varepsilon)$. If the same identity is applied with β instead of α the result is

$$x\varphi_1'(x)\leqslant(\beta-h+\varepsilon)\varphi_1(x)$$

when x is sufficiently large.

φ_1 is clearly positive, increasing and identically zero near the origin. To prove the last part of the lemma it remains to show that in case $\alpha>h$, $\varphi_1(x)\to+\infty$ when $x\to+\infty$. A partial integration of the integral defining φ_1 gives

$$\varphi_1(x)=x^{-h}\varphi(x)+h\int_0^x t^{h+1}\varphi(t)\,dt. \tag{15}$$

We write $x^{-h}\varphi(x)=x^{\alpha-h}(x^{-\alpha}\varphi(x))$ and since $\alpha>h$ and $x^{-\alpha}\varphi(x)$ increases we get $x^{-h}\varphi(x)\to+\infty$ as $x\to+\infty$. Since $h>0$ we diminish $\varphi_1(x)$ by omitting the integral in (15) and hence $\varphi_1(x)\to+\infty$ as $x\to+\infty$.

Lemma 5. *Let $\varphi\in T(\alpha,\beta,a)$. Assume $\alpha>h>0$. Then with a constant C*

$$x^{-h}\varphi(x)\leqslant\varphi_1(x)=\int_0^x t^{-h}d\varphi(t)\leqslant Cx^{-h}\varphi(x) \tag{16}$$

for large values of x.

The first part of the inequality follows from (15). The second part is a consequence of

$$\int_a^x t^{-h-1}\varphi(t)\,dt\leqslant x^{-\alpha}\varphi(x)\int_0^x t^{\alpha-h-1}dt$$

if we observe that $x^{-h}\varphi(x)=x^{\alpha-h}(x^{-\alpha}\varphi(x))$ tends to infinity as $x\to+\infty$.

Lemma 6. *Let $\varphi\in T(\alpha,\beta)$ and let $\psi\in T$. Assume $0<h<\alpha$. If $\varphi_1\sim\psi_1$ where*

$$\varphi_1(x)=\int_0^x t^{-h}d\varphi(t),\ \psi_1(x)=\int_0^x t^{-h}d\psi(t)$$

then $\varphi\sim\psi$.

We normalize ψ and ψ_1 by $\psi(x)=\lim_{t\to x+0}\psi(t)$ and the corresponding for ψ_1. Then

$$\psi(x)=\int_0^x t^h d\psi_1(t)=x^h\psi_1(x)-h\int_0^x t^{h-1}\psi_1(t)\,dt. \tag{17}$$

The continuity of φ_1 shows that we still have $\varphi_1 \sim \psi_1$. This assumption can be written $\psi_1(x) = (1+\eta(x))\varphi_1(x)$ where $\eta(x) \to 0$ as $x \to \infty$. Formula (17) together with the corresponding formula for φ leads to the relation

$$\psi(x) = \varphi(x) + x^h \eta(x) \varphi_1(x) - h\int_0^x t^{h-1}\eta(t)\varphi_1(t)\,dt. \tag{18}$$

Given $\varepsilon > 0$ we get if A is sufficiently large (compare (17))

$$\left|\int_A^x t^{h-1}\eta(t)\varphi_1(t)\,dt\right| < \varepsilon \int_0^x t^{h-1}\varphi_1(t)\,dt = \varepsilon h^{-1}(\varphi(x) - x^h\varphi_1(x)).$$

We know that $\varphi(x) \to +\infty$ when $x \to +\infty$ and according to (16) it follows that (17) can be written

$$\psi(x)/\varphi(x) = 1 + \varepsilon M(x)$$

where $M(x)$ is uniformly $O(1)$ when $x \to \infty$. The conclusion of the theorem follows.

Now we can prove the following theorem.

Theorem 3. *Let $\varphi \in T(\alpha,\beta)$ and $\psi \in T$. Assume $0 < h < \alpha$ and let m be the integral part of $\beta - h$. If*

$$\int_0^\infty u^{-h}(u+x)^{-m-1}d\psi(u) \sim \int_0^\infty u^{-h}(u+x)^{-m-1}d\varphi(u) \tag{19}$$

then $\varphi \sim \psi$.

As was already mentioned in the beginning of this section, (19) can be written

$$\int_0^\infty (u+x)^{-m-1}d\psi_1(u) \sim \int_0^\infty (u+x)^{-m-1}d\varphi_1(u),$$

where $\psi_1(x) = \int_0^x u^{-h}d\psi(u)$ and $\varphi_1(x) = \int_0^x u^{-h}d\varphi(u)$. By lemma 4 we conclude that $\varphi_1 \in T(\alpha',\beta')$ where the integral part of β' equals m and $\alpha' > 0$. Evidently $\psi_1 \in T$ and Theorem A then shows that $\psi_1 \sim \varphi_1$. The theorem then follows from Lemma 6.

4. The bilateral case with the kernel $(u+x)^{-m-1}$

In this section we shall prove the following generalization of Theorem 2.

Theorem 4. *Let $\varphi(x) \in T(\alpha',\beta')$ and $\psi(x) \in T$ for $x \geq 0$. Suppose also that either $\varphi(-x) \in T(\alpha',\beta')$, $\psi(-x) \in T$ or $-\varphi(-x) \in T(\alpha',\beta')$, $-\psi(-x) \in T$ when $x \geq 0$. Assume*

$$A_1 \leq |\varphi(-x)/\varphi(x)| \leq A_2$$

for large values of x, where A_1 and A_2 are positive constants. Suppose that the relation $m < \alpha' < \beta'$ holds where $m = [\beta']$. If

$$\int_{-\infty}^{+\infty} (u-z)^{-m-1}d\varphi(u) \sim \int_{-\infty}^{+\infty} (u-z)^{-m-1}d\psi(u) \tag{20}$$

when $|z|\to\infty$ *along a non-real half-ray from the origin then* $\varphi(x)\sim\psi(x)$ *when* $x\to+\infty$ *and when* $x\to-\infty$.

At first we observe that it is no restriction to assume the half-ray to lie in the upper half-plane. For $m=0$ the theorem is already proved so we can assume $m\geq 1$. We denote the left part of (20) by $f_m(z)$ and the right by $g_m(z)$. Put

$$F_m(z)=\int_0^z (u-z)^{m-1}f_m(u)\,du,$$

$$G_m(z)=\int_0^z (u-z)^{m-1}g_m(u)\,du,$$

where the way of integration does not cut the real axis. If we insert the expressions for f_m and g_m as integrals and invert the order of integration we find

$$F_m(z)=-(-z)^m m^{-1}\int_{-\infty}^{\infty} u^{-m}(u-z)^{-1}d\varphi(u),$$

$$G_m(z)=-(-z)^m m^{-1}\int_{-\infty}^{\infty} u^{-m}(u-z)^{-1}d\psi(u).$$

Put $\varphi_1(x)=\int_0^x u^{-m}d\varphi(u)$ and $\psi_1(x)=\int_0^x u^{-m}d\psi(u)$. Lemma 4 then proves that $\varphi_1(x)\in T(\alpha,\beta,a)$ and either $\varphi_1(-x)$ or $-\varphi_1(-x)$ belongs to $T(\alpha,\beta,a)$ for $x\geq 0$ where $0<\alpha<\beta<1$. We also get $\psi_1(x)\in T$ and we see that either $\psi_1(-x)$ or $-\psi_1(-x)$ belongs to T for $x\geq 0$ and that ψ_1 is monotone in the same way as φ_1. Introducing φ_1 and ψ_1 in the integrals of F_m and G_m we get

$$F_m(z)=-(-z)^m m^{-1}\int_{-\infty}^{\infty}(u-z)^{-1}d\varphi_1(u),$$

$$G_m(z)=-(-z)^m m^{-1}\int_{-\infty}^{\infty}(u-z)^{-1}d\psi_1(u).$$

If we can prove $F_m(z)\sim G_m(z)$ when $|z|\to\infty$ along the given half-ray we can use theorem 2 to get the result. The idea of this proof is the same as in the Tauberian part of Theorem 1.

Let the half-ray be given by re^{iv}, $r>0$. Take $\varepsilon>0$. The assumption (20) can be written

$$g_m(re^{iv})=f_m(re^{iv})+h(re^{iv})f_m(re^{iv}), \tag{21}$$

where $|h(re^{iv})|<\varepsilon$ for $r>c=c(\varepsilon)$. Let $r>2c$ and $c>a$. Put

$$\begin{aligned}F_m(re^{iv})&=F_m^{(1)}(re^{iv})+F_m^{(2)}(re^{iv})\\&=\int_0^{ce^{iv}}(u-re^{iv})^{m-1}f_m(u)\,du+\int_{ce^{iv}}^{re^{iv}}(u-re^{iv})^{m-1}f_m(u)\,du,\end{aligned}$$

$$\begin{aligned}G_m(re^{iv})&=G_m^{(1)}(re^{iv})+G_m^{(2)}(re^{iv})\\&=\int_0^{ce^{iv}}(u-re^{iv})^{m-1}g_m(u)\,du+\int_{ce^{iv}}^{re^{iv}}(u-re^{iv})^{m-1}g_m(u)\,du,\end{aligned}$$

where the way of integration coincides with the given half-ray. Evidently $F_m^{(1)}(re^{iv})$ and $G_m^{(1)}(re^{iv})$ are uniformly $O(r^{m-1})$. We want a lower bound for $F_m^{(2)}(re^{iv})$ and hence we study $F_m(re^{iv})$.

$$F_m(re^{iv}) = (I_1 - rI_2 \cos v + i\,rI_2 \sin v)\,r^m e^{im(v+\pi)+i\pi} m^{-1}, \tag{22}$$

where $$I_1 = \int_{-\infty}^{\infty} u\,|u - re^{iv}|^{-2} d\varphi_1(u), \quad I_2 = \int_{-\infty}^{\infty} |u - re^{iv}|^{-2} d\varphi_1(u).$$

At first the case $d\varphi_1(u) \geqslant 0$ for all u is discussed. Owing to $|u - re^{iv}| \leqslant |u| + r$,

$$I_2 \geqslant \int_{-\infty}^{\infty} (|u| + r)^{-2} d\varphi_1(u)$$

and by the same argument used to prove the left part of (2) we get a positive constant c_1 such that $2c_1(\varphi_1(r) + |\varphi_1(-r)|)r^{-2}$ is a lower bound for I_2. Relation (22) then yields

$$|F_m(re^{iv})| \geqslant 2c_1(\sin v)\,m^{-1} r^{m-1}(\varphi_1(r) + |\varphi_1(-r)|).$$

Since $F_m^{(2)}(re^{iv}) = F_m(re^{iv}) - F_m^{(1)}(re^{iv})$ we get

$$|F_m^{(2)}(re^{iv})| \geqslant c_1(\sin v)\,m^{-1} r^{m-1}(\varphi_1(r) + |\varphi_1(-r)|)$$

for large values of r. To get the relation corresponding to (6) we insert (21) in the expression defining $G_m^{(2)}(re^{iv})$ and obtain

$$G_m^{(2)}(re^{iv}) = F_m^{(2)}(re^{iv}) = \int_{ce^{iv}}^{re^{iv}} (u - re^{iv})^{m-1} f_m(u)\,h(u)\,du.$$

Here $$\left|\int_{ce^{iv}}^{re^{iv}} (u - re^{iv})^{m-1} f_m(u)\,h(u)\,du\right| < \varepsilon \int_c^r (t+r)^{m-1} |f_m(te^{iv})|\,dt.$$

Using the inequality $|u - te^{iv}|^{m+1} \geqslant 2^{-m-1}(\sin v)^{m+1}\sqrt{(u^2+t^2)}\,|u|^m$ we get, by the same methods as in the proof of the right-hand side of (2), $|f_m(te^{iv})|$ uniformly bounded by $(\varphi_1(t) + |\varphi_1(-t)|)t^{-1}$. Since $\varphi_1(t)$ and $|\varphi_1(-t)|$ belong to $T(\alpha,\beta,a)$ the function $t^{-\alpha}(\varphi_1(t) + |\varphi_1(-t)|)$ increases and we obtain with a constant c_2 that

$$\int_c^r (t+r)^{m-1} |f_m(te^{iv})|\,dt \leqslant c_2(\varphi_1(r) + |\varphi_1(-r)|)\,r^{-\alpha} \int_0^r (r+t)^{m-1} u^{\alpha-1}\,dt$$

and the last expression is uniformly $O((\varphi_1(r) + |\varphi_1(-r)|)\,r^{m-1})$. Now $F_m(re^{iv}) \sim G_m(re^{iv})$ when $r \to +\infty$ by the same arguments as in Theorem 1. The result of the theorem then follows from Theorem 2 and Lemma 6.

Corollary 1. *Let m be a non-negative integer and let q be a constant with $0 < q < 1$. Let A and B be two real constants different from zero. For $u \geqslant 0$ we assume that $\psi(u) \in T$ if $A > 0$ and $-\psi(u) \in T$ if $A < 0$ and also $\psi(u) \in T$ if $(-1)^m B > 0$ and $-\psi(-u) \in T$ if $(-1)^m B < 0$. If*

$$\int_{-\infty}^{\infty} (u-z)^{-m-1} d\psi(u) \sim A(-z)^{-1+q} + Bz^{-1+q},$$

where $(-z)^{-1+q}$ *is positive on the negative real axis and* z^{-1+q} *is positive on the positive real axis, then*

$$\psi(u) \sim \frac{Am! \sin \pi q}{\pi q(q+1) \dots (q+m)} u^{q+m}$$

when $u \to +\infty$ *and*

$$\psi(u) \sim \frac{Bm! \sin \pi q}{\pi q(q+1) \dots (q+m)} |u|^{m+q} (-1)^m$$

when $u \to -\infty$.

The corollary is easily proved if we use the asymptotic expressions for ψ above as our φ in Theorem 4.

Remark. A bilateral Tauberian theorem for kernels of the type $u^{-h}(u+x)^{-m-1}$ (compare Theorem 3) can be proved by the same method as in Theorem 4.

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