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Some examples of sets with linear independence

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A necessary and sufficient condition for the real numbers $x_1, x_2, \ldots, x_N$ to be linearly independent (mod 2π) over the rational numbers is, by Kronecker's theorem:

(1) For every $\varepsilon > 0$ and real numbers $\theta_1, \theta_2, \ldots, \theta_N$, there exists a real number t such that

$$|e^{itx_\nu} - e^{i\theta_\nu}| < \varepsilon \quad (\nu = 1, 2, \ldots, N).$$

An equivalent condition is:

(2) For every sequence $a_1, \ldots, a_N$ of complex numbers

$$\sup_t |1 + \sum_{\nu=1}^{N} a_\nu e^{itx_\nu}| = 1 + \sum_1^N |a_\nu|,$$

where t represents a real number.

Instead of using all real numbers t, we might as well use only the positive integers $n > 0$, with the same conclusion.

The two conditions above give rise to generalizations of the notion of linear independence in closed sets.

A. E is a *uniform Kronecker set* if, to every continuous function f on E, of absolute value 1, and to every $\varepsilon > 0$, there exists a real number t such that

$$\sup_{x \in E} |f(x) - e^{itx}| < \varepsilon.$$

B. E is a *Kronecker set* if

$$\inf_{\mu \in \Gamma_0} \sup_n \left| \int_E e^{inx} \, d\mu(x) \right| = 1,$$

where $\Gamma_0(E)$ is the class of functions μ which are constant outside E and satisfy $\int_E |d\mu| = 1$.

For a finite set, $x_1, x_2, \ldots, x_N$, condition B is satisfied if and only if $x_2 - x_1$, $x_3 - x_1, \ldots, x_N - x_1$ are linearly independent (mod 2π).

Thus a finite set E is a uniform K-set if and only if $E \cup \{0\}$ is a K-set. In the general case the following is true:

If E is a uniform K-set, then $E \cup \{0\}$ is a K-set.

Proof. Suppose that E is a uniform K-set and choose an arbitrary $\mu \in \Gamma_0(E \cup \{0\})$ and an $\varepsilon > 0$. We may suppose, without loss of generality, that μ has a real jump a at $x = 0$. We have

$$\sup \left| \int_{E \cup \{0\}} f d\mu \right| = \int_{E \cup \{0\}} |d\mu| = 1,$$

where the supremum is taken over all continuous functions on E, of absolute value 1. Thus there exists a continuous f such that $|\int_{E \cup \{0\}} f d\mu| > 1 - \varepsilon$, and since E is a uniform K-set t_0 exists such that

$$|f(x) - e^{it_0 x}| < \varepsilon \text{ on } E.$$

The triangle inequality gives:

$$\left| \int_{E \cup \{0\}} e^{it_0 x} d\mu(x) \right| = \left| a + \int_E e^{it_0 x} d\mu(x) \right| > 1 - 2\varepsilon.$$

Since ε is arbitrary, $E \cup \{0\}$ is a K-set.

The question, raised by Rudin in [1, p. 113], about the equivalence of K-sets and uniform K-sets is answered negatively by Theorem 2. (If $E \cup \{0\}$ is a K-set, then E is not necessarily a uniform K-set.)

It follows from the Riemann–Lebesgue lemma that a K-set cannot have positive Lebesgue measure. (Choose $d\mu(x) = \exp(iNx) \cdot dx/mE$.) How large can a K-set be? Theorem 1 gives the answer to that question. Finally we prove, in Theorem 3, that a uniform K-set cannot be maximal, i.e. we can always add one point and the set remains a uniform K-set. It is not known whether the same is true for K-sets.

The idea of the construction in Theorem 1 is the following: Let $\{p_N\}_1^\infty$ be a sequenee, such that $p_{N+1}/p_N \to \infty$, and $I_1, I_2, \ldots, I_N$ intervals on $(0, 2\pi)$. A sign $+1$ or -1 is attached to the intervals in every possible way, i.e. 2^N ways. If p_1 is large enough there are intervals in each I_ν where $\exp(ip_1 x)$ is approximately equal to the first combination of signs. The union of these intervals $= E_1$. If p_2/p_1 is large enough there are intervals in each $I_\nu \cap E_1$, where $\exp(ip_2 x)$ is approximately equal to the second combination of signs. The union of these $= E_2 \subset E_1$. Proceeding like that we finally get E_{2^N} which has the property that an arbitrary combination of signs on I_ν is approximated on E_{2^N} by one of the $\exp(ip_\nu x)$, $\nu = 1, 2, \ldots, 2^N$.

The intervals of E_{2^N} are now taken as I_ν's and we proceed as above. If we make the p_ν's grow rapidly, it means that the intervals are partitioned into a great many parts. This makes the Hausdorff measure large. At last we obtain a Cantor set with large Hausdorff measure such that every combination of signs on intervals is approximated by the $\exp(ip_\nu x)$'s. This is roughly speaking the definition of a uniform K-set.

In Theorem 2 we make the same construction but at each partition we leave one "odd" interval large enough to guarantee that the combination of signs is not approximated on it. This is enough for the set not to be a uniform K-set. Since the definition of K-sets involves an integration, the set is a K-set if the "odd" intervals are disjoint and tending to zero.

To facilitate the reading we have omitted some uninteresting details. Therefore the proofs contain a few unprecise statements that can easily be proved.

Theorem 1. *Let $h(r)$ be a continuous positive function defined for $r \geqslant 0$ and such that $h(0)=0$, $h(r)$ is increasing and $\lim_{r\to+0} h(r)/r=+\infty$. Then there exists a perfect uniform K-set E of positive Hausdorff measure with respect to the measure function h. ($M_h(E)>0$).*

Proof. We prove the theorem by constructing a set with the desired property.

If $h(r)/r$ does not tend to infinity monotonously when r tends to $+0$, we study the measure function $h_1(r)$ defined by

$$h_1(r)=r\cdot\inf_{r_1\leqslant r}\frac{h(r_1)}{r_1},$$

which has this property. Since $h_1(r)\leqslant h(r)$, it follows immediately from the definition that $M_{h_1}(E)>0$ implies $M_h(E)>0$. We may thus suppose, without loss of generality, that $h(r)/r$ is decreasing.

We construct the set E as a generalized Cantor set $E=\cap_1^\infty E_N$. Let $\{q_N\}_1^\infty$ be an increasing sequence of positive integers and $\{\delta_N\}_1^\infty$ a decreasing sequence of real numbers with the property $\lim_{N\to\infty}\delta_N=\lim_{N\to\infty} 1/q_N=0$.

Suppose that E_N has been constructed as the union of M_N disjoint closed intervals I_l of length ω_l. We attach to each of these intervals a "sign" $\exp(2\pi ik/q_N)$, where k can be $0,1,2,\ldots,q_N-1$. This can be done in $q_N^{M_N}$ different ways. To each combination of signs we choose a positive integer $p_{\nu,N+1}$, $\nu=1,2,\ldots,q_N^{M_N}$. This choice is made so that the inequalities (1) and (4) are satisfied.

$$\left.\begin{aligned} p_{\nu+1,N+1} &> \frac{20}{\delta_{N+1}}\cdot p_{\nu,N+1} \quad (\nu=1,2,\ldots,q_N^{M_N}-1).\\ p_{1,N=1} &> \frac{20}{\delta_{N+1}}\cdot p_{q_N^{M_N}-1}.\end{aligned}\right\} \tag{1}$$

We construct E_{N+1} as a finite intersection $\bigcap_1^{q_N^{N_M}} E_{\nu,N+1}$, where $E_{\nu+1,N+1}\subset E_{\nu,N+1}$ $E_{\nu+1,N+1}$ is the part of $E_{\nu,N+1}$ where

$$\left|e^{ip_{\nu+1,N+1}x}-e^{2\pi i(k/q_N)}\right|\leqslant\delta_{N+1}. \tag{2}$$

Here k has the different integer values in $I_l\cup E_{\nu,N+1}$ that are associated with ν. $E_N=E_{0,N+1}$. Only intervals of equal lengths $(\sim\delta_{N+1}/p_{\nu+1,N+1})$ are accepted. Then $E_{\nu,N+1}$ consists of $M_{\nu,N+1}$ intervals of lengths $\omega_{\nu,N+1}$. The total length of $E_{\nu+1,N+1}$ is approximately δ_{N+1} times the length of $E_{\nu,N+1}$ and we obtain the following relation:

$$M_{\nu+1,N+1}\cdot\omega_{\nu+1,N+1}>M_{\nu,N+1}\,\omega_{\nu,N+1}\cdot\frac{\delta_{N+1}}{2}. \tag{3}$$

Thus

$$h(\omega_{\nu+1,N+1})\cdot M_{\nu+1,N+1}>\frac{h(\omega_{\nu+1,N+1})}{\omega_{\nu+1,N+1}}\cdot M_{\nu,N+1}\cdot\omega_{\nu,N+1}\cdot\frac{\delta_{N+1}}{2}>1 \tag{4}$$

if $\omega_{\nu+1,N+1}$ is small enough, because $h(r)/r \to \infty$ when $r \to +0$. Since

$$\omega_{\nu+1,N+1} \sim \frac{\delta_{N+1}}{p_{\nu+1,N+1}},$$

the inequality (4) is satisfied if $p_{\nu+1,N+1}$ is chosen large enough.

We have now constructed $E_{\nu+1,N+1}$ starting from $E_{\nu,N+1}$. $E_{1,N+1}$ is constructed in a similar way from E_N. We get $E_{N+1} = \bigcap E_{\nu,N+1}$ and finally $E = \bigcap_1^\infty E_N$ as a perfect set of Cantor type.

We first prove (I) that the Hausdorff measure with respect to h is positive and then (II) that E is a uniform K-set.

I. To prove that $M_h(E) > 0$ we show that the equivalent condition is fulfilled: There exists a non-negative set function $\mu(e)$ with $\mu(E) = 1$ such that $\mu(S) \leqslant h(r)$ for every interval S of length $2r$.

Let $\mu_{\nu,N}$ be continuous, with its total mass 1 equally distributed on the intervals of $E_{\nu,N}$, and constant outside $E_{\nu,N}$. $\mu_{\nu+1,N}$ is equal to $\mu_{\nu,N}$ outside $E_{\nu,N}$ and the mass of μ_N on an interval I_0 of $E_{\nu,N}$ is equally distributed on the intervals of $E_{\nu+1,N}$ that are contained in I_0.

Define
$$\mu_{N+1} = \mu_{0,N+2} = \mu_{\frac{M_N}{q_N},N+1}.$$

Then $\mu_N \to \mu$ when $N \to \infty$ and μ is continuous and distributes its unity mass only on E.

We now prove that $h(mI) \geqslant \mu(I)$ for an arbitrary interval I. For one ν and one N, I satisfies

$$\omega_{\nu+1,N} \leqslant mI \leqslant \omega_{\nu,N}.$$

From the way μ is constructed it follows immediately that

$$\mu(I) \leqslant mI \cdot c_{\nu+1,N}, \tag{5}$$

where
$$c_{\nu+1,N} = \frac{1}{M_{\nu+1,N} \cdot \omega_{\nu+1,N}}$$

and we have from (3)

$$c_{\nu+1,N} < c_{\nu,N} \cdot \frac{2}{\delta_N}.$$

Since $h(r)/r$ is decreasing and (4) holds we obtain

$$\frac{h(mI)}{mI} \geqslant \frac{h(\omega_{\nu,N})}{\omega_{\nu,N}} > \frac{2}{\delta_N \cdot M_{\nu,N}\,\omega_{\nu,N}} > \frac{2}{\delta_N}\, c_{\nu,N} > c_{\nu+1,N}.$$

Thus
$$h(mI) \geqslant mI \cdot c_{\nu+1,N}. \tag{6}$$

(5) and (6) give $h(mI) \geqslant \mu(I)$ and it follows that $M_h(E) > 0$.

II. We now prove that every continuous function φ on E such that $|\varphi| = 1$, can be uniformly approximated on E by characters e^{inx}.

Let φ be an arbitrary function of the described kind and choose $\varepsilon > 0$. φ is uniformly continuous on E and thus

$$|\varphi(x) - \varphi(x_\nu)| < \varepsilon \text{ if } |x - x_\nu| < \delta.$$

Choose an N such that $\omega_N < \delta$ and x_ν as one point in the ν:th interval of E_K, $K > N$. Approximate φ with the step function having the values $\varphi(x_\nu) = \exp(i\theta_\nu)$ on the ν:th interval of E_K. By the triangle inequality we obtain

$$|e^{inx} - e^{i\theta_\nu}| \leqslant |e^{inx} - e^{2\pi i(k/q_K)}| + |e^{2\pi i(k/q_K)} - e^{i\theta_\nu}|.$$

Here the numbers k may be chosen in the best possible way and it follows $|\exp(inx) - \exp(i\theta_\nu)| \leqslant \delta_{K+1} + 2\,|\sin(\pi/q_K)| < \varepsilon$ for $n =$ one of the $p_{\nu, K+1}$, $\nu = 1, 2, \dots, q_K^{M_K}$ and $K > K_0$. But K is arbitrary $> K_0$ and thus

$$\inf_n |\varphi(x) - e^{inx}| < 2\,\varepsilon \text{ on } E.$$

Thus E is a uniform K-set q.e.d.

Lemma. *If E is a uniform K-set there exists, to every $\varepsilon > 0$, a sequence $\{t_\nu\}_0^\infty$, $t_\nu \to \infty$, such that $|\exp(it_\nu x) - 1| < \varepsilon$ on E.*

Proof. Let φ be a continuous function of absolute value 1 on E but not equal to any character on E. Then there exists $\{s_\nu\}_0^\infty$ where $s_\nu \to \infty$ such that

$$|e^{is_\nu x} - \varphi(x)| < \frac{\varepsilon}{2} \text{ on } E.$$

Thus by the triangle inequality:

$$|e^{is_\nu x} - e^{is_\mu x}| < \varepsilon \quad \text{and} \quad |e^{i(s_\nu - s_\mu)x} - 1| < \varepsilon$$

and $\{s_\nu - s_\mu\}$ is a sequence with the desired property.

Theorem 2. *There exists a set E such that $E \cup \{0\}$ is a K-set and E not a uniform K-set.*

Proof. We prove this theorem by a construction similar to that in Theorem 1. Condition (4) need not, however, be fulfilled. But when constructing $E_{\nu, N+1}$ we leave one interval I_1 of length $2\pi/p_{\nu-1, N+1}$ and one interval I_2 of length $2\pi/p_{\nu, N+1}$ unchanged. In $E_{\nu+1, N+1}$ I_2 serves as an I_1 and an interval of length $2\pi/p_{\nu+1, N+1}$ is left unchanged as an I_2. The former I_1 is, however, divided in accordance with the "sign" and so on. These "odd" intervals may be chosen disjoint from all preceeding "odd" intervals. This is easily checked since the number of intervals increase by a factor 3, at least, at each partition because of (1). The set $E_{\nu, N}$ thus consists of one interval of length $2\pi/p_{\nu-1, N}$, one of length $2\pi/p_{\nu, N}$, and all the others of length approximately $\delta_N/p_{\nu, N}$.

$$\bigcap_{\nu=1}^{q_N^{M_N}} E_{\nu, N+1} = E_{N+1} \quad \text{and} \quad \bigcap_1^\infty E_N = E$$

is a perfect set of Cantor type as in Theorem 1.

1. $E \cup \{0\}$ *is a K-set.*

We choose an arbitrary $\varepsilon > 0$ and a function μ of bounded variation with support on $F \cup \{0\}$. We normalize μ so that $\int_{E\cup\{0\}} |d\mu| = 1$, i.e. $E \in \Gamma_0(E \cup \{0\})$, and μ has a real jump a at $x=0$. μ has jumps $=a_n$ at the points x_n. Choose p such that $\sum_{p+1}^{\infty} |a_n| < \varepsilon$ and put $\mu_1(x) = \mu(x) - \sum_{p+1}^{\infty} s_n(x)$ where

$$s_n(x) = \begin{cases} a_n, & x \geqslant x_n \\ 0, & x < x_n. \end{cases}$$

Then
$$\left| \int_{E\cup\{0\}} e^{inx} d\mu(x) - \int_{E\cup\{0\}} e^{inx} d\mu_1(x) \right| < \varepsilon. \tag{7}$$

We make a finite division of $(0, 2\pi)$ in intervals I_ν such that

$$\sum_\nu \left(\int_{I_\nu} |d\mu(x) - e^{i\theta_\nu} |d\mu(x)|| \right) < \varepsilon,$$

where $\bigcup I_\nu \supset E$ and I_ν has its endpoints in the complement of E. For the interval that contains $x=0$ we take $\theta_\nu = 0$. This is possible, for, by the Radon–Nikodym theorem, $d\mu(x) = \exp(ig(x))\,|d\mu(x)|$ where $g(x)$ is measurable with respect to $|d\mu|$ and thus can be approximated by a step function. $\operatorname{Exp}(i\theta_\nu)\,|d\mu_1(x)|$ has constant argument on every interval in E_N for $N > N_0$. For $N > N_1 > N_0$ it is also true that $x_n, n = 1, 2, \ldots, p$, do not lie in any "odd" interval. The continuous part of μ_1 is uniformly continuous and thus $\int |d\mu| < \varepsilon$, where we integrate over the two "odd" intervals of E_N and $N > N_2$.

For $N > \operatorname{Max}(N_1, N_2)$ we obtain

$$\left| \int_{E\cup\{0\}} e^{inx} d\mu_1(x) \right| = \left| \int_{E_N\cup\{0\}} e^{inx} d\mu_1(x) \right| = \left| \sum_{\nu=1}^{2} \int_{I_\nu} e^{inx} d\mu_1(x) + \sum_{\nu=3}^{\nu_0} \int_{I_\nu} e^{inx} d\mu(x) \right|$$

where I_1 and I_2 are the two "odd" intervals of E_N. Hence

$$\left| \int_E e^{inx} d\mu_1(x) \right| \geqslant \left| \sum_3^{\nu_0} \int_{I_\nu} e^{inx} d\mu_1(x) \right| - \varepsilon \geqslant \left| \sum_3^{\nu_0} \int_{I_\nu} e^{i(inx+\theta_\nu)} |d\mu_1(x)| \right| - 2\varepsilon \geqslant$$
$$\geqslant \sum_3^{\nu_0} \int_{I_\nu} |d\mu_1(x)| - \sum_3^{\nu_0} \int_{I_\nu} |1 - e^{i(nx+\theta_\nu)}|\,|d\mu_1(x)| - 2\varepsilon.$$

For $n =$ one of the $p_{\nu, N+1}$, $\nu = 1, 2, \ldots, q_N^{M_N}$ we find

$$\sum_3^{\nu_0} \int_{I_\nu} |1 - e^{i(nx+\nu_\nu)}|\,|d\mu_1(x)| < \delta_{N+1} + 2 \left| \sin \frac{\pi}{q_N} \right| < \varepsilon$$

for $N > N_3$. We thus obtain

$$\sup_n \left| \int_E e^{inx} d\mu_1(x) \right| \geqslant 1 - \varepsilon - \varepsilon - 2\varepsilon = 1 - 4\varepsilon$$

and by (7) and since ε is arbitrary >0 it follows

$$\sup_n \left| \int_E e^{inx} d\mu(x) \right| = 1$$

and E is a K-set.

2. *E is not a uniform K-set.*

Suppose that E is a uniform K-set. Then, by the lemma, there exists t_0 such that

$$|e^{it_0x} - 1| < \varepsilon \text{ on } E. \tag{8}$$

For one ν and one N

$$p_{\nu,N} \leqslant t_0 < p_{\nu+1,N}.$$

On the "odd" interval of length $2\pi/p_{\nu,N}$ there are points from E on every interval of length $2\pi/p_{\nu+2,N}$. But $\exp(it_0x)$ assumes all values of absolute value 1 on an interval of length $2\pi/p_{\nu,N}$. Let ξ be a point from the "odd" interval such that $\exp(it_0\xi) = -1$. x_0 is a point from E such that $|x_0 - \xi| < 2\pi/p_{\nu+2,N}$. Then

$$|e^{it_0x_0} - 1| > |e^{it_0\xi} - 1| - |e^{it_0\xi} - e^{it_0x_0}| > 2 - 2\left|\frac{t_0 \cdot 2\pi}{p_{\nu+2,N}}\right|$$

$$> 2 - 2 \sin \frac{\pi p_{\nu+1,N}}{p_{\nu+2,N}} > 2 - 2 \sin \frac{\pi \delta_N}{20},$$

which is a contradiction to (8) if ε is sufficiently small.

This proves part 2 of the theorem.

Theorem 3. *There exists no maximal uniform K-set.*

Proof. Let E be a uniform K-set. Then, by the lemma, there exists, to an arbitrary $\varepsilon > 0$, a sequence $\{t_\nu\}_0^\infty$ such that $|\exp(it_\nu x) - 1| < \varepsilon$, $\nu = 0, 1, \ldots$. Let $\{\delta_\nu\}_0^\infty$ be a sequence of positive numbers tending to zero. We may take the increasing sequence $\{t_\nu\}_0^\infty$ so thin that

$$\delta_\nu / t_\nu > 2\pi / t_{\nu+1}, \quad \nu = 0, 1, 2, \ldots. \tag{9}$$

Then there exists a point $\xi \in C(E)$ such that $\{\exp(it_\nu \xi)\}_0^\infty$ is everywhere dense on the unit circle.

We construct this point as a $\bigcap_0^\infty E_\nu$, where $E_{\nu+1} \subset E_\nu$. The complement of E contains an interval I and $mI > 2\pi/t_N$ for some N.

E_0 = interval in $C(E)$ of length $2\pi/t_N$.

E_1 = one interval where $|e^{it_{N+1}x} - 1| < \delta_1$.

E_ν = one interval where $|\exp(it_{N+\nu}x) - \exp(2\pi i r_\nu)| < \delta_\nu$, $\nu = 2, 3, \ldots$, where r_ν runs through the rational numbers between 0 and 1.

(9) secures that this can be done.

Let f be an arbitrary continuous function, $|f|=1$, on $E \cup \{\xi\}$ and $f(\xi)=\exp(i\varphi_0)$. Since E is a uniform K-set there exists, to an arbitrary $\varepsilon>0$, a real number s_0 such that $|f(x)-\exp(is_0x)|<\varepsilon$ on E. But $\exp(is_0\xi)=\exp(i\varphi_1)$, where φ_1 may be different from φ_0. The way we constructed ξ gives us a number t_{ν_0} such that $|\exp(it_{\nu_0}\xi)-\exp(i(\varphi_0-\varphi_1))|<\varepsilon$ and $|\exp(it_{\nu_0}x)-1|<\varepsilon$ on E. Hence

$$|f(x)-\exp(i(s_0+t_{\nu_0})x)|<|f(x)-e^{is_0x}|+|e^{is_0x}-e^{is_0x}\cdot\exp(it_{\nu_0}x)|<2\varepsilon$$

on E.

$$|f(\xi)-\exp(i(s_0+t_{\nu_0})\xi)|=|e^{i\varphi_0}-e^{i\varphi_1}\cdot\exp(it_{\nu_0}\xi)|<\varepsilon.$$

Thus $s_0+t_{\nu_0}$ is a t such that $|\exp(itx)-f(x)|<2\varepsilon$ on $E\cup\{\xi\}$. Since ε is arbitrary $E\cup\{\xi\}$ is a uniform K-set, which proves Theorem 3.

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