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Studies on a convolution inequality

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Introduction

Let μ be a positive, regular measure with total mass one on a locally compact Abelian group G (we refer to Appendix E 1 in Rudin [12] for the definition of regular measure). For certain classes of regular measures ν the operation $\nu * \mu$ can be defined and gives a new regular measure. We consider classes such that, if ν is in a certain class, then the same is true for (cf. Rudin [12] 1.3.4) $k * \nu$, where k is any continuous function with compact support, and

$$(k * \nu) * \mu = k * (\nu * \mu). \tag{0.1}$$

The measure $\nu * \mu$ can be interpreted as a "weighted mean value" of ν. The starting point of this paper is the following problem:

Let $\{\nu\}$ be a given class and consider the inequality

$$\nu - \nu * \mu \geqslant 0. \tag{0.2}$$

Which are the solutions in the given class and what properties do they have?

Suppose ν is such a solution. Let k be an arbitrary non-negative continuous function with compact support and form $\varphi = k * \nu$. It easily follows from (0.1) and (0.2) that

$$\varphi - \varphi * \mu \geqslant 0. \tag{0.3}$$

Obviously ν can be completely described by varying the function k. Hence the solution of the original problem can be characterized using the continuous solutions of (0.3). This gives a reason for our choice to confine the investigations of this paper to classes of continuous solutions of (0.3). At some instances in the forthcoming discussions, however, we shall mention the implications on the original problem.

In § 2, we let μ be arbitrary and study a class $\{\varphi\}$ of continuous functions which satisfy (0.3) and which are bounded from below. In Theorem 2.2, conditions are given which are necessary and sufficient for the existence of such solutions of (0.3), non-trivial in the sense that the strict inequality holds in a set of positive Haar measure. An equivalent criterion is given in Lemma 2.1, namely that for some neighborhood $\hat{O}$ of zero in the dual group $\hat{G}$ of G there exists a constant C such that

$$\int_{\hat{O}} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x} \leqslant C \quad \text{for all} \quad \varepsilon > 0. \tag{0.4}$$

In special cases (0.4) has been considered by many authors, cf. e.g. Chung and Fuchs [4]. We find that in this case we always have non-trivial solutions which are bounded.

In § 3, the criterion (0.4) is used to give more direct conditions on μ when G is n-dimensional Euclidean space R^n or the space of points with integral coordinates in R^n, Z^n. We have only been able to treat the case when μ has moments of a certain order. Since the existence of moments implies that $\mu(\hat{x})$ approaches 1 rather quickly as $\hat{x}$ approaches zero, the case treated is unfavorable for the fulfilment of the criterion (0.4) and hence also unfavorable for the existence of bounded, non-trivial solutions of (0.3).

In paragraphs 4, 5 and 6 Tauberian methods are used for the study of properties of continuous solutions of (0.3) in R^n and Z^n. Many proofs have been simplified by application of results from Domar [5]. One of the problems we consider is to find a connection between the growth of a solution φ at infinity and the magnitude of $\int_G(\varphi - \varphi * \mu)(x)dx$. A survey of the results obtained is given in § 7. In the paragraphs 4–6, we also study the equation

$$\varphi - \varphi * \mu = g \tag{0.5}$$

for g in certain classes of continuous functions. Measures μ such that bounded, non-trivial solutions of the inequality (0.3) exist are treated in § 4. Theorems 4.1 c and 4.3 c which give properties of solutions of (0.5) should be compared with a theorem by Feller [8]. The corollaries of Theorems 4.2 and 4.3 are related to results on the renewal equation earlier obtained by Karlin [9] by an application of the Wiener Tauberian theorem. Although the same theorem is applied in § 4, our approach to (0.5) is different from the one of Karlin and in many cases, we obtain more general results. Measures μ such that no bounded, non-trivial solutions of (0.3) exist are treated in § 5 and § 6. It follows from the results of § 3 that this can only occur in one and two dimensions. The one-dimensional case is considered in § 5 and the two-dimensional case in § 6.

In § 8, we study functions φ on R^2 such that (0.3) is true for a certain sequence of measures $\{\mu_n\}$ and prove that $\{\varphi\}$ is a class of superharmonic functions.

The main tools used in this paper are taken from Fourier analysis and from the theory of Banach algebras. The use of probabilistic methods has been avoided. For the basic definitions of Fourier transforms, convolutions, etc., we refer for instance to Rudin [12].

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1. Definitions and assumptions

Let G be a locally compact Abelian group (cf. e.g. Appendix B 4 in Rudin [12]). As group operation we choose addition. We assume that μ is a positive regular measure with total mass one on G and that μ is not the Dirac measure δ. We shall study the class $\{\varphi\}$ of real-valued, continuous functions on G such that for every x

$$\varphi * \mu(x) = \int_G \varphi(x-y)\, d\mu(y)$$

converges absolutely and such that on G

$$\varphi - \varphi * \mu \geqslant 0. \tag{1.1}$$

We call such functions *solutions* of the inequality (1.1).

We shall also consider the corresponding class $\{\varphi\}$ of functions which, instead of the inequality (1.1), satisfy the equation

$$\varphi - \varphi * \mu = g, \tag{1.2}$$

where g is chosen in certain classes of continuous functions on G. We call such a function φ a *solution* of (1.2).

A special class of solutions of (1.1) is formed by the solutions of

$$\varphi - \varphi * \mu = 0. \tag{1.3}$$

We call such solutions of (1.1) *trivial*. In this paper, we are chiefly interested in the non-trivial solutions of (1.1).

We also assume that μ has no mass at zero. We shall prove that this is no essential restriction.

Let μ have mass a at zero (where $0 \leqslant a < 1$) and define μ_1 by the formula

$$\mu = a\delta + (1-a)\,\mu_1.$$

Since

$$\varphi - \varphi * \mu = (1-a)\,(\varphi - \varphi * \mu_1)$$

the class of solutions of the inequality (1.1) coincides with the class of solutions of the inequality

$$\varphi - \varphi * \mu_1 \geqslant 0 \tag{1.4}$$

and it is sufficient to consider (1.4). Since μ_1 has no mass at zero, our assertion is proved.

Starting from μ we form measures $\mu^{(n)}$, $n = 0, 1, 2, \dots$. We interpret $\mu^{(0)}$ as the Dirac measure, $\mu^{(1)}$ as μ and define for $n \geqslant 1$

$$\mu^{(n+1)} = \mu^{(n)} * \mu,$$

where the existence of $\mu^{(2)}$, $\mu^{(3)}$ etc. follows from Theorem 1.3.2 in Rudin [12]. Obviously all the measures μ are positive and regular and

$$\int_G d\mu^{(n)} = 1. \tag{1.5}$$

We introduce the measures

$$F_n = \sum_{k=0}^{n} \mu^{(k)}.$$

They form a monotonically increasing sequence. Thus for every open set O the limit

$$F(O) = \lim_{n\to\infty} F_n(O)$$

exists, finite or infinite.

We denote by $G(\mu)$ the closed subgroup of G which is generated by the elements in the support of μ. We shall often assume that $G(\mu) = G$. The main reason for this is the result stated in Theorem 2.3 b. We want to mention another important consequence of this assumption, namely that the only solution of the equation $\hat{\mu}(\hat{x}) - 1 = 0$ in $\hat{G}$ is the zero element.

Constants are often denoted by C, or, if several constants appear in the same relation, by C_1, C_2 etc. These symbols may represent different numbers in different formulas.

2. Necessary and sufficient conditions for the existence of bounded non-trivial solutions of the inequality $\varphi - \varphi * \mu \geqslant 0$

Before stating our results, we have to discuss the properties of the bounded trivial solutions of the inequality (1.1), i.e. the solutions of the equation (1.3). It is easily seen that the equation (1.3) is satisfied by all constant functions. Under certain conditions, no other bounded solutions exist. This can be shown to be a consequence of the fact that if $\varphi \in L^\infty$ and if the spectrum of φ (in the sense of Beurling [2]) only contains one point, then φ is a character. On R^1, this was first proved by Beurling [1]. At the beginning of § 5, we shall use this method for solving the equation (1.3).

The bounded solutions of the equation (1.3) can also be found by use of the following theorem by Choquet and Deny [3]. Their proof does not use Fourier methods.

Theorem 2.1. *The bounded solutions of the equation (1.3) are periodic functions whose group of periods contains the support of* μ.

Consequently, all the elements in the subgroup $G(\mu)$ are periods of a bounded solution of (1.3). If $G = G(\mu)$, all such solutions are identically constant.

We can now state the main theorem of this section.

Theorem 2.2. *The following properties of* μ *are equivalent:*

A. *(1.1) has a non-trivial solution which is bounded from below and such that the left member in (1.1) is positive in a set of positive Haar measure.*

B. *There exists an open, relatively compact set* O *such that* $0 < F(O) < \infty$.

C. $F(x+O)$ *is a bounded function of* x *for every open, relatively compact set* O.

D. *(1.1) has a bounded, non-trivial solution such that the left member in (1.1) is positive in a set of positive Haar measure.*

Remark 1. Obviously $D \Rightarrow A$ and $C \Rightarrow B$.

Remark 2. It is mentioned but not proved in Lemma 1 in Choquet and Deny [3] that A implies the weak convergence of F_n to a measure, which is finite on every compact set.

Remark 3. In Theorem 2.2, we have excluded non-trivial solutions of (1.1) which are bounded from below and such that the left member in (1.1) is zero except in a set of Haar measure zero. The behavior of such functions is given by Theorem 3 in Choquet and Deny [3].

We can immediately state a corollary of Theorem 2.2. The total mass of F_n is $n+1$, hence $F(G) = \infty$. If G is compact, every continuous function on G is bounded. Since G is always open, the existence of a non-trivial solution would contradict C. Hence we have proved the following

Corollary. *If G is compact, all solutions of the inequality (1.1) are trivial.*

Before proving Theorem 2.2, we want to state Theorem 2.3. We start by observing that we can prove the existence of a non-trivial solution of the inequality (1.1) by showing that the equation

$$\varphi - \varphi * \mu = g \tag{2.1}$$

has a solution for some non-negative, not identically vanishing function g. When considering this equation, it is natural to introduce a class of functions M.

Definition 2.1. *$g \in M$ if*

(a) *g is continuous,*

(b) *there exists an open, relatively compact set O and a sequence $\{x_\nu\}_1^\infty$ such that the support of g is covered by $\bigcup_1^\infty (x_\nu + O)$ and such that*

$$\sum_1^\infty \operatorname*{Max}_{x \in x_\nu + O} |g(x)| < \infty.$$

It is easy to show that if $g \in M$, then for every open relatively compact set O there exists a sequence $\{x_\nu\}_1^\infty$ such that the conditions in (b) are fulfilled. On R_1, the class M was introduced by Wiener ([15], § 10).

Theorem 2.3. (a) *Let $g \in M$ and let condition C in Theorem 2.2 be true for F. Then $g * F$ is a bounded, uniformly continuous function which satisfies the equation (2.1).*

(b) *We assume that the conditions in (a) are fulfilled and that $G(\mu) = G$. Then the general bounded solution of the equation (2.1) has the form $A + g * F$, where A is an arbitrary constant.*

Proof of Theorem 2.2. The theorem follows if we prove that $A \Rightarrow B \Rightarrow C \Rightarrow D$. For technical reasons, we introduce a proposition E and prove that $A \Rightarrow B \Rightarrow E \Rightarrow C \Rightarrow D$.

E. *There exists a neighborhood O of zero such that $F(O)$ is finite.*

$A \Rightarrow B$. Let the inequality (1.1) have the non-trivial solution φ_0 with properties as those assumed in A. Let k be a non-negative, continuous function with compact support. Then $\varphi = \varphi_0 * k$ is also a solution of (1.1), which is bounded from below, and

$$k * (\varphi_0 * \mu) = (k * \varphi_0) * \mu$$

is a continuous function. Since $\varphi_0 - \varphi_0 * \mu$ is positive in a set of positive measure, the continuous function $\varphi - \varphi * \mu$ is not identically zero and there exists an open, relatively compact set N such that

$$(\varphi - \varphi * \mu)(x) > k > 0 \text{ for } x \in -N. \tag{2.2}$$

Now consider the algebraic identity.

$$\varphi = (\varphi - \varphi * \mu) * (\delta + \mu + \dots + \mu^{(n)}) + \varphi * \mu^{(n+1)} \tag{2.3}$$

which is valid for all positive integers n. Since the total mass of $\mu^{(n+1)}$ is one, $\varphi \geqslant -a$ implies

$$\varphi * \mu^{(n+1)} \geqslant -a. \tag{2.4}$$

It now follows from (2.2), (2.3) and (2.4) that

$$k \cdot \int_{x-y \in -N} dF_n(y) \leqslant a + \varphi(x)$$

for all x and all positive integers n, i.e.

$$F(x+N) \leqslant \frac{a+\varphi(x)}{k}$$

and B is proved.

$B \Rightarrow E$. The proof will follow from the algebraic identity

$$F_n = F_{n_0-1} + \mu^{(n_0)} * F_n - \mu^{(n+1)} - \ldots - \mu^{(n+n_0)} = H_n + \mu^{(n_0)} * F_n, \tag{2.5}$$

which is valid for all positive integers n and n_0. Using (1.5) we find that the total variation of H_n (cf. e.g. Rudin [12] p. 265) is at most $2n_0$.

Let O be an open relatively compact set such that $0 < F(O) < \infty$. Then there exists a positive integer n_0 such that $\mu^{(n_0)}(O)$ is positive. Now take $x_1 \in O$ such that for all neighborhoods N of zero $\mu^{(n_0)}(x_1+N) > 0$. It is easy to see that we can choose open, relatively compact neighborhoods of zero O_1, O_2 and O_3 such that

(a) $\{x_1 + O_1\} \subset O$,

(b) $O_2 \subset O_1$,

(c) $\bigcap_{y \in O_2} (O_1 - y) \supset O_3$.

Using (c) in the third inequality, we obtain from (2.5) that there exists a constant C such that

$$\int_{y \in x_1 + O_1} dF_n(y) \geqslant C + \iint_{\substack{x \in x_1 + O_1 \\ y \in G}} d_x F_n(x-y)\, d\mu^{(n_0)}(y) \geqslant$$

$$\geqslant C + \int_{y \in x_1 + O_2} d\mu^{(n_0)}(y) \int_{u \in x_1 + O_1 - y} dF_n(u) \geqslant$$

$$\geqslant C + \int_{y \in x_1 + O_2} d\mu^{(n_0)}(y)\, F_n(O_3) = C + \mu^{(n_0)}(x_1 + O_2)\, F_n(O_3).$$

Since $\mu^{(n_0)}(x_1+O_2)$ is positive and (a) is true, it follows that there exist constants C_1 and C_2 such that

$$F_n(O_3) \leqslant C_1 \cdot F_n(x_1 + O_1) + C_2 \leqslant C_1 \cdot F(O) + C_2.$$

Thus $F(O_3)$ is finite, and E is proved.

$E \Rightarrow C$. We first introduce certain concepts, which we shall need in the proof.

Definition 2.2. $h \in H$ *if*

(a) *the support of h is a compact neighborhood of zero,*

(b) *h is continuous, non-negative and not identically zero,*

(c) *$\hat{h}$ is non-negative and $\hat{h} \in L^1(\hat{G})$.*

Obviously the class H is non-empty. Since $h \in L^1(G)$ and $\hat{h} \in L^1(\hat{G})$ we have

$$\int_G h(x-y)\, d\mu(y) = \int_G \int_{\hat{G}} \hat{h}(\hat{x})\, (x-y, \hat{x})\, dx\, d\mu(y) = \int_{\hat{G}} \hat{h}(\hat{x})\, (x, \hat{x})\, \hat{\mu}(\hat{x})\, d\hat{x}. \tag{2.6}$$

(The integral in the middle is absolutely convergent.) We also introduce the weak limit

$$F_\varepsilon = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mu^{(k)}}{(1+\varepsilon)^k},$$

which exists if $\varepsilon > 0$. For every open set O, $F_\varepsilon(O)$ is a decreasing function of ε for $\varepsilon > 0$ and

$$\lim_{\varepsilon\to+0} F_\varepsilon(O) = F(O),$$

where the right member can be finite or infinite. Now let $h \in H$. Using (2.6), we find that

$$h * F_\varepsilon(x) = \int_{\hat{G}} \hat{h}(\hat{x})\,(x,\hat{x})\,\frac{1+\varepsilon}{1+\varepsilon-\hat{\mu}(\hat{x})}\,d\hat{x}.$$

Since $h * F_\varepsilon$ and $\hat{h}$ are real-valued, we can also write

$$h * F_\varepsilon(x) = \int_{\hat{G}} \hat{h}(\hat{x})\,\mathrm{Re}\left\{\frac{(1+\varepsilon)\,(x,\hat{x})}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x}. \tag{2.7}$$

We observe that for $\varepsilon > 0$

$$\mathrm{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} > 0. \tag{2.8}$$

We are now in a position to prove that $E \Rightarrow C$. Let O be an open, relatively compact neighborhood of zero such that $F(O)$ is finite. We choose $h \in H$ such that the support of h is contained in $-O$ and such that

$$O_1 = \{x \mid h(-x) > 1\}$$

is non-empty. Obviously $O_1 \subset O$. Hence

$$F_\varepsilon(x+O_1) \leqslant \int_{x-y\in -O_1} h(x-y)\,dF_\varepsilon(y) \leqslant h * F_\varepsilon(x).$$

Using (2.7) and (2.8), we find

$$\begin{aligned} h * F_\varepsilon[(x)+(-x)] &= \int_{\hat{G}} \hat{h}(\hat{x})\,(1+\varepsilon)\,\mathrm{Re}\left\{\frac{(x,\hat{x})+\overline{(x,\hat{x})}}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x} \\ &\leqslant 2\int_{\hat{G}} \hat{h}(\hat{x})\,(1+\varepsilon)\,\mathrm{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x} = 2h * F_\varepsilon(0). \end{aligned}$$

Hence

$$F_\varepsilon(x+O_1) \leqslant h * F(x) \leqslant 2h * F_\varepsilon(0) \leqslant 2\,\|h\|_\infty\,F_\varepsilon(O) \leqslant 2\,\|h\|_\infty\,F(O) < \infty.$$

It follows that $F(x+O_1)$ is a bounded function of x. Since an arbitrary open relatively compact set O_2 can be covered by $\{(x_n+O_1)\}_{n=1}^{N}$, the same conclusion is true for $F(x+O_2)$ and C is proved.

$C \Rightarrow D$. Let g be a continuous, non-negative function with compact support. It follows from C that $g \ast F$ is defined and is a bounded, continuous function of x. We want to prove that $g \ast F$ is a solution of the equation (2.1). Since $g \ast F_n - g \ast F_n \ast \mu = g - g \ast \mu^{(n+1)}$ and $g \ast F_n$ is uniformly bounded, this assertion follows if

$$\text{(a)} \qquad \lim_{n\to\infty} g \ast F_n(x) = g \ast F(x)$$

uniformly on every compact set and

$$\text{(b)} \qquad \lim_{n\to\infty} g \ast \mu^{(n+1)}(x) = 0 \text{ for every } x \in G.$$

(a) $\{g \ast F_n\}_{n=1}^{\infty}$ is an increasing sequence of continuous functions with the continuous limit $g \ast F$. Hence (a) follows from Dini's theorem.

(b) Since the series

$$\sum_0^\infty g \ast \mu^{(n)}(x) = g \ast F(x)$$

is convergent, we have

$$\lim_{n\to\infty} g \ast \mu^{(n+1)}(x) = 0.$$

Thus D is proved and the proof of Theorem 2.2 is complete.

The following remark is needed in the proof of Theorem 2.3.

Remark. Let g be a continuous function with compact support and assume that C in Theorem 2.2 is true. The function g is uniformly continuous, and it is easy to prove that $g \ast F$ is also uniformly continuous.

Proof of Theorem 2.3. (a) There is no essential restriction in assuming that g is non-negative. Since $g \in M$, there exists an open, relatively compact set O and a corresponding sequence $\{x_n\}_1^\infty$ with properties according to Definition 2.1. Since C in Theorem 2.2 is true, there exists a constant C_1 such that

$$0 \leqslant g \ast F(x) \leqslant \sum_1^\infty \int_{y\in x_\nu + O} g(y)\, dF(x-y) \leqslant \sum_1^\infty \max_{y\in x_\nu+O} g(x) \int_{y\in x_\nu+O} dF(x-y) \leqslant C_1.$$

Thus $g \ast F$ is a bounded function of x. It follows in the same way as in the proof of $C \rightarrow D$ in Theorem 2.2, that $g \ast F$ is a solution of the equation (2.1).

It remains to prove that $g \ast F$ is uniformly continuous. Take $\varepsilon > 0$. We choose an integer n_0 such that

$$\sum_{n_0+1}^\infty \max_{y\in x_\nu+O} g(y) < \varepsilon$$

and define $\bigcup_1^{n_0} (x_\nu + O) = O_0$. O_0 is an open, relatively compact set. Let for some neighborhood O_1 of zero k be continuous and fulfil

$$\text{(a)} \qquad k(x) = 1 \quad \text{for } x \in O_0,$$

$$\text{(b)} \qquad 0 \leqslant k(x) \leqslant 1 \quad \text{for all } x,$$

$$\text{(c)} \qquad k(x) = 0 \quad \text{for } x \in \text{compl. } (O_0 + O_1).$$

We have
$$g = k \cdot g + (1-k) g = g_1 + g_2$$
and
$$g * F = g_1 * F + g_2 * F.$$

The function g_1 is continuous and has compact support. According to the remark after the proof of Theorem 2.2 $g_1 * F$ is uniformly continuous. For $g_2 * F$, we have that there exists a constant C such that

$$|g_2 * F(x)| \leqslant C \sum_{n_0+1}^{\infty} \operatorname*{Max}_{y \in x_\nu + O} g(y) \leqslant C\varepsilon$$

and the uniform continuity of $g * F$ follows.

(b) Let φ be an arbitrary bounded solution of the equation (2.1). For the function $\psi = \varphi - g * F$, we have $\psi - \psi * \mu = 0$. Since we have assumed that $G = G(\mu)$ it follows from Theorem 2.1 that ψ is identically constant. Conversely, it is easily seen that $A + g * F$ satisfies (2.1) for arbitrary constants A, and hence Theorem 2.3 is proved.

It is difficult to see when the conditions of Theorem 2.2 are fulfilled. Lemma 2.1 gives an equivalent, more convenient condition.

Lemma 2.1. *The following property of $\hat{\mu}$ is equivalent to the conditions of Theorem 2.2.*

There exists a neighborhood $\hat{O}$ of zero in the dual group $\hat{G}$ of G and a constant C such that

$$\int_{\hat{O}} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x} \leqslant C < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{2.9}$$

Remark. Chung and Fuchs [4] have similar conditions in an n-dimensional Euclidean space. They treat the convergence problem for $\{F_n\}_1^\infty$ with combinations of probabilistic and Fourier arguments.

Proof of Lemma 2.1. We first define a class of functions H'.

Definition 2.3. $h \in H'$ *if*

(a) *the support of $\hat{h}$ is a compact neighborhood of the zero element in $\hat{G}$,*
(b) *$\hat{h}$ is continuous, non-negative and not identically zero,*
(c) *h is non-negative and $h \in L^1(G)$.*

Obviously the class H' is non-empty.

Using the same type of argument as the one used in the proof of (2.7), we see that if h belongs to H or H' and if $\varepsilon > 0$,

$$\int_G h(x)\, dF_\varepsilon(x) = \int_{\hat{G}} \hat{h}(\hat{x}) \operatorname{Re}\left\{\frac{1+\varepsilon}{1+\varepsilon-\hat{\mu}(\hat{x})}\right\} d\hat{x}. \tag{2.10}$$

Let the conditions of Theorem 2.2 be true. We take $h \in H$ (Definition 2.2). Since the left member of (2.10) is a bounded function of ε for $\varepsilon > 0$ and $\hat{h}$ is positive in some neighborhood $\hat{O}$ of the zero element of $\hat{G}$, (2.9) follows. If conversely (2.9) is true for some neighborhood $\hat{O}$ of zero, we take $h \in H'$ such that the support of $\hat{h}$ is in $\hat{O}$. Since the right member of (2.10) is bounded for $\varepsilon > 0$ and h is positive in some neighborhood of the zero element of G, B of Theorem 2.2 follows and we have proved Lemma 2.1.

3. The existence of bounded, non-trivial solutions in R^n and Z^n.

From now on, we shall only be concerned with the groups R^n and Z^n under their usual topologies. We introduce a coordinate system and define, if x belongs to R^n or Z^n

$$|x| = \left\{\sum_1^n x_\nu^2\right\}^{\frac{1}{2}}$$

and

$$tx = \sum_1^n t_\nu x_\nu.$$

Theorem 3.1. *Let G be R^n or Z^n and assume that $G(\mu) = G$. Then the following conditions for the existence of bounded, non-trivial solutions of the inequality (1.1) can be stated.*

(a) *if $n = 1$ and $\int_G |x|\, d\mu(x) < \infty$, a necessary and sufficient condition is that*

$$\int_G x\, d\mu(x) \neq 0.$$

(b) *If $n = 2$ and if for some positive number δ*

$$\int_G |x|^{1+\delta} d\mu(x) < \infty. \tag{3.1}$$

a sufficient condition is that

$$\int_G l(x)\, d\mu(x) \neq 0 \text{ for some linear function } l(x) = ax_1 + bx_2. \tag{3.2}$$

If $\int_G |x|^2 d\mu(x) < \infty$, condition (3.2) is also necessary.

(c) *If $n \geq 3$, such solutions always exist.*

Corollary. *Let μ satisfy the conditions of Theorem 3.1. Then the equation*

$$\varphi - \varphi * \mu = g$$

has bounded solutions for all $g \in M$ (Definition 2.1). These solutions are uniformly continuous and are, except for an additive constant, unique.

Proof of Theorem 3.1. According to Lemma 2.1, condition (2.9) is necessary and sufficient for the existence of bounded, non-trivial solutions of (1.1), and it thus suffices to consider (2.9).

We observe that the assumption $G(\mu) = G$ implies that

$$\hat{\mu}(t) \neq 1 \quad \text{if} \quad t \neq 0. \tag{3.3}$$

This is important for certain estimates in the proof.

We also have

$$\mathrm{Re}\,\{1 - \hat{\mu}(t)\} \geq 0. \tag{3.4}$$

Thus for every $\varepsilon > 0$

$$2(\varepsilon^2 + |1 - \hat{\mu}(t)|^2) \geqslant |1 + \varepsilon - \hat{\mu}(t)|^2 \geqslant \varepsilon^2 + |1 - \hat{\mu}(t)|^2 \geqslant |1 - \hat{\mu}(t)|^2. \tag{3.5}$$

(a) *The sufficiency.* We assume that

$$\hat{\mu}'(0) = -i \int_G x \, d\mu(x) = -im \neq 0.$$

We have

$$\operatorname{Re}\{1 - \hat{\mu}(t)\} = \int_G (1 - \cos tx) \, d\mu(x) = 2 \int_G \sin^2 \frac{tx}{2} \, d\mu(x). \tag{3.6}$$

Let the number b satisfy $0 < b < \pi$. It follows from the assumption $m \neq 0$ and (3.3) that there exists a constant C such that

$$0 < C|t| < |1 - \hat{\mu}(t)| \quad \text{if} \quad 0 < |t| < b. \tag{3.7}$$

Now consider

$$I = \int_{-b}^{b} \operatorname{Re}\left\{\frac{1}{1 + \varepsilon - \hat{\mu}(t)}\right\} dt = \int_{-b}^{b} \frac{\operatorname{Re}\{1 - \hat{\mu}(t)\}}{|1 + \varepsilon - \hat{\mu}(t)|^2} dt + \int_{-b}^{b} \frac{\varepsilon}{|1 + \varepsilon - \hat{\mu}(t)|^2} dt = I_1 + I_2.$$

Using (3.5), (3.6), and (3.7) and changing the order of integration, we see that there exist constants C_1 and C_2 such that

$$0 \leqslant I_1 \leqslant C_1 \int_G d\mu(x) \int_{-b}^{b} \frac{\sin^2 \frac{tx}{2}}{t^2} dt \leqslant C_2 \int_G |x| \, d\mu(x).$$

Using (3.4) and (3.7), we see that there exists a constant C_3 such that

$$0 \leqslant I_2 \leqslant \int_{-b}^{b} \frac{\varepsilon}{\varepsilon^2 + |1 - \hat{\mu}(t)|^2} dt \leqslant C_3 \int_{-b}^{b} \frac{\varepsilon}{\varepsilon^2 + t^2} dt \leqslant C_3 \cdot \pi.$$

Thus I is bounded and the sufficiency is proved.

The necessity. We assume $m = 0$. Then to every $\varepsilon_1 > 0$ there exists $\delta > 0$ such that $|1 - \hat{\mu}(t)| < \varepsilon_1 |t|$ for $|t| < \delta$. If $\delta < b$, it follows from (3.5) that

$$\lim_{\varepsilon \to +0} I_2 \geqslant \lim_{\varepsilon \to +0} \int_{-\delta}^{\delta} \frac{\varepsilon}{\varepsilon^2 + \varepsilon_1^2 t^2} dt = \frac{\pi}{\varepsilon_1}.$$

Hence $\lim_{\varepsilon \to +0} I_2 = \infty$, I is not a bounded function of ε, and the necessity is proved.

(b) All constants used in this proof are positive. Let (3.1) and (3.2) be true. It follows from (3.1) that the partial derivatives of $\hat{\mu}$ satisfy a Lipschitz condition of order δ, We choose the coordinate system in the t-plane so that

$$\frac{\partial \hat{\mu}}{\partial t_1}(0) = 2ai \neq 0 \quad (a > 0),$$

$$\frac{\partial \hat{\mu}}{\partial t_2}(0) = 0.$$

Two estimates of $|1-\hat{\mu}(t)|$ are needed. The first one is obtained by use of the mean value theorem and of the Lipschitz condition for the partial derivatives of $\hat{\mu}$ and gives the existence of a positive number $b<\pi$ and constants C_1 and K such that

$$\begin{aligned}(1)\qquad |1-\hat{\mu}(t)| &\geqslant |\operatorname{Im}\{\hat{\mu}(0)-\hat{\mu}(t)\}| \\ &= \left|\operatorname{Im}\left\{t_1\frac{\partial\hat{\mu}}{\partial t_1}(\theta t)+t_2\frac{\partial\hat{\mu}}{\partial t_2}(\theta t)\right\}\right| \\ &\geqslant 2a|t_1|-C_1\{|t_1||t|^{\delta}+|t_2||t|^{\delta}\} \\ &\geqslant 2a|t_1|-K|t|^{1+\delta}\quad\text{for}\quad |t|<b.\end{aligned}$$

Our second estimate is obtained by use of a method due to Chung and Fuchs [4]. We observe that it follows from the assumption $G(\mu)=G$ that the mass of μ cannot lie on a straight line through the origin. Hence there exists a constant C_2 such that

$$(2)\qquad |1-\hat{\mu}(t)|\geqslant \operatorname{Re}\{1-\hat{\mu}(t)\}=\iint_G 2\sin^2\frac{tx}{2}\,d\mu(x)\geqslant |t|^2C_2\quad\text{for}\quad |t|<b.$$

We shall prove that there exists a constant C such that

$$\iint_{|t|<b}\operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(t)}\right\}dt\leqslant C<\infty\quad\text{for all}\quad \varepsilon>0. \tag{3.8}$$

It follows from (3.5) that

$$\left|\operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(t)}\right\}\right|\leqslant\frac{1}{|1+\varepsilon-\hat{\mu}(t)|}\leqslant\frac{1}{|1-\hat{\mu}(t)|}.$$

Hence it is sufficient to prove that

$$\iint_{|t|<b}\frac{dt}{|1-\hat{\mu}(t)|}<\infty.$$

We have

$$I=\iint_{|t|<b}\frac{dt}{|1-\hat{\mu}(t)|}=\left\{\iint_{D_1}+\iint_{D_2}\right\}\frac{dt}{|1-\hat{\mu}(t)|}=I_1+I_2.$$

Here

$$D_1=\{t\,|\,a|t_1|-K|t|^{1+\delta}\geqslant 0;\quad |t|\leqslant b\}$$

and

$$D_2=\{t\,|\,a|t_1|-K|t|^{1+\delta}\leqslant 0;\quad |t|\leqslant b\}.$$

The estimates (1) and (2) are used for I_1 and I_2 respectively. There exist constants k and $\{C_i\}^4$ such that

$$\begin{aligned}0\leqslant I_1 &\leqslant C_1\int_0^{2\pi}\int_0^{k|\cos\varphi|^{1/\delta}}\frac{r\,dr\,d\varphi}{ar|\cos\varphi|}= \\ &=C_2\int_0^{2\pi}|\cos\varphi|^{1/\delta-1}\,d\varphi<\infty.\end{aligned}$$

$$0 \leqslant I_2 \leqslant C_3 \int_0^{2\pi} \int_{k|\cos\varphi|^{1/\delta}}^{b} \frac{r\,dr\,d\varphi}{r^2} \leqslant$$
$$\leqslant C_4 \int_0^{2\pi} \left\{1 + \frac{1}{\delta} \left| \log |\cos\varphi| \right| \right\} d\varphi < \infty.$$

Hence I is convergent. Thus, if for some positive number δ

$$\int_G |x|^{1+\delta} d\mu(x) < \infty,$$

it is true that (3.2)$\to$(3.8).

We now assume that

$$\int_G |x|^2 d\mu(x) < \infty.$$

Obviously, it is still true that (3.2)$\to$(3.8). We shall prove that (3.8)$\to$(3.2).

If (3.2) is false, then

$$\frac{\partial \hat{\mu}}{\partial t_1}(0) = \frac{\partial \hat{\mu}}{\partial t_2}(0) = 0.$$

We conclude, in the same way as in the discussion of estimate 2 that there exist constants C_1 and C_2 such that

$$\operatorname{Re}\{1 - \hat{\mu}(t)\} \geqslant C_1 |t|^2 \quad \text{for} \quad |t| < b.$$
$$|1 - \hat{\mu}(t)| \leqslant C_2 |t|^2 \quad \text{for} \quad |t| < b.$$

Hence it follows by use of (3.5) that there exist constants C_3 and C_4 such that

$$\iint_{|t|<b} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(t)}\right\} dt \geqslant \iint_{|t|<b} \frac{\operatorname{Re}\{1-\hat{\mu}(t)\}}{|1+\varepsilon-\hat{\mu}(t)|^2} dt \geqslant$$
$$\geqslant C_3 \int_0^b \int_0^{2\pi} \frac{r^3\,dr\,d\varphi}{\varepsilon^2 + r^4} = C_4 \int_0^{b/\sqrt{\varepsilon}} \frac{u^3\,du}{1+u^4} \to \infty$$

when $\varepsilon \to +0$, (3.8) is false and the proof of (b) is complete.

(c) Let the number b satisfy $0 < b < \pi$. According to Theorem 6 in Chung and Fuchs [4], there exists a positive constant C such that

$$|1 - \hat{\mu}(t)| \geqslant C|t|^2 \quad \text{for} \quad |t| < b.$$

Since

$$\int_{|t|<b} \operatorname{Re}\left\{\frac{1}{1+\varepsilon-\hat{\mu}(t)}\right\} dt \leqslant \int_{|t|<b} \frac{dt}{|1-\hat{\mu}(t)|} \leqslant$$
$$\leqslant \frac{1}{C} \int_0^b r^{n-3} dr < \infty$$

if $n \geqslant 3$, (c) follows.

4. Properties of bounded non-trivial solutions of $\varphi - \varphi * \mu \geqslant 0$

The main part of this paragraph is devoted to a study of the bounded solutions of the equation

$$\varphi - \varphi * \mu = g, \tag{4.1}$$

where $g \in M$ (Definition 2.1). If g is non-negative, these solutions also satisfy

$$\varphi - \varphi * \mu \geqslant 0. \tag{4.2}$$

The one-dimensional cases (R^1 and Z^1) are treated in Theorems 4.1, 4.2 and 4.3 and the multi-dimensional cases (R^n and Z^n for $n \geqslant 2$) in Theorem 4.4. We especially mention the following result: Consider the class $\{\mu\}$ of measures with the property that if φ is a bounded, non-trivial solution of (4.2), then $\varphi - \varphi * \mu \in L^1(G)$. On R^1 and Z^1, this class is non-empty since it contains every measure μ which satisfies the conditions of Theorems 3.1 a (cf. Theorems 4.1 a and 4.3 a). If $n \geqslant 2$ there exists no measure in this class on R^n and Z^n (cf. Theorem 4.4 b).

Theorems 4.1 *Let $G = G(\mu) = R^1$ and assume that*

$$\int_{-\infty}^{\infty} |x| \, d\mu(x) < \infty$$

and that

$$\int_{-\infty}^{\infty} x \, d\mu(x) = m \neq 0.$$

(a) *If the bounded function φ is a solution of the inequality (4.2), then $\varphi - \varphi * \mu \in L^1(R^1)$.*

(b) *Let φ be as in* (a). *If φ is slowly decreasing (cf. e.g. Definition 9 b, Ch. V in Widder [14]), then $\lim_{x\to\infty}\varphi(x)$ and $\lim_{x\to-\infty}\varphi(x)$ exist and*

$$\varphi(\infty) - \varphi(-\infty) = \frac{\int_{-\infty}^{\infty} (\varphi - \varphi * \mu)(x) \, dx}{m}.$$

(c) *Let φ be as in (a). If $\lim_{x\to\infty}\varphi(x) = \lim_{x\to-\infty}\varphi(x)$, then $\varphi - \varphi * \mu = 0$.*

(d) *Let $g \in M$ (Definition 2.1). If φ is a bounded solution of the equation (4.1), then $\lim_{x\to\infty}\varphi(x)$ and $\lim_{x\to-\infty}\varphi(x)$ exist and*

$$\varphi(\infty) - \varphi(-\infty) = \frac{\hat{g}(0)}{m}.$$

Remark 1. Let ν be a bounded measure satisfying $\nu - \nu * \mu \geqslant 0$. Then results corresponding to those in (a) and (c) are valid for ν. In (c), we assume that

$$\lim_{x\to\infty} \nu * k(x) = \lim_{x\to-\infty} \nu * k(x)$$

for every continuous and non-negative function k with compact support. The results are

(a) $$\int_{-\infty}^{\infty} d(\nu - \nu * \mu)(x) < \infty$$

and

(c) $$\nu - \nu * \mu = 0.$$

This follows immediately from the formula

$$\int_{-\infty}^{\infty}(\nu-\nu*\mu)*k(x)\,dx=\int_{-\infty}^{\infty}k(x)\,dx\cdot\int_{-\infty}^{\infty}d(\nu-\nu*\mu)(x).$$

Remark 2. Results similar to those in Theorem 4.1 are found in Feller [8] for right members g with compact support but with weaker conditions on μ. Feller only assumes the existence of F. His methods are probabilistic. Similar result can also be found in Karlin [9], and his methods of proof resemble ours. Karlin only treats measures μ such that $\overline{\lim}_{|t|\to\infty}|\hat{\mu}(t)|<1$.

Before proving Theorem 4.1, we state two other theorems. The proofs of these three theorems are closely related and it is natural to treat them simultaneously.

The following Banach algebra A_n will be used in Theorems 4.2 and 4.3.

Definition 4.1. *Let G be R^1 or Z^1, let n be a non-negative and C_0 a positive number. The function $g\in A_n$ if*

$$\|g\|=C_0\int_G(1+|x|^n)|g(x)|\,dx<\infty.$$

With suitable choice of C_0, A_n is a Banach algebra of functions on R^1 and Z^1.

Theorem 4.2. *Let μ and g be as in Theorem 4.1, let for some non-negative number n*

$$\int_{-\infty}^{\infty}|x|^{n+1}\,d\mu(x)<\infty \tag{4.3}$$

and let $g\in A_n$ be such that $\hat{g}$ has compact support. If the bounded function φ is a solution of the equation (4.1), then φ' exists and $\varphi'\in A_n$.

Corollary. *Let φ be as in Theorem 4.2. If $n\geqslant 1$, then*

$$\int_0^{\infty}\{|\varphi(\infty)-\varphi(x)|+|\varphi(-\infty)-\varphi(-x)|\}|x|^{n-1}\,dx<\infty.$$

We introduce the following notation. If φ is a function on Z^1, φ' is defined by the formula

$$\{\varphi_k'\}_{k=-\infty}^{\infty}=\{\varphi_k-\varphi_{k-1}\}_{k=-\infty}^{\infty}.$$

Theorem 4.3. *Let $G=G(\mu)=Z^1$ and assume that for some non-negative number n*

$$\sum_{-\infty}^{\infty}|k|^{n+1}\mu k<\infty \tag{4.4}$$

and that

$$\sum_{-\infty}^{\infty}k\mu_k=m\neq 0.$$

(a) *Assume that $n=0$. If the bounded sequence φ is a solution of the inequality (4.2), then $\varphi-\varphi*\mu\in L^1(Z^1)$.*

(b) *Let φ and n be as in (a). If $\lim_{k\to\infty}\varphi_k=\lim_{k\to-\infty}\varphi_k$, then $\varphi-\varphi*\mu=0$.*

(c) *Assume that* $n=0$ *and that* $g \in L^1(Z^1)$. *If* φ *is a bounded solution of the equation (4.1), then* $\lim_{k\to\infty}\varphi_k$ *and* $\lim_{k\to-\infty}\varphi_k$ *exist and*

$$\varphi_\infty - \varphi_{-\infty} = \frac{\hat{g}(0)}{m}.$$

(d) *Let the sequence* $g \in A_n$. *If* φ *is a bounded solution of the equation (4.1), then* $\varphi' \in A_n$.

Corollary. *Let* φ *be as in Theorem 4.3 d. If* $n \geq 1$, *then*

$$\sum_1^\infty k^{n-1}\{|\varphi_\infty - \varphi_k| + |\varphi_{-\infty} - \varphi_{-k}|\} < \infty.$$

Remark 1. Theorem 4.3 is the analogue on Z^1 of Theorems 4.1 and 4.2. Since on Z^1 every function is slowly decreasing, the analogue of Theorem 4.1 b is contained in Theorem 4.3 c.

The results for Z^1 are more complete than those for R^1. This is due to the fact that the dual group of Z^1, the unit circle, is compact.

Remark 2. The corollaries of Theorems 4.2 and 4.3 are related to results in Karlin [9], obtained by use of an induction argument. On R^1, Karlin only treats the case when μ is absolutely continuous and the right member g in the equation (4.1) is monotonic at infinity. Our corresponding restriction is the assumption that $\hat{g}$ has compact support.

The following lemma is used in the proofs of Theorems 4.1, 4.2, and 4.3.

Lemma 4.1. *Let the function* p *satisfy conditions 2.12 and 2.13 in Domar [5] and consider the Banach algebra* $A(p)$ *of functions on* G *under the norm*

$$\|f\| = \int_G |f(x)|\,p(x)\,dx.$$

If C *is a compact set in* $\hat{G}$ *and* $f \in A(p)$ *is such that* $\hat{f}(\hat{x}) \neq 0$ *on* C, *then there exists* $g \in A(p)$ *such that* $\hat{f}(\hat{x})\hat{g}(\hat{x}) = 1$ *on* C.

Proof of Lemma 4.1. $A(p)$ is a regular Banach algebra and $\hat{f}$ is the Gelfand representation of $f \in A(p)$ (Domar [5] Theorem 2.11, Lemma 1.24 and p. 15). Let $k(C)$ be the ideal of functions in $A(p)$ whose Fourier transforms are zero on C. Since $k(C)$ is a closed ideal, the quotient algebra $A(p)/k(C)$ is a Banach algebra with identity and the result follows (Loomis [10] 6 B, 23 B, 25 B).

Proof of Theorems 4.1 and 4.3 a, b. For the proof of Theorem 4.1 a, b, we refer to Essén [7]. It is there assumed that μ is absolutely continuous, but the generalization to measures μ satisfying the conditions of Theorem 4.1 is easy. Theorem 4.1 c is a direct consequence of Theorem 4.1 b. The results in Theorem 4.3 a, b follow in the same way, if partial integrations are replaced by partial summations.

To prove Theorem 4.1 d, we observe that the general solution of the equation (4.1) has, according to Theorem 2.3, the form $A + g * F$. Hence our solution φ is uniformly continuous. In particular it is slowly decreasing and hence our result follows from Theorem 4.1 b.

Proof of Theorem 4.2. We shall use Lemma 4.1 with $p(x) = C_0(1 + |x|^n)$, which satisfies the conditions 2.12 and 2.13 in Domar [5] if the constant C_0 is suitably chosen. Obviously $A(p) = A_n$ (Definition 4.1). Let

$$N_1(x) = \begin{cases} \int_x^{\infty} d\mu(y) & (x > 0). \\ -\int_{-\infty}^{x} d\mu(y) & (x < 0). \end{cases}$$

Condition (4.3) implies that $N_1 \in A_n$. Since μ has no mass at zero,

$$\begin{cases} \hat{N}_1(t) = \dfrac{1 - \hat{\mu}(t)}{it} & (t \neq 0), \\ \hat{N}_1(0) = m. \end{cases}$$

We assume that the function $[1 - \hat{\mu}(t)]/it$ is defined by continuity for $t = 0$. Since $\hat{N}_1(t) \neq 0$ for all t, there exists for every compact set C a function in A_n whose Fourier transform is $(it)/[1 - \hat{\mu}(t)]$ on C. Now consider the equation (4.1) where $g \in A_n$, $g \in M$ (Definition 2.1) and $\hat{g}$ has compact support. There exists $h \in A_n$ such that

$$\hat{h}(t) = \frac{it\hat{g}(t)}{1 - \hat{\mu}(t)}.$$

Here $\hat{h}(0) = \hat{g}(0)/m$ and $h - h * \mu = g'$. Integrating, we obtain

$$\int_a^x h(t)\,dt - \int_{-\infty}^{\infty} \int_{a-y}^{x-y} h(t)\,dt\,d\mu(y) = g(x) - g(a).$$

Let
$$H(x) = \int_{-\infty}^{x} h(t)\,dt.$$

If $a \to -\infty$, we obtain

$$H - H * \mu = g.$$

If φ is a bounded solution of (4.1), it follows from Theorem 2.3 that there exists a constant C such that

$$\varphi = H + C.$$

Thus $\varphi' = h \in A_n$ and Theorem 4.2 is proved.

Proof of the corollary. Since

$$\begin{cases} \varphi(\infty) - \varphi(x) = \int_x^{\infty} h(t)\,dt, \\ \varphi(x) - \varphi(-\infty) = \int_{-\infty}^{x} h(t)\,dt \end{cases}$$

and $h \in A_n$, the corollary follows.

Proof of Theorem 4.3 c, d. The same principles as in the discussion on R^1 will be used, but on Z^1 we have to replace derivation by the formation of differences. We put

$$\hat{a}(t) = \sum_{-\infty}^{\infty} a_k e^{-ikt}.$$

The sequence $a' = \{a_k - a_{k-1}\}_{-\infty}^{\infty}$ has the Fourier transform $\hat{a}(t)(1-e^{-it})$. Hence $[1-\hat{\mu}(t)]/(1-e^{-it})$ (defined by continuity for $t=0$) is the Fourier transform of c, where

$$\begin{cases} c_\nu = \sum_{\nu+1}^{\infty} \mu_k & \text{for} \quad \nu \geq 0, \\ c_\nu = -\sum_{-\infty}^{\nu} \mu_k & \text{for} \quad \nu \leq -1. \end{cases}$$

It follows from (4.4) that $c \in A_n$ (Definition 4.1). Since $\hat{c}(t) \neq 0$ for all t (in particular we have $\hat{c}(0) = m \neq 0$), Lemma 4.1 implies that $(1/\hat{c}) \in A_n$ (the compact set C of the lemma is chosen to be the whole unit circle). There exists $h \in A_n$ such that

$$\hat{h}(t) = \frac{\hat{g}(t)(1-e^{-it})}{1-\hat{\mu}(t)}.$$

Here $\hat{h}(0) = \hat{g}(0)/m$ and $h - h \ast \mu = g'$. A summation gives

$$\sum_{\nu=n_0}^{n} h_\nu - \sum_{k=-\infty}^{\infty} \sum_{n_0-k}^{n-k} h_\nu \mu_k = g_n - g_{n_0-1}.$$

Let
$$\{H_n\}_{-\infty}^{\infty} = \left\{\sum_{-\infty}^{n} h_\nu\right\}_{-\infty}^{\infty}$$

If $n_0 \to -\infty$, we obtain

$$H - H \ast \mu = g.$$

If φ is a bounded solution of (4.1), it follows from Theorem 2.3 that there exists a constant C such that

$$\varphi = H + C.$$

Thus $\lim_{n\to\infty} \varphi_n$ and $\lim_{n\to-\infty} \varphi_n$ exist,

$$\varphi_\infty - \varphi_{-\infty} = \sum_{-\infty}^{\infty} h_n = \hat{h}(0) = \frac{\hat{g}(0)}{m},$$

and Theorem 4.3 *c* is proved. Theorem 4.3 *d* and the corollary follow in the same way as in Theorem 4.2.

It remains to investigate what happens in R^n and Z^n when $n \geq 2$.

Theorem 4.4. *Let $G = G(\mu) = R^n$, where $n \geq 2$. If $n = 2$, we assume that μ satisfies conditions* (3.1) *and* (3.2).

(a) *Let $g \in M$ (Definition 2.1). If φ is a bounded solution of the equation (4.1), then $\lim_{x\to\infty} \varphi(x)$ exists.*

(b) *There exists a bounded solution φ of the inequality* (4.2) *such that $\varphi - \varphi * \mu \notin L^1(R^n)$ and $\lim_{x\to\infty} \varphi(x)$ does not exist.*

Remark. The analogous theorem on Z^n is proved in the same way.

Proof of Theorem 4.4 a. In the proof we shall use the Wiener Tauberian theorem. We wish to consider the class M as a Banach algebra under convolution. As norm we chose

$$\|g\| = \sum_{\nu=1}^{\infty} \operatorname{Max}_{x\in I_\nu} |g(x)|,$$

where $\{I_\nu\}_1^\infty$ is e.g. a partition of R^n into "cubes" with side 1. It is clear that there exists a constant C, which only depends on the dimension n such that

$$\|g * f\| \leqslant C \|g\| \, \|f\|.$$

We can equip the algebra with an equivalent norm so that the constant is replaced by 1 (Loomis [10] § 18).

Now the Gelfand transforms of the functions in M form an algebra belonging to the class of Banach algebras considered in Domar [5]. It is easy to verify that the assumptions I and II in [5], Ch. I, § 1 are satisfied. Hence the Wiener Tauberian theorem [ibid. Theorem 1.53] is valid in this algebra.

In the special case $G = R^1$, this has also been found by Edwards [6].

The general solution of the equation (4.1) has, according to Theorem 2.3, the form $A + g * F$. Consider the set I of functions $g \in M$ such that $\lim_{x\to\infty} g * F(x) = 0$. It follows in the same way as in the proof of Corollary 2 in 37 A in Loomis [10] that I is a closed ideal. We shall prove that $\lim_{x\to\infty} g * F(x) = 0$ for every $g \in M$ such that $\hat{g}$ has compact support. It then follows from the Wiener Tauberian theorem that $\lim_{x\to\infty} g * F(x) = O$ for all $g \in M$ and the proof of Theorem 4.4 a will be complete.

In the same way as in the proofs of Theorem 3.1 b or c, we conclude that the following integrals are absolutely convergent. The constant C_n is only dependent on the dimension n.

$$\varphi(x) = C_n \int_{\hat{G}} e^{itx} \frac{\hat{g}(t)}{1 - \hat{\mu}(t)} \, dt =$$

$$= \lim_{\varepsilon\to+0} C_n \int_{\hat{G}} e^{itx} \frac{\hat{g}(t)\,(1+\varepsilon)}{1 + \varepsilon - \hat{\mu}(t)} \, dt = \lim_{\varepsilon\to+0} g * F_\varepsilon(x) = g * F(x).$$

Since φ is the Fourier transform of $\hat{g}/(1-\hat{\mu}) \in L^1(\hat{G})$, $\lim_{x\to\infty} \varphi(x) = 0$ and Theorem 4.4 a is proved.

Proof of Theorem 4.4 b. Let q be a continuous, non-negative function with compact support. Theorem 2.3 and Theorem 4.4 a imply that $a = q * F$ is uniformly continuous and that $\lim_{x\to\infty} a(x) = 0$. (This conclusion is not true in one dimension.) We shall construct a non-negative function g such that

(a) $a * g(x)$ is bounded function of x,

(b) $g \notin L^1$,

(c) $\lim_{x\to\infty} a * g(x)$ does not exist.

The existence of such a function proves Theorem 4.4 b.

Construction of g: Suppose that $a(x) < 2^{-n}$ for $|x| > R_n$, where the sequence $\{R_n\}_1^\infty$ is chosen so that

$$R_1 + R_2 + \ldots + R_{n-1} < R_n. \quad (n = 2, 3, \ldots). \tag{4.5}$$

We put
$$Q_n = \{x \mid R_{n-1} \leqslant |x| < R_n\} \quad (n \geqslant 2),$$
$$Q_1 = \{x \mid |x| < R_1\}.$$

In each annulus Q_{4n} we take a sphere S_{4n} (all with the same radius). Let now g be continuous and such that

(a) $g(x) = 1$ when $x \in S_{4n}$ $(n = 1, 2, \ldots)$,

(b) $0 \leqslant g(x) \leqslant 1$,

(c) $\int_{Q_{4n}} g(x)\,dx = 1$ $(n = 1, 2, \ldots)$,

(d) $g(x) = 0$ if $x \notin Q_{4n}$ $(n = 1, 2, \ldots)$.

It is clear that $g \notin L^1$.

First we have to show that $a \ast g$ is defined and bounded. Let $x \in Q_N$. Using (4.5) we obtain

$$|x - y| \geqslant R_{N-3} \quad \text{if} \quad x \in Q_N, \quad y \in Q_k \quad (k \leqslant N-2),$$
$$|x - y| \geqslant R_{k-3} \quad \text{if} \quad x \in Q_N, \quad x \in Q_k \quad (k \geqslant N+2)$$

and that there exist constants $\{C_i\}_1^3$ such that

$$\begin{aligned}\int_{R^n} a(x-y)g(y)\,dy &= \sum_{k=1}^{\infty} \int_{Q_k} a(x-y)\,g(y)\,dy \leqslant \\ &\leqslant \sum_{k=1}^{N-2} 2^{-N+3} \int_{Q_k} g(y)\,dy + C_1 \sum_{N-1}^{N+1} \int_{Q_k} g(y)\,dy + \sum_{N+2}^{\infty} 2^{-k+3} \int_{Q_k} g(y)\,dy \leqslant \\ &\leqslant (N-2)\,2^{-N+3} + C_2 + 2^{-N+2} \leqslant C_3.\end{aligned} \tag{4.6}$$

Thus $a \ast g$ is a bounded function. Since for all positive integers $g(x) = 1$ for $x \in S_{4n}$ n and the function a is non-negative and not identically zero,

$$\overline{\lim_{x\to\infty}}\, a \ast g(x) > 0.$$

It remains to prove that

$$\underline{\lim_{x\to\infty}}\, a \ast g(x) = 0.$$

Take $N \equiv 2 \pmod 4$ and $x \in Q_N$. Since $\int_{Q_k} g(y)\,dy = 0$ if $k \not\equiv 0 \pmod 4$, it follows from (4.6) that

$$a \ast g(x) \leqslant (N-2)\,2^{-N+3} + 2^{-N+2} \to 0 \quad \text{as} \quad N \to \infty,$$

and the proof of Theorem 4.4 b is concluded.

5. Properties of unbounded solutions of the inequality $\varphi - \varphi * \mu \geqslant 0$ in R^1 and Z^1 when no bounded non-trivial solutions exist

In the remaining part of the paper we are going to deal with the case when no bounded, non-trival solutions of the inequality

$$\varphi - \varphi * \mu \geqslant 0 \tag{5.1}$$

exist. According to Theorem 3.1, this means that the groups considered will be R^n and Z^n with $n = 1$ and $n = 2$.

In this paragraph, we deal with measures μ in R^1 or Z^1 satisfying $\int_G d\mu = 1$ and $\int_G x\, d\mu(x) = 0$ and possessing moments of a certain order. We investigate properties of solutions not growing faster than linear functions (Theorem 5.1). Since it is easily proved that every concave function satisfies the inequality, such solutions exist. The existence of other solutions than the concave ones will follow from the discussion of the equation

$$\varphi - \varphi * \mu = g \tag{5.2}$$

(Theorems 5.2 a and 5.3 a) where we assume that $g \in L^1$ and that $\hat{g}$ has compact support. We also show that if a solution φ of the inequality does not grow too fast, then $\varphi - \varphi * \mu \in L^1$ (Theorem 5.1). Summing up these results, we obtain regularity properties of functions satisfying the inequality and a certain growth condition (Theorems 5.2 b and 5.3 b).

In order to find all the solutions of the inequality, we have to solve the equation

$$\varphi - \varphi * \mu = 0. \tag{5.3}$$

Now unbounded solutions can occur and these will be found by use of the theory of Beurling [2] and Domar [5]. (The result of Choquet and Deny mentioned in Theorem 2.1 only gives us the bounded solutions.) For simplicity the following lemma will be proved only for the case $G = R^1$, but analogous results with analogous proofs are true in Z^1, R^2 and Z^2.

Lemma 5.1. *Let $G = G(\mu) = R^1$ and assume that for some positive integer n*

$$\int_{-\infty}^{\infty} |x|^n \, d\mu(x) < \infty.$$

If φ is a solution of the equation (5.3) such that $\varphi(x) = O(|x|^n)$ at infinity, then φ is a polynomial of degree not exceeding n.

Corollary. *Let μ and φ be as in Lemma 5.1. We further assume that*

$$\int_{-\infty}^{\infty} x\, d\mu(x) = 0.$$

Then there exist constants A and B such that $\varphi(x) = Ax + B$. Conversely, every such function satisfies (5.3).

Proof of Lemma 5.1. Let $f \in A_n$ (Definition 4.1). Consider $0 = (\varphi - \varphi * \mu) * f = \varphi * (f - f * \mu)$. Here $f - f * \mu \in A_n$ and $\varphi \in A_n^*$ (the class of linear functionals on A_n). Let

Λ_φ be the spectrum of φ (cf. e.g. Definition 3.21 in Domar [5]). We here have that $\Lambda_\varphi \subset \{t \mid \hat{f}(t)(1-\hat{\mu}(t))=0\}$. Since f is arbitrary in A_n, it follows that $\Lambda_\varphi \subset \{t \mid \hat{\mu}(t)=1\}$. Now, according to Definition 3.41 in Domar [5], $\{\hat{f} \mid f \in A_n\}$ has polynomial growth $< n+1$ for a compact neighborhood of zero in $\hat{G}$ and $x_0 \neq 0$ (and growth <1 for $x_0=0$). Hence Theorem 3.42 in Domar [5] implies that φ is a polynomial of degree not exceeding n.

Proof of the corollary. Lemma 5.1 implies that φ is a polynomial. Let $\varphi_k(x)=x^k$. We obtain $(\varphi_k - \varphi_k * \mu)(x) = -\binom{k}{2} x^{k-2} \int_{-\infty}^{\infty} y^2 \, d\mu + \ldots$ If P_k is a polynomial of degree $k \geqslant 2$, then $P_k - P_k * \mu$ contains a term of degree $k-2$, which cannot be compensated by any other term. Thus $P_k - P_k * \mu \neq 0$ if $k \geqslant 2$. It is easy to see that $\varphi(x) = Ax+B$ satisfies the equation (5.3) and the corollary follows.

We shall now investigate the connection between the growth of a solution φ of (5.1) and the magnitude of $\int_{-\infty}^{\infty} (\varphi - \varphi * \mu)(x)\,dx$.

Theorem 5.1. *Let $G = G(\mu) = R^1$ and assume that for some α satisfying $0 \leqslant \alpha \leqslant 1$*

$$\int_{-\infty}^{\infty} |x|^{1+\alpha} d\mu(x) < \infty$$

and that

$$\int_{-\infty}^{\infty} x \, d\mu(x) = 0.$$

If φ is a solution of the inequality (5.1), then

(a) $\varphi(x) = O(|x|^\alpha) \Rightarrow \varphi - \varphi * \mu \in L^1(R^1)$,

(b) $\varphi(x) = o(|x|^\alpha) \Rightarrow \varphi - \varphi * \mu = 0$.

Remark 1. The corresponding theorem on Z^1 is true. The proof of this will be discussed in the proof of Theorem 5.3.

Remark 2. Let α be a given number satisfying $0 < \alpha \leqslant 1$. Then there exists a measure μ and a solution φ of the corresponding inequality (5.1) such that $\lim_{|x| \to \infty} \varphi(x)/|x|^\alpha$ exists different from zero. We can for instance take $d\mu(x) = f(x)\,dx$, where

$$f(x) = \begin{cases} \dfrac{1+\alpha}{2} \dfrac{1}{|x|^{2+\alpha}} & \text{for } |x| \geqslant 1 \\ 0 & \text{for } |x| < 1. \end{cases}$$

Then we have

$$\lim_{t \to 0} \frac{1-\hat{\mu}(t)}{|t|^{1+\alpha}} \neq 0,$$

and, if $g \in H'$ (Definition 2.3), that the function

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(t)(e^{itx}-1)}{1-\hat{\mu}(t)} dt$$

is a solution of (5.1) with the desired property.

Remark 3. Let μ be as in the theorem, let μ satisfy

$$\nu - \nu * \mu \geqslant 0$$

and assume that for some continuous, non-negative function k with compact support

$$\nu * k(x) = O(|x|^{\alpha}) \quad \text{at infinity}$$

or

$$\nu * k(x) = o(|x|^{\alpha}) \quad \text{at infinity.}$$

Then results corresponding to those in Theorem 5.1 are true for ν. Confer Remark 1 of Theorem 4.1!

We shall need the following lemma in the proof of Theorem 5.1.

Lemma 5.2. *Let $h \in L^1(R^1)$ be even, bounded and non-negative with two continuous, bounded derivatives. We further assume that $h(0) = 1$, that h'' is monotonic for $|x| > B > O$ and that for some number α satisfying $0 \leqslant \alpha \leqslant 1$*

$$\int_{-\infty}^{\infty} |y|^{\alpha} h(y)\, dy < \infty,$$

$$\int_{-\infty}^{\infty} |y|^{\alpha} |h'(y)|\, dy < \infty$$

and

$$\int_{-\infty}^{\infty} |y|^{\alpha} |h''(y)|^{(1+\alpha)/2}\, dy < \infty.$$

Then there exists a constant C such that

$$|r|^{-1-\alpha} \int_{-\infty}^{\infty} |y|^{\alpha} |h(y) - h(y+r) + rh'(y)|\, dy \leqslant C \quad \text{for} \quad r \neq 0. \tag{5.4}$$

Proof of Lemma 5.2. We first assume that $|r| \leqslant 1$. Using

$$|h(y) - h(y+r) + rh'(y)| \leqslant r^2 |h''(y + \theta_y r)|$$

we obtain that there exist constants C_1 and C_2 such that

$$\begin{aligned} |r|^{-1-\alpha} \int_{-\infty}^{\infty} |y|^{\alpha} |h(y) - h(y+r) + rh'(y)|\, dy \leqslant \\ \leqslant \int_{-\infty}^{\infty} |y|^{\alpha} |h(y) - h(y+r) + rh'(y)|^{(1-\alpha)/2} \cdot |h''(y+\theta_y r)|^{(1+\alpha)/2} dy \leqslant \\ \leqslant C_1 \left\{ \int_{|y| < B+1} + \int_{|y| > B+1} \right\} |y|^{\alpha} |h''(y+\theta_y r)|^{(1+\alpha)/2} dy \leqslant \\ \leqslant C_2 \left\{ 1 + \int_{|y| > B+1} |y|^{\alpha} \{ |h''(y-1)| + |h''(y+1)| \}^{(1+\alpha)/2} dy \right\} = C. \end{aligned}$$

Thus (5.4) is proved for $|r| \leqslant 1$.

If $|r|\geqslant 1$, the contribution of every term in the integrand in (5.4) is bounded (division by $r^{1+\alpha}$) and the result follows.

Proof of Theorem 5.1. We shall show that it suffices to prove the theorem for differentiable solutions of the inequality (5.1). Let φ be a solution of (5.1) and choose a non-negative function $k\in C^{\infty}$ with compact support. Then $\varphi_0=\varphi * k$ is a differentiable solution of (5.1), and it follows by the same argument as in Remark 1 of Theorem 4.1 that if the theorem is true for the solution φ_0, it is also true for the solution φ.

We assume $\varphi_0(0)=0$. Hence for every positive number b there exist constants k_0 and k_1 such that

$$\begin{cases} |\varphi_0(x)|\leqslant k_0|x|^{\alpha}, & (|x|\leqslant b),\\ |\varphi_0(x)|\leqslant k_1|x|^{\alpha}, & (|x|\geqslant b).\end{cases}$$

We put $\varphi_0-\varphi_0*\mu=g$ and investigate $\int_{-\infty}^{\infty} g(x)\,h(ax)\,dx$ with h chosen as in Lemma 5.2. If there exists a positive constant C such that

$$\int_{-\infty}^{\infty} g(x)\,h(ax)\,dx\leqslant C$$

for a near to 0, then $g\in L^1(R^1)$. Since g is non-negative and continuous, we also have that $g(x)=0$ for all x if C can be taken arbitrarily small.

$$\begin{aligned}\left|\int_{-\infty}^{\infty} g(x)\,h(ax)\,dx\right| &= \left|\int_{-\infty}^{\infty}(\varphi_0-\varphi_0*\mu)(x)\,h(ax)\,dx\right|\\ &= \left|\int_{-\infty}^{\infty}\varphi_0(x)\left\{h(ax)-\int_{-\infty}^{\infty} h(ax+ay)\,d\mu(y)\right\}dx\right|\\ &= \left|\int_{-\infty}^{\infty}\varphi_0(x)\int_{-\infty}^{\infty}\{h(ax)-h(ax+ay)+ayh'(ax)\}\,d\mu(y)\,dx\right|\\ &\leqslant \int_{-\infty}^{\infty} d\mu(y)\int_{-\infty}^{\infty}|\varphi_0(x)\{h(ax)-h(ax+ay)+ayh'(ax)\}|\,dx.\end{aligned}$$

We divide the inner integral into $\int_{|x|<b}+\int_{|x|>b}$ and use Lemma 5.2 and our estimates of φ_0.

$$\begin{aligned}&\int_{|x|<b} k_0|x|^{\alpha}|h(ax)-h(ax+ay)+ayh'(ax)|\,dx\\ &\quad = k_0|y|^{1+\alpha}\frac{1}{|ay|^{1+\alpha}}\int_{|t|<ba}|t|^{\alpha}|h(t)-h(t+ay)+ayh'(t)|\,dt\\ &\quad = o(a)\,|y|^{1+\alpha},\quad \text{as}\quad a\to 0.\end{aligned}$$

There exists a constant C_1 such that

$$\int_{|x|>b} k_1|x|^{\alpha}|h(ax)-h(ax+ay)+ayh'(ax)|\,dx< C_1 k_1|y|^{1+\alpha}.$$

Hence $$\int_{-\infty}^{\infty} g(x)\,h(ax)\,dx \leqslant \int_{-\infty}^{\infty} |y|^{1+\alpha}\,d\mu(y)\,\{o(a)+k_1\cdot C_1\}.$$

This implies Theorem 5.1 a. Since we can chose $k_1>0$ arbitrarily small, Theorem 5.1 b follows.

Theorem 5.2. *Let $G=G(\mu)=R^1$ and assume that*

$$\int_{-\infty}^{\infty} x^2\,d\mu(x)<\infty$$

and that $$\int_{-\infty}^{\infty} x\,d\mu(x)=0.$$

(a) *Let $g\in L^1(R^1)$ be such that $\hat{g}$ has compact support. Solutions of the equation (5.2) are unique up to addition of linear functions. If φ is a solution of (5.2) such that $\varphi(x)=O(x^2)$ at infinity, then $\varphi''\in L^1(R^1)$, $\lim_{x\to\infty}\varphi(x)/x$ and $\lim_{x\to-\infty}\varphi(x)/x$ exist and*

$$\lim_{|x|\to\infty}\frac{\varphi(x)+\varphi(-x)}{|x|}=-\frac{2\hat{g}(0)}{\sigma^2},$$

where $$\sigma^2=\int_{-\infty}^{\infty} x^2\,d\mu(x).$$

(b) *If φ is a solution of the inequality (5.1) such that $\varphi(x)/x$ is bounded for $|x|\geqslant 1$ and slowly oscillating* (cf. e.g. Rudin [12] 7.2.7), *then* $\lim_{x\to\infty}\varphi(x)/x$ *and* $\lim_{x\to-\infty}\varphi(x)/x$ *exist and*

$$\lim_{|x|\to\infty}\frac{\varphi(x)+\varphi(-x)}{|x|}=-\frac{2\int_{-\infty}^{\infty}(\varphi-\varphi*\mu)(x)\,dx}{\sigma^2}, \tag{5.5}$$

where the right member is finite.

The following corollary follows directly from Theorem 5.2 b. It can also be obtained from Theorem 5.1 b.

Corollary. *Let μ be as in Theorem 5.2. If φ is a solution of the inequality (5.1) such that $\varphi(x)=o(|x|)$ at infinity, then $\varphi-\varphi*\mu=0$.*

Remark. Assume that for some positive integer n

$$\int_{-\infty}^{\infty} |x|^{n+2}\,d\mu(x)<\infty,$$

that $\hat{g}$ has compact support and that $g\in A_n$ (Definition 4.1). Then, with methods similar to those we shall use in the proof of Theorem 5.2, we can find the following property of a solution φ of the equation (5.2).

There exist linear functions Q_+ and Q_- such that

$$\varphi(x)-Q_+(x)=o(|x|^{-n+1})\quad\text{as}\quad x\to\infty,$$
$$\varphi(x)-Q_-(x)=o(|x|^{-n+1})\quad\text{as}\quad x\to-\infty.$$

Before proving Theorem 5.2, we state the corresponding theorem on Z^1. The proofs of these two theorems are closely related, and it is natural to treat them simultaneously.

We introduce the following notation. If φ is a function on Z^1, we define φ'' by the formula

$$\{\varphi_k''\}_{-\infty}^{\infty} = \{\varphi_k - 2\varphi_{k-1} + \varphi_{k-2}\}_{-\infty}^{\infty}.$$

φ' was defined before Theorem 4.3 and obviously

$$\varphi'' = (\varphi')'.$$

Theorem 5.3. *Let $G = G(\mu) = Z^1$ and assume that*

$$\sum_{-\infty}^{\infty} k^2 \mu_k < \infty$$

and that

$$\sum_{-\infty}^{\infty} k \mu_k = 0.$$

(a) *Let $g \in L^1(Z^1)$. Solutions of the equation (5.2) are unique up to addition of linear functions. If φ is a solution of (5.2) such that $\varphi_n = O(n^2)$ at infinity, then $\varphi'' \in L^1(Z^2)$,* $\lim_{n\to\infty} \varphi_n/n$ and $\lim_{n\to-\infty} \varphi_n/n$ *exist and*

$$\lim_{|n|\to\infty} \frac{\varphi_n + \varphi_{-n}}{|n|} = -\frac{2\hat{g}(0)}{\sigma^2},$$

where

$$\sigma^2 = \sum_{-\infty}^{\infty} k^2 \mu_k.$$

(b) *If φ is a solution of the inequality (5.1) such that φ_n/n is bounded for $n \neq 0$, then $\lim_{n\to\infty} \varphi_n/n$ and $\lim_{n\to-\infty} \varphi_n/n$ exist and*

$$\lim_{|n|\to\infty} \frac{\varphi_n + \varphi_{-n}}{|n|} = -\frac{2\sum_{-\infty}^{\infty} (\varphi - \varphi * \mu)_n}{\sigma^2},$$

where the right member is finite.

The following corollary is a direct consequence of Theorem 5.3 b.

Corollary. *Let μ be as in Theorem 5.3. If φ is a solution of the inequality (5.1) such that $\varphi(n) = o(|n|)$ at infinity, then $\varphi - \varphi * \mu = 0$.*

Remark. The same remark as the one of Theorem 5.2 applies here. Since $\hat{Z}^1$ is compact, $\hat{g}$ always has compact support.

In the proof of Theorem 5.2, we shall need the following lemma.

Lemma 5.3. *Let μ be as in Theorem 5.2. We define*

$$N_2(x) = \begin{cases} \displaystyle\int_x^{\infty} \int_y^{\infty} d\mu(t)\,dy & (x > 0), \\ \displaystyle\int_{-\infty}^{x} \int_{-\infty}^{y} d\mu(t)\,dy & (x < 0). \end{cases}$$

If g is such that

(a) $\int_{-\infty}^{\infty} |g(x)|\,d\mu(x) < \infty$,

(b) $\lim_{x\to\infty} g(x)\int_x^{\infty} d\mu(y) = 0$

$\lim_{x\to-\infty} g(x)\int_{-\infty}^{x} d\mu(y) = 0$,

(c) $\lim_{|x|\to\infty} g'(x)\,N_2(x) = 0$,

(d) *g' is absolutely continuous and g'' a.e. bounded from above or below, then*

$$g(0) - \int_{-\infty}^{\infty} g(x)\,d\mu(x) = -\int_{-\infty}^{\infty} g''(x)\,N_2(x)\,dx. \tag{5.6}$$

Corollary, (a) *Let g satisfy the conditions in Lemma 5. Then*

$$g - g * \mu = -g'' * N_2.$$

(b) *If we assume that the right member is defined by continuity for $t=0$,*

$$\hat{N}_2(t) = \frac{\hat{\mu}(t)-1}{(it)^2}.$$

Proof of the corollary. (a) Apply (5.6) to g_1, where $g_1(x) = g(a-x)$.

(b) Apply (5.6) to $g(x) = e^{-itx}$.

Proof of Lemma 5.3. We use partial integrations. The function N_1 is defined in the proof of Theorem 4.2.

$$\int_0^{\infty} g(x)\,d\mu(x) = -\int_0^{\infty} g(x)\,dN_1(x) = g(0)\,N_1(+0) + \int_0^{\infty} g'(x)\,N_1(x)\,dx =$$
$$= g(0)\,N_1(+0) - \int_0^{\infty} g'(x)\,dN_2(x) =$$
$$= g(0)\,N_1(+0) + g'(0)\,N_2(+0) + \int_0^{\infty} g''(x)\,N_2(x)\,dx.$$

In the same way

$$\int_{-\infty}^{0} g(x)\,d\mu(x) = -g(0)\,N_1(-0) - g'(0)\,N_2(-0) - \int_{-\infty}^{0} g''(x)\,N_2(x)\,dx.$$

Adding these equations, we obtain an expression where the coefficient of $g(0)$ is $\int_{-\infty}^{\infty} d\mu(x) = 1$ and the one of $g'(0)$ is $\int_{-\infty}^{\infty} x\,d\mu(x) = 0$. Thus the new equation is (5.6) and Lemma 5.3 is proved.

Proof of Theorem 5.2. It follows from the corollary of Lemma 5.1 that two solutions differ by a linear function. We shall find the solutions of the equation (5.2) by inverting $(1-\hat{\mu}(t))$ on the support of $\hat{g}$. It follows from the corollary of Lemma 5.3 that $(\hat{\mu}(t)-1)/(it)^2$ is the Fourier transform of $N_2 \in L^1$. Since $\hat{N}_2(t) \neq 0$ for all t, we can apply Lemma 4.1 where we choose p in such a way that $p(x)=1$ for all x. Hence there exists $h \in L^1$, so that

$$\frac{(it)^2 \hat{g}(t)}{(1-\hat{\mu}(t))} = \hat{h}(t), \tag{5.7}$$

i.e. $h - h * \mu = g''$. We put

$$H(x) = \int_0^x (x-y)\, h(y)\, dy$$

and assert that H is a solution of the equation (5.2). The assertion follows from the corollary of Lemma 5.3, since

$$H - H * \mu = -H'' * N_2$$

and the Fourier transform of the right member is $\hat{g}$. Thus the general solution of (5.2) can be written $\varphi(x) = H(x) + Ax + B$.

From this the existence of $\lim_{x\to\infty} \varphi(x)/x$ and$_{x\to-\infty} \varphi(x)/x$ follows and we have

$$\lim_{|x|\to\infty} \frac{\varphi(x)+\varphi(-x)}{|x|} = \int_{-\infty}^{\infty} h(y)\, dy = \hat{h}(0) = -\frac{2\hat{g}(0)}{\sigma^2}. \tag{5.8}$$

The last equality is implied by (5.7). Hence (a) is proved.

(b) Let φ be as in Theorem 5.1 b. We shall prove that $\lim_{x\to\infty} \varphi(x)/x$ exists. The proof for $x \to -\infty$ is analogous.

We put $\varphi - \varphi * \mu = g$. It follows from Theorem 5.1 a that $g \in L^1$. Now take $q \in H'$ (Definition 2.3) such that $\hat{q}(0)=1$ and

$$\int_{-\infty}^{\infty} |x|\, q(x)\, dx < \infty.$$

Let a be a positive number. We define $q_a(x) = aq(ax)$. Then $\varphi_1 = \varphi * q_a$ satisfies

$$\varphi_1 - \varphi_1 * \mu = g * q_a \geqslant 0.$$

Since the Fourier transform of $g * q_a$ has compact support, it follows from (a) that $\lim_{x\to\infty} \varphi_1(x)/x$ exists.

It may occur that $\varphi(x)/x$ is not a bounded function of x. Therefore we write $\varphi = \psi_0 + \psi_1$ where, if b and δ are positive numbers,

$$\psi_0(x) = \begin{cases} \varphi(x) & \text{for } |x| \leqslant b \\ 0 & \text{for } |x| \geqslant b+\delta \end{cases}$$

and

$$\psi_1(x) = \begin{cases} 0 & \text{for } |x| \leqslant b \\ \varphi(x) & \text{for } |x| \geqslant b+\delta. \end{cases}$$

For $b < |x| < b+\delta$, we define ψ_0 and ψ_1 so that they are continuous for all x. Hence

$$\lim_{x\to\infty}\frac{1}{x}\int_{-\infty}^{\infty}\varphi_0(x-y)\,q_a(y)\,dy=\lim_{x\to\infty}\frac{1}{x}\int_{-\infty}^{\infty}[\psi_0(x-y)+\psi_1(x-y)]\,q_a(y)\,dy.$$

Now
$$\lim_{x\to\infty}\frac{1}{x}\int_{-\infty}^{\infty}\psi_0(x-y)\,q_a(y)\,dy=0$$

and thus, if we define $\psi_1(x)/x=0$ if $\psi_1(x)=0$,

$$\lim_{x\to\infty}\frac{1}{x}\int_{-\infty}^{\infty}\frac{\psi_1(x-y)}{x-y}(x-y)\,q_a(y)\,dy=$$
$$=\lim_{x\to\infty}\int_{-\infty}^{\infty}\frac{\psi_1(x-y)}{x-y}\,q_a(y)\,\mathrm{dy}=\lim_{x\to\infty}\frac{\varphi_1(x)}{x}.$$

Since $\varphi(x)/x$ is slowly oscillating, the same is true for $\nu(x)=\psi_1(x)/x$. We know that $\nu\in L^\infty$ and that $\lim_{x\to\infty}\nu\ast q_a(x)$ exists for all positive numbers a, and thus we can apply the Wiener Tauberian theorem in the same way as in the proof of Theorem 4.4 a. This implies the existence of $\lim_{x\to\infty}\nu\ast f(x)$ for every $f\in L^1$. In particular, we see that this is true for every function $f\in L^1$ with the property $\hat{f}(t)\neq 0$ for all t.

Hence a Tauberian theorem of Pitt (cf. e.g. Rudin [12] 7.2.7) implies that $\lim_{x\to\infty}\nu(x)=\lim_{x\to\infty}\varphi(x)/x$ exists. It remains to prove the formula (5.5). We define $\check{\nu}(x)=\nu(-x)$. It follows from (5.8) that

$$\lim_{x\to\infty}\frac{\varphi_1(x)+\varphi_1(-x)}{x}=-\frac{2\hat{g}(0)}{\sigma^2}.$$

This implies that for all a

$$\lim_{x\to\infty}\left(\nu-\check{\nu}+\frac{2\hat{g}(0)}{\sigma^2}\right)\ast q_a(x)=0$$

and it follows from Pitt's theorem that

$$\lim_{x\to\infty}\left(\frac{\varphi(x)+\varphi(-x)}{x}+\frac{2\hat{g}(0)}{\sigma^2}\right)=\lim_{x\to\infty}\left(\nu(x)-\check{\nu}(x)+\frac{2\hat{g}(0)}{\sigma^2}\right)=0$$

and the proof is complete.

We shall prove theorem 5.3 in the same way as Theorem 5.2. Before doing this, a preliminary discussion is needed. We first observe that the solutions of the equation (5.3) are obtained from the corollary of Lemma 5.1, which is also valid on Z^1. Secondly, let the function h be defined on R^1 and satisfy the conditions of Lemma 5.2. The theorem on Z^1 corresponding to Theorem 5.1 on R^1 is true, and this follows if we use the summation sequence

$$\{h_k\}_{-\infty}^{\infty}=\{h(ak)\}_{-\infty}^{\infty}.$$

In the proof of Theorem 5.3, we modify the method, used for obtaining Theorem 5.2, in the same way as the proof of Theorem 4.2 was modified in the demonstration of Theorem 4.3 c, d. The sequence g'' has the Fourier transform $\hat{g}(t)\,(1-e^{-it})^2$ which can be divided by $(1-\hat{\mu}(t))$. We obtain a sequence $h\in L^1(Z^1)$ which satisfies the equation $h-h\ast\mu=g''$.

Now we can proceed as in the proof of Theorem 5.2, and Theorem 5.3 follows.

6. Properties of unbounded solutions of the inequality $\varphi - \varphi * \mu \geq 0$ in R^2 and Z^2 when no bounded non-trivial solutions exist

In this paragraph we consider the two-dimensional analogues of Theorems 5.1, 5.2, and 5.3. The measure μ is assumed to have properties such that no bounded non-trivial solutions of the inequality exist. Conditions implying this are deduced from Theorem 3.1 b. We deal with solutions which are $O(\log|x|)$ at infinity. The existence of solutions satisfying this growth condition can be proved in the following way. Take $g \in H'$ (Definition 2.3) and put

$$\varphi_1(x) = \frac{1}{(2\pi)^2}\iint_{\hat{G}} \hat{g}(t)\frac{e^{itx}-1}{1-\hat{\mu}(t)}\,dt_1\,dt_2.$$

The function φ_1 satisfies the equation

$$\varphi - \varphi * \mu = g \tag{6.1}$$

and it will follow from Theorem 6.1 that $\varphi_1(x) = O(\log|x|)$ at infinity.

The solutions of the equation

$$\varphi - \varphi * \mu = 0$$

in R^2 and Z^2 are found in the same way as those in R^1 (Lemma 5.1).

We introduce certain notations. Let G be R^2 or Z^2, let $G = G(\mu)$ and assume that

$$\int_G x_1\,d\mu(x) = \int_G x_2\,d\mu(x) = 0 \tag{6.2}$$

and that

$$\int_G |x|^2\,d\mu(x) < \infty. \tag{6.3}$$

We put

$$\int_G x_1^2\,d\mu(x) = a,$$

$$\int_G x_2^2\,d\mu(x) = b,$$

$$\int_G x_1 x_2\,d\mu(x) = c.$$

Since $G(\mu)$ is two-dimensional, the polynomial of the second degree

$$at_1^2 + bt_2^2 + 2ct_1t_2 = \int_G (tx)^2\,d\mu(x)$$

is positive definite. Hence there exists an affine transformation $t = Au$ such that

$$at_1^2 + bt_2^2 + 2ct_1t_2 \to u_1^2 + u_2^2.$$

Let $|A|$ be the determinant of the matrix A and let A^* be the adjoint of A. We choose A so that $|A| > 0$.

Theorem 6.1. *Let $G = G(\mu) = R^2$ and assume that the conditions (6.2) and (6.3) are fulfilled.*

(a) *Let $g \in L^1(R^2)$ be such that $\hat{g}$ has compact support. Solutions of the equation (6.1) are unique up to addition of linear functions. If φ is a solution of (6.1) such that $\varphi(x) = O(\log|x|)$ at infinity, then*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\log|A^*x|} = -\frac{\hat{g}(0)}{\pi}|A|. \tag{6.4}$$

(b) *Let φ be a solution of the inequality*

$$\varphi - \varphi * \mu \geqslant 0. \tag{6.5}$$

If $\varphi(x) = O(\log|x|)$, then $\varphi - \varphi * \mu \in L^1(R^2)$.

(c) *Let φ be as in (b) and furthermore let $\varphi(x)/\log|A^*x|$ be slowly oscillating* (cf. e.g. Rudin [12] 7.2.7). *Then*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\log|A^*x|} = -\frac{\iint_{R^2}(\varphi - \varphi * \mu)(x)\,dx}{\pi}|A|.$$

Remark 1. The result in (b) can be generalized. Let μ be as in the theorem, let the measure ν satisfy

$$\nu - \nu * \mu \geqslant 0$$

and assume that for some continuous, non-negative function k with compact support

$$\nu * k(x) = O(\log|x|)$$

at infinity. If (b) is true, then $\nu * k - \nu * k * \mu \in L^1(R^2)$, and it follows that the total mass of the measure $\nu - \nu * \mu$ is finite. Confer Remark 1 of Theorem 4.1 and Remark 3 of Theorem 5.1.

Remark 2. If the coordinate system is chosen such that $a = b = \sigma^2$ and $c = 0$, we obtain in (c)

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\log|x|} = -\frac{\iint_{R^2}(\varphi - \varphi * \mu)(x)\,dx}{\pi\sigma^2}.$$

Remark 3. The methods used in the proof of (b) could also have been applied in the one-dimensional case.

Theorem 6.1 is also valid on Z^2. We can, however, simplify the statement in (a) and combine (b) and (c).

Theorem 6.2. *Let $G = G(\mu) = Z^2$ and assume that the conditions (6.2) and (6.3) are fulfilled.*

(a) *Let $g \in L^1(Z^2)$. Solutions of the equation (6.1) are unique up to addition of linear functions. If φ is a solution of (6.1) such that $\varphi(x) = O(\log|x|)$ at infinity, then*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\log|A^*x|} = -\frac{\hat{g}(0)}{\pi}|A|.$$

b, c) *Let φ be a solution of the inequality (6.5) such that $\varphi(x) = O(\log|x|)$ at infinity. Then $\varphi - \varphi * \mu \in L^1(Z^2)$ and*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\log|A^*x|} = -\frac{\iint_{Z^2} (\varphi - \varphi * \mu)(x)\,dx}{\pi} |A|.$$

Remark. Corollaries of Theorems 6.1 and 6.2 analogous to the corollaries of Theorems 5.2 and 5.3 are true. The proof is easy.

The following lemma is needed in the proofs of Theorems 6.1 and 6.2.

Lemma 6.1. *Let μ and A be as in Theorem 6.1 or Theorem 6.2 and let the number b satisfy $0 < b < \pi$. Then*

$$\lim_{|x|\to\infty} \frac{1}{\log|A^*x|} \iint_{|t|<b} \frac{e^{itx}-1}{1-\hat{\mu}(t)}\,dt_1\,dt_2 = -4\pi|A|.$$

Proof of Lemma 6.1. We approximate $(1-\hat{\mu}(t))$ by $\frac{1}{2}(at_1^2 + bt_2^2 + 2ct_1t_2)$. It is easily shown that

$$(\hat{\mu}(t) - 1) + \tfrac{1}{2}(at_1^2 + bt_2^2 + 2t_1t_2c) = \varepsilon(t)|t|^2, \tag{6.6}$$

where $\lim_{t\to 0} \varepsilon(t) = 0$.

$$\iint_{|t|<b} \frac{e^{itx}-1}{1-\hat{\mu}(t)}\,dt_1\,dt_2 = \iint_{|t|<b} (e^{itx}-1)\left\{\frac{1}{1-\hat{\mu}(t)} - \frac{2}{at_1^2+bt_2^2+2ct_1t_2}\right\}dt_1\,dt_2 + $$
$$+ \iint_{|t|<b} \frac{(e^{itx}-1)2}{at_1^2+bt_2^2+2ct_1t_2}\,dt_1\,dt_2 = I_1 + I_2.$$

We write the integrand in I_1 with a common denominator and use the estimate (6.6). Assuming that $|\varepsilon(t)| < \varepsilon$ for $|t| < \delta < 1$, we obtain that there exists a constant $C(\delta)$ such that

$$|I_1| \leq C(\delta) + \iint_{|t|<\delta} \frac{|e^{itx}-1|}{|t|^2} |\varepsilon(t)| 2dt_1\,dt_2$$

and that there exists a constant C such that for every $\varepsilon > 0$

$$\varlimsup_{|x|\to\infty} \frac{|I_1|}{\log|A^*x|} \leq \varlimsup_{|x|\to\infty} \frac{2\varepsilon}{\log|A^*x|} \int_0^{2\pi}\int_0^{\delta} \frac{|e^{ir|x|\cos\varphi}-1|}{r}\,dr\,d\varphi =$$
$$= \varlimsup_{|x|\to\infty} \frac{2\varepsilon}{\log|A^*x|} \int_0^{2\pi}\int_0^{\delta|x||\cos\varphi|} \frac{|e^{iu}-1|}{u}\,du\,d\varphi \leq$$
$$\leq \varlimsup_{|x|\to\infty} \frac{2\varepsilon}{\log|A^*x|} \int_0^{2\pi}\int_0^{|x|} \frac{|e^{iu}-1|}{u}\,du\,d\varphi \leq C\cdot\varepsilon.$$

Hence
$$\lim_{|x|\to\infty} \frac{I_1}{\log|A^*x|} = 0.$$

In the second integral we have for some $c > 0$

$$I_2 = \iint_{|t|<b} \frac{2(e^{itx} - 1}{at_1^2 + bt_2^2 + 2ct_1 t_2} dt_1 dt_2 =$$

$$= \iint_{|Au|<b} \frac{2(e^{ix(Au)} - 1)}{|u|^2} |A| du_1 du_2 =$$

$$= \left\{ \iint_{|u|<c} + \iint_{\substack{|u|>c \\ |Au|<b}} \right\} \frac{2(e^{iu(A^*x)} - 1)}{|u|^2} |A| du_1 du_2$$

and that

$$\lim_{|x|\to\infty} \frac{I_2}{\log|A^*x|} = \lim_{|x|\to\infty} \frac{|A|}{\log|A^*x|} \int_{-\pi}^{\pi} \int_0^c \frac{2(e^{ir|A^*x|\cos\varphi} - 1)}{r} dr\, d\varphi =$$

$$= \lim_{|x|\to\infty} \frac{4|A|}{\log|A^*x|} \int_{-\pi/2}^{\pi/2} \int_0^c \frac{\cos(r|A^*x|\cos\varphi) - 1}{r} dr\, d\varphi =$$

$$= \lim_{|x|\to\infty} \frac{4|A|}{\log|A^*x|} \int_{-\pi/2}^{\pi/2} \int_0^{c|A^*x||\cos\varphi|} \frac{\cos u - 1}{u} du\, d\varphi =$$

$$= \lim_{|x|\to\infty} \frac{4|A|}{\log|A^*x|} \int_{-\pi/2}^{\pi/2} (-\log c - \log|A^*x| - \log|\cos\varphi|)\, d\varphi =$$

$$= -4\pi|A|$$

and Lemma 6.1 is proved.

Proof of Theorem 6.1. (a) The equation (6.1) is satisfied by

$$\varphi_1(x) = \frac{1}{(2\pi)^2} \iint_{R^2} \frac{e^{itx} - 1}{1 - \hat{\mu}(t)} \hat{g}(t)\, dt.$$

Since the support of $\hat{g}$ is compact, the integral is absolutely convergent and we can calculate $\varphi_1 - \varphi_1 * \mu$ by an inversion of the order of integration. We shall show that $\lim_{|x|\to\infty} [\varphi_1(x)]/\log|A^*x|$ exists and is finite. Hence if φ is a solution of (6.1) satisfying $\varphi(x) = O(\log|x|)$ at infinity, there exists a constant C such that $\varphi(x) = \varphi_1(x) + C$. Thus (6.4) is proved if we can show that

$$\lim_{|x|\to\infty} \frac{\varphi_1(x)}{\log|A^*x|} = -\frac{\hat{g}(0)}{\pi}|A|.$$

Since $g \in L^1$, $\hat{g}$ is continuous. Assuming that $|\hat{g}(t) - \hat{g}(0)| < \varepsilon$ for $|t| < \delta$, we have

$$\varphi_1(x) = \frac{1}{(2\pi)^2} \iint_{|t|<\delta} \frac{e^{itx} - 1}{1 - \hat{\mu}(t)} \{\hat{g}(t) - \hat{g}(0)\}\, dt +$$

$$+ \frac{\hat{g}(0)}{(2\pi)^2} \iint_{|t|<\delta} \frac{e^{itx} - 1}{1 - \hat{\mu}(t)} dt + \frac{1}{(2\pi)^2} \iint_{|t|>\delta} \frac{e^{itx} - 1}{1 - \hat{\mu}(t)} \hat{g}(t)\, dt =$$

$$= I_1 + I_2 + I_3.$$

There exist constants C_1 and C_2 such that for every $\varepsilon > 0$

$$|I_1| \leqslant C_1 \cdot \varepsilon \iint_{|t|<\delta} \frac{|e^{itx}-1|}{|t|^2}\,dt \leqslant C_2 \varepsilon \log|x|.$$

Lemma 6.1 implies that

$$\lim_{|x|\to\infty} \frac{I_2}{\log|A^* x|} = -\frac{\hat{g}(0)\,|A|}{\pi}.$$

We further have that $|I_3|$ is a bounded function of x. Hence

$$\lim_{|x|\to\infty} \frac{\varphi_1(x)}{\log|A^* x|} = -\frac{|A|\,\hat{g}(0)}{\pi}$$

and the proof of (a) is complete.

(b) We put
$$\varphi - \varphi * \mu = g = h_R + g_R,$$

where
$$h_R = \begin{cases} g & |x| < R, \\ 0 & |x| \geqslant R \end{cases}$$

and
$$g_R = \begin{cases} 0 & |x| < R, \\ g & |x| \geqslant R. \end{cases}$$

Now take $q \in H'$ (Definition 2.3) such that $\hat{q}(0) = 1$ and

$$\iint_{R^2} \log^+ |x|\, q(x)\,dx < \infty.$$

Consider the equation

$$\varphi = \varphi * \mu = h_R * q. \tag{6.7}$$

It follows from (a) that there exists a solution ψ_R of (6.7) such that $\psi_R(x) = O(\log|x|)$ at infinity and such that

$$\lim_{|x|\to\infty} \frac{\psi_R(x)}{\log|A^* x|} = -\frac{\hat{h}_R(0)\,\hat{q}(0)}{\pi}|A| = -\frac{\hat{h}_R(0)}{\pi}|A|.$$

We put $\varphi_1 = \varphi * q$. Hence $\varphi_1 - \psi_R$ is a solution of the equation

$$\varphi - \varphi * \mu = g_R * q.$$

Since $\varphi_1(x) = O(\log|x|)$ at infinity, there exists a constant C such that for large x

$$(\varphi_1 - \psi_R)(x) \geqslant \log|A^* x| \left\{ C + \frac{\hat{h}_R(0)}{\pi}|A| \right\}.$$

If $g \notin L^1(R^2)$, we can by choosing R large attain that

$$C + \frac{\hat{h}_R(0)}{\pi}|A| > 0.$$

Hence there exists a number R_0 such that

$$\lim_{|x|\to\infty} (\varphi_1 - \psi_{R_0})(x) = \infty \tag{6.8}$$

and the continuous function $(\varphi_1 - \psi_{R_0})$ has an absolute minimum which is assumed at some point x_0. Since $\varphi_1 - \psi_{R_0}$ satisfies the inequality (6.5) it follows that

$$(\varphi_1 - \psi_{R_0})(x) = (\varphi_1 - \psi_{R_0})(x_0)$$

for $x \in x_0 - S(\mu)$, where $S(\mu)$ is the semigroup generated by the support of μ. This contradicts (6.8), it follows that $g \in L^1(R^2)$ and (b) is proved.

(c) Applying the same method as the one used in the proof of Theorem 5.2 b, we put $\varphi - \varphi * \mu = g$. It follows from (b) that $g \in L^1(R^2)$. Now take $q \in H'$ (Definition 2.3) such that $\hat{q}(0) = 1$ and

$$\iint_{R^2} \log^+ |x|\, q(x)\, dx < \infty.$$

We define $q_a(x) = a^2 q(ax)$ and $\varphi_1 = \varphi * q_a$. It follows in the same way as in the proof of (a) that

$$\lim_{|x|\to\infty} \frac{\varphi_1(x)}{\log|A^* x|} = -\frac{\hat{g}(0)}{\pi}|A|.$$

Keeping the method and the notations from the proof of Theorem 5.2 b we put $\varphi = \psi_0 + \psi_1$. Choose b such that $|A^* x| \geqslant 2$ for $|x| \geqslant b$ and define $\psi_1(x)/(\log|A^* x|) = 0$ if $\psi_1(x) = 0$. We have

$$\lim_{|x|\to\infty} \frac{1}{\log|A^* x|} \psi_0 * q_a(x) = 0$$

and hence

$$\lim_{|x|\to\infty} \frac{1}{\log|A^* x|} \iint_{R^2} \psi_1(x-y)\, q_a(y)\, dy = -\frac{\hat{g}(0)}{\pi}|A|.$$

We shall prove that this implies that

$$\lim_{|x|\to\infty} \iint_{R^2} \frac{\psi_1(x-y)}{\log|A^*(x-y)|} q_a(y)\, dy = -\frac{\hat{g}(0)}{\pi}|A|.$$

If this is true, the same argument as in the proof of Theorem 5.2 b with Wiener's and Pitt's theorems implies that

$$\lim_{|x|\to\infty} \left\{\frac{\varphi(x)}{\log|A^* x|} + \frac{\hat{g}(0)}{\pi}|A|\right\} = \lim_{|x|\to\infty} \left\{\frac{\psi_1(x)}{\log|A^* x|} + \frac{\hat{g}(0)}{\pi}|A|\right\} = 0.$$

Since

$$\hat{g}(0) = \iint_{R^2} (\varphi - \varphi * \mu)(y)\, dy,$$

this is the result we want. It thus remains for us to prove that

$$\lim_{|x|\to\infty} \frac{1}{\log|A^*x|} \iint_{|x-y|\geqslant b} \psi_1(x-y)\, q_a(y)\, dy =$$
$$= \lim_{|x|\to\infty} \iint_{|x-y|\geqslant b} \frac{\psi_1(x-y)}{\log|A^*(x-y)|}\, q_a(y)\, dy.$$

If we form the difference of the two integrals and define $\nu(x) = \psi_1(x)/(\log|A^*x|)$, we obtain

$$\left|\iint_{|x-y|\geqslant b} \nu(x-y)\left\{\frac{\log|A^*(x-y)|}{\log|A^*x|} - 1\right\} q_a(y)\, dy\right| =$$
$$= \left|\left(\iint_{\substack{|x-y|\geqslant b\\ |y|\leqslant R}} + \iint_{\substack{|x-y|\geqslant b\\ |y|\geqslant R}}\right) \nu(x-y) q_a(y)\left\{\frac{\log|A^*(x-y)|}{\log|A^*x|} - 1\right\} dy\right| = \left|I_1 + I_2\right|.$$

In I_1,
$$\lim_{|x|\to\infty} \frac{\log|A^*(x-y)|}{\log|A^*x|} = 1$$

uniformly. Since ν is a bounded function, we have $\lim_{|x|\to\infty} I_1 = 0$. There exists a constant C and a number R_0 such that for $|x|\geqslant b$, for $R > R_0$, and for y in the domain of integration of the integral I_2

$$0 \leqslant \log|A^*(x-y)| \leqslant C(\log|A^*x| + \log|A^*y|).$$

Hence there exists a constant C_1 such that for every $\varepsilon > 0$

$$|I_2| \leqslant C_1\left\{\iint_{|y|\geqslant R}\left(1 + \frac{\log|A^*y|}{\log|A^*x|}\right) q_a(y)\, dy + \iint_{|y|\geqslant R} q_a(y)\, dy\right\} < \varepsilon$$

for $R \geqslant R(\varepsilon)$ and $|x| \geqslant b$. Thus $\lim_{|x|\to\infty} |I_1 + I_2| = 0$ and the proof of Theorem 6.1 is complete.

Proof of Theorem 6.2 The proof is the same as the proof of Theorem 6.1, except that it is unecessary to introduce a function $q \in H'$, since $\hat{Z}^2$ is compact.

7. Summary of certain results on R^n and Z^n

We here summarize those results on R^n and Z^n which deal with the connection between the growth of a solution φ of the inequality (1.1) and the magnitude of $\int_G (\varphi - \varphi * \mu)(x)\, dx$.

I. Let $G = G(\mu)$ be R^1 or Z^1 and assume that

$$\int_G |x|\, d\mu(x) < \infty$$

and
$$\int_G x\, d\mu(x) \neq 0.$$

Then bounded, non-trivial solutions of the inequality (1.1) exist (Theorem 3.1). Let φ be a bounded function. We have the following properties of μ.

(a) If φ is a solution of (1.1), then $\varphi - \varphi * \mu \in L^1(G)$ (Theorems 4.1 a and 4.3 a).

(b) If φ is a solution of (1.1) and $\lim_{x\to\infty} \varphi(x) = \lim_{x\to-\infty} \varphi(x)$, then $\varphi - \varphi * \mu = 0$ (Theorems 4.1 c and 4.3 b).

(c) Let $G = R^1$. If φ is a solution of (1.1) and φ is slowly decreasing, then $\lim_{x\to\infty} \varphi(x)$ and $\lim_{x\to-\infty} \varphi(x)$ exist (Theorem 4.1 b).

Let $G = Z^1$. If φ is a solution of (1.1), then $\lim_{x\to\infty} \varphi(x)$ and $\lim_{x\to-\infty} \varphi(x)$ exist (Theorem 4.3 a, c).

II. Let $G = G(\mu)$ be R^n or Z^n where $n \geq 2$ and assume that a bounded, non-trivial solution of the inequality (1.1) exists (cf. the sufficient conditions of Theorem 3.1). Then the results in I a, b and c are not true (Theorem 4.4).

III. Let $G = G(\mu)$ be R^n or Z^n. We consider the case when no bounded, non-trivial solutions of the inequality (1.1) exist. This can only occur if $n = 1$ or $n = 2$ and is in particular true if

$$\int_G |x|^2 \, d\mu(x) < \infty$$

and if for every linear function l

$$\int_G l(x) \, d\mu(x) = 0$$

(Theorem 3.1).

Under these conditions, we have the following properties of μ if $n = 1$.

(a) If φ is a solution of (1.1) and $\varphi(x) = O(|x|)$ at infinity, then $\varphi - \varphi * \mu \in L^1(G)$ (Theorem 5.1 a).

(b) If φ is a solution of (1.1) and if $\varphi(x) = o(|x|)$ at infinity, then $\varphi - \varphi * \mu = 0$ (corollaries of Theorems 5.2 and 5.3).

If $n = 2$, we have analogous results with $|x|$ replaced by $\log |x|$ (Theorems 6.1 and 6.2).

8. Properties of functions satisfying a sequence of convolution inequalities

In this paragraph, we consider a new problem which connects the results of this paper with the theory of subharmonic functions (cf. Radó [11]). The emphasis here is on this connection, and therefore the treatment of the problem is not complete. The author hopes that he will be able to return to this subject.

It is well known that the mean value of a continuous, superharmonic function over e.g. the interior of a circle is smaller than the value that the function assumes at the centre of the circle. This means that the function is a solution of a convolution inequality of our type, namely the one that occurs if the measure μ is chosen as the uniform distribution of the unit mass over the circle. Obviously this is true for a whole class of measures. Thus we are led to the following problem.

Let n be a natural number. We define the measure μ_n by setting

$$\mu_n(O) = \mu(nO)$$

for every open set O. Let A_μ be the class of all functions φ that are solutions of

$$\varphi - \varphi * \mu_n \geqslant 0 \tag{8.1}$$

for all natural numbers n in an increasing sequence $\{n_k\}_1^\infty$. What functions belong to A_μ? It is clear that there exist measures μ such that A_μ is non-empty. In the special case when μ is a uniform distribution of the unit mass over the perimeter or the interior of a circle, it is well known that A_μ is a class of functions, superharmonic in the whole plane (cf. Radó [11] 3.7).

Theorem 8.1 *Let $G = G(\mu) = R^2$ and assume that μ is as in Theorem 6.1, that $a = b = \sigma^2$, that $c = 0$ and that for some positive number a*

$$\int_{R^2} |x|^{a+3}\, d\mu(x) < \infty.$$

Let $\varphi \in A_\mu$ be such that $\varphi(x) = O(|x|^a)$ at infinity. Then φ is a superharmonic function.

Remark. If we cancel the assumptions $a = b = \sigma^2$ and $c = 0$, we obtain that there exists a superharmonic function s and an affine transformation B such that $\varphi(x) = s(Bx)$.

Proof of Theorem 8.1 All integrals in the proof are absolutely convergent, and we can therefore change the order of integration as we like. Let the positive function $\psi \in C^\infty$ have compact support. Multiplying (8.1) with ψ and integrating, we obtain

$$\int_{R^2} \varphi(u) \int_{R^2} (\psi(u) - \psi(u+y))\, d\mu(ny)\, du \geqslant 0. \tag{8.2}$$

It follows from (6.2) that

$$\int_{R^2} (\psi(u) - \psi(u+y))\, d\mu(ny) = \int_{R^2} \left(\psi(u) - \psi(u+y) + y_1 \frac{\partial \psi}{\partial u_1}(u) + y_2 \frac{\partial \psi}{\partial u_2}(u)\right) d\mu(ny) \tag{8.3}$$

and from Taylor's theorem that

$$\psi(u+y) - \left(\psi(u) + y_1 \frac{\partial \psi}{\partial u_1}(u) + y_2 \frac{\partial \psi}{\partial u_2}(u)\right) =$$

$$= \frac{1}{2!}\left\{y_1^2 \psi_{11}''(u+\theta y) + 2 y_1 y_2 \psi_{12}''(u+\theta y) + y_2^2 \psi_{22}''(u+\theta y)\right\}. \tag{8.4}$$

Here $\psi_{ik}'' = \partial^2\psi / \partial u_i\, \partial u_k$ $(i, k = 1,2)$ and θ is a function of u and y such that $0 \leqslant \theta \leqslant 1$. Using (8.3) and (8.4), we obtain from (8.2) after a change of the order of integration and the change of variables $ny = v$ that

$$\int_{R^2} d\mu(v) \int_{R^2} -\frac{1}{2n^2}\left(v_1^2 \psi_{11}''\left(u + \frac{\theta v}{n}\right) + 2 v_1 v_2 \psi_{12}''\left(u + \frac{\theta v}{n}\right) + v_2^2 \psi_{22}''\left(u + \frac{\theta v}{n}\right)\right) \varphi(u)\, du \geqslant 0.$$

Multiply by n^2 and let $n \to \infty$. Here we will only consider the term

$$\int_{R^2} d\mu(v) \int_{R^2} v_1^2 \psi_{11}''\left(u + \frac{\theta v}{n}\right) \varphi(u)\, du. \tag{8.5}$$

The discussions of the two other terms are similar. We need an estimate of

$$\int_{R^2} \psi_{11}''\left(u+\frac{\theta v}{n}\right)\varphi(u)\,du. \tag{8.6}$$

Let S_φ be the support of φ. The domain of integration in (8.6) is contained in

$$D=\bigcup_{0\leqslant\alpha\leqslant 1}\left(S_\varphi-\frac{\alpha v}{n}\right).$$

Hence there exist constants $\{C_i\}_1^3$ such that

$$|(8.6)|\leqslant C_1\int_D|\varphi(u)|\,du\leqslant C_2\int_D(1+|u|^a)\,du\leqslant C_3\left(1+\left|\frac{v}{n}\right|^{a+1}\right).$$

Since
$$\left(1+\left|\frac{v}{n}\right|^{a+1}\right)\leqslant(1+|v|^{a+1})$$

and
$$\int_{R^2}|v|^{a+3}\,d\mu(v)<\infty,$$

we can use Lebesgue's theorem on dominated convergence twice and obtain

$$\lim_{n\to\infty}(8.5)=\int_{R^2}d\mu(v)\int_{R^2}v_1^2\,\psi_{11}''(u)\,\varphi(u)\,du=\int_{R^2}\varphi(u)\,\psi_{11}''(u)\,\sigma^2\,du.$$

Hence it follows that

$$-\int_{R^2}\varphi(u)\Delta\psi(u)\,du\geqslant 0$$

and by Schwartz ([13] p. 76) φ is superharmonic a.e. Since φ is continuous, we conclude that φ is superharmonic, and the theorem is proved.

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