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A characteristic property of Euclidean spaces

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Let B be a real normed linear space. The well-known Jordan–von Neumann characterization of Euclidean spaces states that if B satisfies

$$\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2) \quad \text{for all} \quad u, v \in B \tag{1}$$

then B must be Euclidean. An improvement of this result is due to Schoenberg [3], who proved that in (1) the sign of equality may be replaced by either $\geqslant$ or $\leqslant$, the same throughout B. In a previous paper [1] the author proved the following generalization of the Jordan–von Neumann result: If B satisfies

$$\sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu v\|^2 = 0 \quad \text{for all} \quad u, v \in B, \tag{2}$$

where $a_\nu \neq 0, b_\nu, c_\nu, \nu=1, \ldots, m$, are real numbers such that (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu \neq \mu$, then B is a Euclidean space. The proof of this depended on the theory of F-series. The purpose of the present note is to give a quite elementary proof of the fact that condition (2) remains characteristic for Euclidean spaces if the sign of equality is replaced by $\geqslant$ (and then also if it is replaced by $\leqslant$). However, we now have to assume explicitly that

$$\sum_{\nu=1}^{m} a_\nu b_\nu^2 = \sum_{\nu=1}^{m} a_\nu b_\nu c_\nu = \sum_{\nu=1}^{m} a_\nu c_\nu^2 = 0. \tag{3}$$

More precisely, we are going to prove the following theorem:

Theorem: *Let $a_\nu \neq 0, b_\nu, c_\nu, \nu=1, \ldots, m$, be a fixed collection of real numbers satisfying (3) and such that (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu \neq \mu$. If B is a real normed linear space satisfying the condition*

$$\sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu v\|^2 \geqslant 0 \quad \textit{for all} \quad u, v \in B \tag{4}$$

then B is a Euclidean space.

The proof of the theorem depends on two lemmas.

Lemma 1. *If B satisfies* (4) *then B also satisfies a condition of the form*

$$\sum_{\nu=1}^{m} p_\nu \| u + q_\nu v \|^2 \geqslant 0 \quad \textit{for all} \quad u, v \in B, \tag{5}$$

where

$$p_\nu \neq 0 \quad \textit{for} \quad \nu = 1, \ldots, m, \qquad q_\nu \neq q_\mu \quad \textit{for} \quad \nu \neq \mu, \ \sum_{\nu=1}^{m} p_\nu = \sum_{\nu=1}^{m} p_\nu q_\nu = \sum_{\nu=1}^{m} p_\nu q_\nu^2 = 0 \tag{6}$$

and

$$\sum_{q_\nu > 0} p_\nu q_\nu > 0. \tag{7}$$

Proof of Lemma 1. Replace v in (4) by $\lambda u + v$, where λ is so chosen that $b'_\nu = b_\nu + c_\nu \lambda \neq 0$ for $\nu = 1, \ldots, m$. Then we see that (4) is equivalent to the condition

$$\sum_{\nu=1}^{m} a_\nu \| b'_\nu u + c_\nu v \|^2 \geqslant 0 \quad \text{for all} \quad u, v \in B, \tag{8}$$

where $b'_\nu \neq 0$ for $\nu = 1, \ldots, m$. We now write condition (8) in the form

$$\sum_{\nu=1}^{m} r_\nu \| u + s_\nu v \|^2 \geqslant 0 \quad \text{for all} \quad u, v \in B, \tag{9}$$

where we have put

$$r_\nu = a_\nu b_\nu'^2, \; s_\nu = c_\nu / b'_\nu \quad \text{for} \quad \nu = 1, \ldots, m. \tag{10}$$

From the properties of a_ν, b_ν, c_ν and the choice of λ it follows that

$$r_\nu \neq 0 \quad \text{for} \quad \nu = 1, \ldots, m, \qquad s_\nu \neq s_\mu \quad \text{for} \quad \nu \neq \mu, \ \sum_{\nu=1}^{m} r_\nu = \sum_{\nu=1}^{m} r_\nu s_\nu = \sum_{\nu=1}^{m} r_\nu s_\nu^2 = 0. \tag{11}$$

For some $\tau, 1 \leqslant \tau \leqslant m$, we have $r_\tau > 0$. Choose $\varepsilon > 0$ so that $|s_\tau - s_\nu| > \varepsilon$ for $\nu \neq \tau$. Now replace u and v in (9) by $(\varepsilon - s_\tau) u + (\varepsilon + s_\tau) v$ and $u - v$ respectively. Then (9) gives

$$\sum_{\nu=1}^{m} r_\nu \| (\varepsilon - s_\tau) u + (\varepsilon + s_\tau) v + s_\nu (u - v) \|^2 \geqslant 0 \quad \text{for all} \quad u, v \in B$$

or

$$\sum_{\nu=1}^{m} p_\nu \| u + q_\nu v \|^2 \geqslant 0 \quad \text{for all} \quad u, v \in B,$$

where

$$p_\nu = r_\nu (\varepsilon - s_\tau + s_\nu)^2, \quad q_\nu = (\varepsilon + s_\tau - s_\nu)(\varepsilon - s_\tau + s_\nu)^{-1}, \quad \nu = 1, \ldots, m.$$

If we use (11) it is easy to verify that the numbers p_ν, q_ν satisfy relations (6). Moreover, we have by the choice of ε that

$$q_\nu < 0 \quad \text{for} \quad \nu \neq \tau, \quad q_\tau > 0.$$

Hence we have
$$\sum_{q_\nu > 0} p_\nu q_\nu = p_\tau q_\tau = r_\tau \varepsilon^2 > 0,$$

so that the inequality (7) is fulfilled. This completes the proof of Lemma 1.

Now suppose that B is two-dimensional and put $C = \{u \mid u \in B, \|u\| = 1\}$. Then C is a closed convex curve, symmetrical about the origin. It is a well-known fact that there exists a unique ellipse E (with its center at the origin) of maximal area inside C and that this ellipse touches C in at least four different points (see for instance Day [2], pp. 117–118). Let u_0 and v_0 be two of these points such that $u_0 \neq \pm v_0$. We denote by $\|\cdot\|_1$ the norm in B for which E is the unit circle, that is $E = \{u \mid u \in B, \|u\|_1 = 1\}$. Because E is inside C we have $\|u\| \leqslant \|u\|_1$ for all $u \in B$, and because E is an ellipse $\|\lambda u_0 + \mu v_0\|_1^2 = \lambda^2 + 2a\lambda\mu + \mu^2$ for all λ, μ (a is a constant).

Lemma 2. *Put* $\varphi(x) = \|u_0 + x v_0\|^2 - \|u_0 + x v_0\|_1^2$. *Then*

$$\varphi(x) = O(x^2), \quad \textit{when} \quad x \to 0, \tag{12}$$

$$\varphi(x) = O(1), \quad \textit{when} \quad x \to \pm\infty. \tag{13}$$

Proof of Lemma 2. Let w be a vector lying along the common tangent line to C and E at u_0. Then we have

$$1 \leqslant \|u_0 + xw\|^2 \leqslant \|u_0 + xw\|_1^2 = 1 + x^2 \|w\|_1^2$$

from which it follows that

$$\|u_0 + xw\|^2 - \|u_0 + xw\|_1^2 = O(x^2), \quad \text{when} \quad x \to 0. \tag{14}$$

Writing $v_0 = \lambda u_0 + \mu w$ and using (14) we get

$$\|u_0 + x v_0\|^2 - \|u_0 + x v_0\|_1^2 = \|u_0 + x(\lambda u_0 + \mu w)\|^2 - \|u_0 + x(\lambda u_0 + \lambda w)\|_1^2$$
$$= (1 + \lambda x)^2 [\|u_0 + \mu x(1 + \lambda x)^{-1} w\|^2 - \|u_0 + \mu x (1 + \lambda x)^{-1} w\|_1^2] = O(x^2), \quad \text{when} \quad x \to 0$$

so that (12) is proved. For symmetry reasons we also have

$$\|v_0 + x u_0\|^2 - \|v_0 + x u_0\|_1^2 = O(x^2), \quad \text{when} \quad x \to 0.$$

Hence it follows that

$$\|u_0 + x v_0\|^2 - \|u_0 + x v_0\|_1^2 = x^2 (\|v_0 + x^{-1} u_0\|^2 - \|v_0 + x^{-1} u_0\|_1^2)$$
$$= x^2 O(x^{-2}) = O(1), \quad \text{when} \quad x \to \pm\infty$$

and Lemma 2 is proved.

Proof of the theorem. It is sufficient to prove the theorem in the case that B is two-dimensional. Let u_0 and v_0 be the elements introduced above. According to Lemma 1 we have

$$\sum_{\nu=1}^{m} p_\nu \| u_0 + q_\nu x v_0 \|^2 \geqslant 0 \quad \text{for} \quad -\infty < x < +\infty.$$

From relations (6) it follows that

$$\sum_{\nu=1}^{m} p_\nu \| u_0 + q_\nu x v_0 \|_1^2 = 0 \quad \text{for} \quad -\infty < x < +\infty.$$

Hence the function $\varphi(x)$ satisfies

$$\sum_{\nu=1}^{m} p_\nu \varphi(q_\nu x) \geqslant 0 \quad \text{for} \quad -\infty < x < +\infty. \tag{15}$$

If we use Lemma 2 we see that $x^{-2}\varphi(x)$ is integrable over the interval $(-\infty, +\infty)$. Thus we may divide the inequality (15) by x^2 and integrate from $-\infty$ to $+\infty$. Then we get

$$0 \leqslant \int_{-\infty}^{\infty} \sum_{\nu=1}^{m} p_\nu \varphi(q_\nu x) x^{-2} dx = \left(\sum_{q_\nu > 0} p_\nu q_\nu - \sum_{q_\nu < 0} p_\nu q_\nu \right) \cdot \int_{-\infty}^{\infty} x^{-2} \varphi(x) dx$$

$$= 2 \sum_{q_\nu > 0} p_\nu q_\nu \int_{-\infty}^{\infty} x^{-2} \varphi(x) dx.$$

But since $\sum_{q_\nu > 0} p_\nu q_\nu > 0$ and $\varphi(x) \leqslant 0$ for all x it follows that

$$\varphi(x) = 0 \quad \text{for} \quad -\infty < x < +\infty$$

or

$$\| u_0 + x v_0 \| = \| u_0 + x v_0 \|_1 \quad \text{for} \quad -\infty < x < +\infty.$$

This means that C and E coincide or that B is a Euclidean space, which was to be proved.

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