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A theorem of the Phragmén–Lindelöf type för second-order elliptic operators

By LARS LITHNER

1. Introduction and notations

Let R^n be the real n-dimensional Euclidean space with coordinates $x=(x_1, x_2, \dots, x_n)$, $|x|=\sqrt{x_1^2+x_2^2+\dots+x_n^2}$. C denotes the set of all complex-valued infinitely differentiable functions on R^n with compact supports and $L^2(\Omega)$ is the Hilbert space of all complex-valued square integrable functions on the set Ω.

In R^1 let D be the domain $\{x \mid x \geqslant a\}$ where a is arbitrary and let L be the differential operator

$$-\left(\frac{d}{dx}\right)^2+\lambda, \lambda>0.$$

The solutions of $Lu=0$ are

$$u(x)=C_1 e^{\sqrt{\lambda}x}+C_2 e^{-\sqrt{\lambda}x},$$

where C_1 and C_2 are arbitrary constants. From this we conclude that if a solution is bounded in the domain D or if it belongs to $L^2(D)$ then it decreases like $e^{-\sqrt{\lambda}x}$ when x tends to infinity and the same holds for its derivative. In particular, $ue^{\mu x}$ and $(du/dx)e^{\mu x}$ belong to $L^2(D)$ if $\mu<\sqrt{\lambda}$.

In this paper we shall extend this result to second-order elliptic differential operators in R^n,

$$L=-\sum_{i,k=1}^{n} a_{ik}(x) D_{ik}+\sum_{k=1}^{n} b_k(x) D_k+a(x),$$

where $D_{ik}=\partial^2/\partial x_i \partial x_k$, $D_k=\partial/\partial x_k$ and $a_{ik}(x)=a_{ki}(x)$ (for simplicity we confine ourselves to the real domain).

Giving the result of the general case at the end of the paper we start with the operator $L=-\Delta+a(x)$, where Δ is the Laplace operator in R^n and where a is positive and continuous or, more generally, locally bounded and Borel measurable. Then we can prove that if u is a solution of $Lu=0$ outside some compact set K and if u belongs to $L^2(R^n-K)$ then, in the same sense as above, u and its first derivatives decrease exponentially like $e^{-\varphi(x)}$ when $|x|$ tends to infinity. $\varphi(x)$ is the geodetic distance from the origin to the point x in the metric $ds^2=a(x)(dx_1^2+dx_2^2+\dots+dx_n^2)$.

2. The special case

Let D be the domain $\{x \mid |x| \geqslant R\}$, where R is a positive number and let B be the boundary of D. L is the operator $-\Delta + a$ where the function a is strictly positive in R^n. Let $\varphi(x)$ be the geodetic distance from the origin to the point x in the metric $ds^2 = a(x)(dx_1^2 + dx_2^2 + \ldots + dx_n^2)$ that is, $\varphi(x)$ is the greatest lower bound of

$$\left| \int_\Gamma \sqrt{a(y)} \sqrt{dy_1^2 + dy_2^2 + \ldots + dy_n^2} \right|, \; y = (y_1, y_2, \ldots, y_n),$$

where Γ is a piecewise continuously differentiable curve starting at the origin and ending at x. Putting further conditions on a φ will be continuously differentiable.

Lemma 1. $|\text{grad } \varphi(x)| \leqslant \sqrt{a(x)}$.

Proof. It is evident from the definition of φ that

$$|\varphi(x + \Delta x) - \varphi(x)| \leqslant \left| \int_{\Gamma_0} \sqrt{a(y)} \sqrt{dy_1^2 + dy_2^2 + \ldots + dy_n^2} \right|,$$

where Γ_0 is the straight line segment joining x and $x + \Delta x$. This gives the inequality.

Lemma 2. (*Carleman* [1].) *If u belongs to $L^2(D)$ and is a solution of $Lu = 0$ then $\sqrt{a}u$ and $|\text{grad } u|$ belong to $L^2(D)$.*

Proof. Let ψ be a positive function in C. Then we have

$$0 = \int_D u(x)\psi(x)Lu(x)\,dx = \int_D a(x)\psi(x)u^2(x)\,dx - \int_D \sum_1^n u_{ii}(x)u(x)\psi(x)\,dx,$$

where $$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}.$$

By partial integration we get

$$0 = \int_B M(u)\,ds + \int_D \sum_1^n u_i^2(x)\psi(x)\,dx + \\ + \int_D \sum_1^n u_i(x)\psi_i(x)u(x)\,dx + \int_D a(x)\psi(x)u^2(x)\,dx, \quad (1)$$

where $M(u)$ contains u and first derivatives of u and where ds denotes the surface element. In the third integral we can integrate by part once more and get

$$\int_D \sum_1^n u_i(x)\psi_i(x)u(x)\,dx = \int_B M'(u)\,ds - \tfrac{1}{2}\int_D \sum_1^n \psi_{ii}(x)u^2(x)\,dx, \quad (2)$$

where M' is an analogue of M.

From (1) and (2) we get the following estimates

$$\int_D \psi(x)\,\mathrm{grad}^2 u(x)\,dx + \int_D a(x)\,\psi(x)\,u^2(x)\,dx \leqslant$$

$$\leqslant \left| \int_B (M(u) + M'(u))\,ds \right| + \tfrac{1}{2} \int_D \sum_1^n |\psi_{ii}(x)|\,u^2(x)\,dx.$$

Letting ψ tend to 1 in such a way that ψ_{ii} is bounded we conclude that the lemma is true.

We are now going to prove Theorem 1 below. As before, let $u \in L^2(D)$ be a solution of $Lu = 0$ and let v be a function with locally square integrable first derivatives such that $\sqrt{a}v$ and $|\mathrm{grad}\, v|$ belong to $L^2(D)$. An integration by parts gives

$$0 = \int_D v(x)\,Lu(x)\,dx = \int_B M(u, v)\,ds +$$

$$+ \int_D \mathrm{grad}\, u(x) \cdot \mathrm{grad}\, v(x)\,dx + \int_D a(x)\,u(x)\,v(x)\,dx, \qquad (3)$$

where $M(u, v)$ contains u, grad u and v. In fact, this identity is true if we replace v by ψv, where $\psi \in C$. Letting ψ tend to 1 in such a way that $|\mathrm{grad}\, \psi|$ tends to zero, we get the general case.

Now let $0 < \varepsilon < 1$ and put

$$f_N(x) = \min\,(e^{2(1-\varepsilon)\varphi(x)}, N),$$

where N is a positive number and where φ is the function in Lemma 1. It follows from Lemma 1 and Lemma 2 that $\sqrt{a} f_N u$ and $|\mathrm{grad}\,(f_N u)|$ belong to $L^2(D)$. Thus we can substitute $f_N u$ for v in the formula (3) and get

$$0 = \int_B M(u, f_N u)\,ds + \int_D f_N(x)\,\mathrm{grad}^2 u(x)\,dx +$$

$$+ \int_D u(x)\,\mathrm{grad}\, u(x) \cdot \mathrm{grad}\, f_N(x)\,dx + \int_D a(x)\,f_N(x)\,u^2(x)\,dx. \qquad (4)$$

From Lemma 1 it follows

$$|\mathrm{grad}\, f_N(x)| \leqslant 2(1-\varepsilon)\,|\mathrm{grad}\, \varphi(x)|\,f_N(x) \leqslant 2(1-\varepsilon)\sqrt{a(x)}\,f_N(x). \qquad (5)$$

From (4) and (5) we get

$$\int_D f_N(x)\,\mathrm{grad}^2 u(x)\,dx + \int_D a(x)\,f_N(x)\,u^2(x)\,dx \leqslant$$

$$\leqslant \left| \int_B M(u, f_N \cdot u)\,ds \right| + 2(1-\varepsilon) \int_D f_N(x)\sqrt{a(x)}\,|u(x)|\,|\mathrm{grad}\, u(x)|\,dx.$$

Using Schwarz's inequality we get

$$\varepsilon\left(\int_D f_N(x)\,\mathrm{grad}^2 u(x)\,dx + \int_D f_N(x)\,a(x)\,u^2(x)\,dx\right) \leqslant \left|\int_B M(u, f_N\cdot u)\,ds\right|.$$

For large N the right member is a constant and thus, letting N tend to infinity, we have proved the following

Theorem 1. *If u belongs to $L^2(D)$ and is a solution of $Lu=0$, then for every positive number ε*

$$\sqrt{a(x)}\,u(x)\,e^{(1-\varepsilon)\varphi(x)} \quad \textit{and} \quad |\mathrm{grad}\,u(x)|\,e^{(1-\varepsilon)\varphi(x)}$$

belong to $L^2(D)$.

3. The general case

Consider the operator

$$L = -\sum_{i,k=1}^{n} a_{ik}(x)\,D_{ik} + \sum_{1}^{n} b_k(x)\,D_k + a(x),$$

$a_{ik}(x)=a_{ki}(x)$. We suppose that a, besides the conditions in Theorem 1, satisfies $a(x)\geqslant d>0$ outside some compact set.

The operator L is supposed to be uniformly elliptic in D, that is

$$\sum_{i,k=1}^{n} a_{ik}(x)\,\xi_i\,\xi_k \geqslant \alpha\sum_{1}^{n}\xi_i^2 \text{ for all } x \text{ in } D,$$

where α is a positive number. Further we suppose that $\partial a_{ik}(x)/\partial x_k$ and $b_k(x)$, i, $k=1,2,\ldots,n$, tend to zero when $|x|$ tends to infinity.

With the same technique as in the proof of Theorem 1 we can prove

Theorem 2. *If u belongs to $L^2(D)$ and is a solution of $Lu=0$, then for every positive ε*

$$ue^{(1-\varepsilon)\varphi} \quad \textit{and} \quad |\mathrm{grad}\,u|\,e^{(1-\varepsilon)\varphi} \quad \textit{belong to} \quad L^2(D),$$

where $\varphi(x)$ is the geodetic distance from the origin to x in the metric

$$ds^2 = a(x)\sum_{i,k=1}^{n} a'_{ik}(x)\,dx_i\,dx_k$$

$(a'_{ik}(x))$ is the matrix which is inverse to $(a_{ik}(x))$.

To show that our theorems in a certain sense are the best possible we give the following

Example. Let b and c be two positive numbers. The function

$$u(x,y) = e^{-\frac{1}{2}(\sqrt{b}\,x^2+\sqrt{c}\,y^2)}$$

satisfies an equation $-\Delta u + a(x, y)\,u = 0$, where $a(x, y)/bx^2 + cy^2 \to 1$ when $\sqrt{x^2 + y^2}$ tends to infinity. If $\varphi(x, y)$ denotes the geodetic distance from (0, 0) to (x, y) in the metric $ds^2 = (bx^2 + cy^2)(dx^2 + dy^2)$ it is easy to see that $\varphi(x, y) = \frac{1}{2}(\sqrt{b}x^2 + \sqrt{c}y^2)$.

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