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On Lerch's class number formulae for binary quadratic forms

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Let $f(-d)$ be the number of classes of positive quadratic forms $ax^2+2bxy+cy^2$ with negative determinant $-d=b^2-ac$, where a and c are not both even and $(a, b, c)=1$. It is well known that if $d\neq 1$,

$$f(-d)=\frac{2\sqrt{d}}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{1}{n}\left(\frac{-d}{n}\right), \tag{1}$$

and that when d is a prime $p\equiv 3 \pmod 4$, this can be written as

$$f(-p)=\sum_{n=1}^{\frac{1}{2}(p-1)}\left(\frac{n}{p}\right). \tag{2}$$

Here (n/p) is the Legendre-Jacobi symbol. There are other formulae expressing partial sums of quadratic residues in terms of $f(-p)$, $f(-2p)$, $f(-3p)$. Such results were known to Gauss.

Many more interesting formulae of this kind have been given by Lerch [1] in his Paris Academy prize memoir for 1900. He, however, deals with Kronecker forms $ax^2+bxy+cy^2$ whose discriminant b^2-4ac, when positive is written as D, and when negative as $-\Delta$. The corresponding class number $h(-\Delta)$ of forms with $(a, b, c)=1$ can be written, when $-\Delta$ is a fundamental discriminant and $\Delta>4$, as

$$h(-\Delta)=-\frac{1}{\Delta}\sum_{n=1}^{\Delta}\left(\frac{-\Delta}{n}\right)n. \tag{3}$$

The symbol $(-\Delta/n)$ is now and hereafter the Kronecker symbol. We note that $(-\Delta/m)=(-\Delta/n)$ sgn mn if $m\equiv n \pmod{\Delta}$ and $-\Delta$ is a negative discriminant, and that $(D/m)=(D/n)$ if $m\equiv \pm n \pmod{D}$ and D is a positive discriminant.

He then deduces many formulae of the type (2) from his

Theorem I. *Let $D>0$, and $-\Delta<0$ be two fundamental discriminants with $(D, \Delta)=1$. Then*

$$h(-\Delta D)=2\sum_{a=1}^{\frac{1}{2}\Delta}\left(\frac{-\Delta}{a}\right)\sum_{n=1}^{aD/\Delta}\left(\frac{D}{n}\right), \tag{4}$$

$$-h(-\Delta D)=2\sum_{a=1}^{\frac{1}{2}D}\left(\frac{D}{a}\right)\sum_{n=1}^{a\Delta/D}\left(\frac{-\Delta}{n}\right). \tag{5}$$

A sum such as $\sum_{n=1}^{N}$ means $1\leqslant n\leqslant [N]$ where $[N]$ is the integer part of N.

Theorem II. *Let $D_1, D_2, \ldots, D_r$ denote fundamental discriminants relatively prime in pairs with absolute values $\Delta_1, \ldots, \Delta_r$ and with $2s+1$ of the D negative. Then*

$$h(D_1D_2\ldots D_r)=(-1)^s\sum\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{u_2}\right)\cdots\left(\frac{D_r}{n_r}\right)\left[\frac{n_1}{\Delta_1}+\frac{n_2}{\Delta_2}+-+\frac{n_r}{\Delta_r}\right], \tag{6}$$

where the square bracket denotes the integer part and the summation is taken over

$$0<n_1<\Delta_1,\ \ldots,\ 0<n_r<\Delta_r.$$

In the enunciation of Theorem I, Lerch does not state emplicitly that $(D,\Delta)=1$, but this is tacitly assumed on page 405 [2] in his formula for $h(-\Delta D)$ as a Fourier series. Similarly in Theorem II, he omits the conditions $(D_1, D_2)=1$, etc., but without these, his argument on page 374 [3], stating that a certain sum is zero, is not valid.

Lerch's proof of Theorem I is rather complicated involving various transformations of infinite series. His proof of Theorem II is also much longer than seems to be necessary. It may be worth while pointing out that (4,) (5) are particular cases of (6), and that (6) is a simple consequence of (3). In (3), write $-\Delta=D_1D_2\ldots D_r$ where the D are as in Theorem II. In (3), n runs through a complete set of positive residues mod Δ. A complete set of positive residues mod Δ is given by

$$R=\Delta\left(\frac{n_1}{\Delta_1}+\frac{n_2}{\Delta_2}+\ldots+\frac{n_r}{\Delta_r}\right),\quad 0\leqslant n_1<\Delta_1,\ \ldots,\ 0\leqslant n_r<\Delta_r.$$

These numbers, however, are not all less than Δ and so we cannot replace n by R. We can, however, put

$$n=R-\varepsilon_t\Delta,$$

where $\varepsilon_t=t$, $(t=0, 1, \ldots, r-1)$ when $t\leqslant R/\Delta<t+1$, i.e., $\varepsilon_t=[R/\Delta]$.

Now
$$\left(\frac{-\Delta}{n}\right)=\left(\frac{D_1}{n}\right)\cdots\left(\frac{D_r}{n}\right)=\prod\left(\frac{D_1}{n}\right),$$

say. Also when $A\Delta_1+B>0, B>0$,

$$\left(\frac{D_1}{A\Delta_1+B}\right)=\left(\frac{D_1}{B}\right).$$

Hence
$$\left(\frac{-\Delta}{n}\right)=\prod\left(\frac{D_1}{\Delta_2\ldots\Delta_r}\right)\left(\frac{D_1}{n_1}\right)=\prod\left(\frac{D_1}{\Delta_2}\right)\left(\frac{D_2}{\Delta_1}\right)\left(\frac{D_1}{n_1}\right),$$

say. By the law of quadratic reciprocity, this becomes

$$\left(\frac{-\Delta}{n}\right)=\prod(-1)^{\frac{1-\operatorname{sgn} D_1}{2}\cdot\frac{1-\operatorname{sgn} D_2}{2}}\prod\left(\frac{D_1}{n_1}\right)$$
$$=(-1)^{\frac{(2s+1)2s}{2}}\prod\left(\frac{D_1}{n_1}\right)$$
$$=(-1)^s\prod\left(\frac{D_1}{n_1}\right).$$

Hence $$-\Delta h(-\Delta)=(-1)^s\sum\left(\frac{D_1}{n_1}\right)\dots\left(\frac{D_r}{n_r}\right)\left(\Delta\left(\frac{n_1}{\Delta_1}+\dots+\frac{n_2}{\Delta_2}\right)-\varepsilon_t\Delta\right).$$

Since $\sum(D_1/n_1)=0$, etc., the first part of the sum is zero and we have (6), namely,

$$h(-\Delta)=\sum\left(\frac{D_1}{n_1}\right)\dots\left(\frac{D_r}{n_r}\right)\left[\frac{n_1}{\Delta_1}+\dots+\frac{n_r}{\Delta_r}\right].$$

We now deduce (4), (5). Take $r=2$, $-\Delta=D_1D_2$ and so $s=0$.

Then $$h(D_1D_2)=\sum\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{n_2}\right)\left[\frac{n_1}{\Delta_1}+\frac{n_2}{\Delta_2}\right]. \tag{7}$$

Replace n_1 by Δ_1-n_1. Then (7) becomes

$$h(D_1D_2)=\sum\left(\frac{D_1}{\Delta-n_1}\right)\left(\frac{D_2}{n_2}\right)\left(\frac{n_2}{\Delta_2}-\frac{n_1}{\Delta_1}+1\right)$$
$$=\operatorname{sgn} D_1\sum\left(\frac{D_1}{n_1}\right)\left(\frac{D_2}{n_2}\right)\left[\frac{n_2}{\Delta_2}-\frac{n_1}{\Delta_1}+1\right].$$

The sum is zero unless $n_1/\Delta_1<n_2/\Delta_2$, and then we have

$$h(D_1D_2)=\operatorname{sgn} D_1\sum_{n_2=1}^{\Delta_2}\left(\frac{D_2}{n_2}\right)\sum_{n_1=1}^{\Delta_1 n_2/\Delta_2}\left(\frac{D_1}{n_1}\right).$$

For $D_1=D$, $D_2=-\Delta$, this becomes

$$h(-D\Delta)=\sum_{n_2=1}^{\Delta}\left(\frac{-\Delta}{n_2}\right)\sum_{n_1=1}^{n_2D/\Delta}\left(\frac{D}{n_1}\right). \tag{8}$$

For $D_1=-\Delta$, $D_2=D$, this becomes

$$h(-D\Delta)=-\sum_{n_2=1}^{D}\left(\frac{D}{n_2}\right)\sum_{n_1=1}^{n_2\Delta/D}\left(\frac{-\Delta}{n_1}\right). \tag{9}$$

We show now that the n_2 summation in (8) and (9) need only be taken to $\frac{1}{2}\Delta$, $\frac{1}{2}D$ respectively if we insert a factor 2 on the right-hand sides of (8) and (9). We have then the results (4) and (5). It suffices to show that the sum (8) remains unaltered if we replace n_2 by $\Delta-n_2$. For then with $(\Delta/0)=0$, etc.,

$$\sum_{n_2=\frac{1}{2}\Delta}^{\Delta} = \sum_{n_2=0}^{\frac{1}{2}\Delta}.$$

Also
$$\left(\frac{-\Delta}{\Delta-n_2}\right)=\left(\frac{-\Delta}{n_2}\right),$$

and the n_1 sum in (8) becomes

$$\sum_{n_1=1}^{D-n_2D/\Delta}\left(\frac{D}{n_1}\right).$$

Replace now n_1 by $D-n_1$. Then

$$\left(\frac{D}{D-n_1}\right)=\left(\frac{D}{n_1}\right)$$

and the sum becomes

$$\sum_{n_1=n_2D/\Delta}^{D}\left(\frac{D}{n_1}\right)=-\sum_{n_1=1}^{n_2D/\Delta}\left(\frac{D}{n_1}\right)$$

since
$$\sum_{n_1=1}^{D}\left(\frac{D}{n_1}\right)=0.$$

This finishes the proof.

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