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The Frobenius–Nirenberg theorem

By Lars Hörmander

The extension of the classical Frobenius theorem given by Nirenberg [5], combined with the Poincaré lemma for the operators d and $\bar\partial$, gives conditions on a first order system of differential equations

$$P_j u = f_j,\ j = 1, \ldots, N, \tag{1}$$

for one unknown, which guarantee the existence of local solutions when $f = (f_1, \ldots, f_N)$ satisfies the obvious integrability conditions. In fact, using the classical Frobenius theorem and the theorem of Newlander and Nirenberg [4], Nirenberg determined when it is possible to reduce (1) by a change of variables to a system of equations where each P_j is either $\partial/\partial x^j$ or $\partial/\partial x^j + i\,\partial/\partial x^{j+1}$ for some j. Now Kohn [3] has given a proof of the Newlander–Nirenberg theorem which is based on L^2 estimates. We shall show here that a modified form of his approach leads to a direct proof of existence theorems for the system of equations (1). These results are global and they require only very light smoothness assumptions on the coefficients.

Le P_j, $j = 1, \ldots, N$, be first order differential operators in an open set $\Omega \subset \mathbf{R}^n$,

$$P_j = \sum_{k=1}^{n} a_j^k\, \partial/\partial x_k + a_j^0,\ j = 1, \ldots, N.$$

(No additional difficulty arises if Ω is a manifold and instead of the operators $(P_1, \ldots, P_N)$ we have a first order differential operator P between two complex vector bundles over Ω, with fibers of dimension 1 and N respectively. However, this more general framework would make the notations somewhat heavier.) We denote the principal parts by p_j,

$$p_j = \sum_{k=1}^{n} a_j^k\, \partial/\partial x_k,$$

and we write $\bar p_j$ for the operator obtained by complex conjugation of the coefficients.

Since equations of the form (1) allow us to obtain additional first order equations by forming brackets,

$$[P_j, P_k]u = (P_j P_k - P_k P_j)u = P_j f_k - P_k f_j,$$

it is natural to assume that this procedure has already been exhausted. More precisely, we require that

(i) $a_j^k \in C^1(\Omega)$ *if* $k \neq 0$, *and* $a_j^0 \in C(\Omega)$;

(ii) *For all* j *and* k *there exist functions* $c_{jk}^l \in C(\Omega)$ *such that*

$$[P_j, P_k] = P_j P_k - P_k P_j = \sum_{l=1}^N c_{jk}^l P_l \quad (j, k = 1, \ldots, N).$$

If $u, f_1, \ldots, f_N \in L^2_{\mathrm{loc}}(\Omega)$ and (1) is satisfied, we must have

$$P_j f_k - P_k f_j = \sum_{l=1}^N c_{jk}^l f_l \quad (j, k = 1, \ldots, N). \tag{2}$$

(Note that the product of a C^1 function and the derivative of a function in L^2 is well defined in the distribution sense.) In fact, (2) is obvious if $u \in C^\infty$ and follows in general by regularization and application of Friedrichs' lemma (see e.g. Hörmander [2]).

Following Nirenberg [5] we also introduce the hypothesis

(iii) *There exist functions* d_{jk}^l *and* e_{jk}^l *in* $C^1(\Omega)$ *such that*

$$[p_j, \bar{p}_k] = \sum_{l=1}^N d_{jk}^l p_l - \sum_{l=1}^N e_{jk}^l \bar{p}_l.$$

The coefficients d_{jk}^l and e_{jk}^l need not be uniquely determined. In fact, matrices (δ_{jk}^l) and (ε_{jk}^l) may be added to them provided that

$$\sum_{l=1}^N \delta_{jk}^l p_l = \sum_{l=1}^N \varepsilon_{jk}^l \bar{p}_l = p_{jk},$$

where the last equality is a definition. Both p_{jk} and $\bar{p}_{jk}$ are then in the linear space spanned by the operators $p_1, \ldots, p_N$. Hence the operators with real coefficients $(p_{jk} + \bar{p}_{jk})/2$ and $(p_{jk} - \bar{p}_{jk})/2i$ are also in the span of these operators. Thus p_{jk} is in the space spanned by the real operators which are linear combinations of $p_1, \ldots, p_N$, which proves that if $\varphi \in C^1(\Omega)$ then $p_{jk}\varphi = 0$ at every point in Ω where

$$\sum_1^N b_1 p_1 = \sum_1^N \bar{b}_1 \bar{p}_1 \Rightarrow \sum_1^N b_1 p_1 \varphi = 0. \tag{3}$$

Also note that since

$$[p_j, \bar{p}_k] + \overline{[p_k, \bar{p}_j]} = 0,$$

we have

$$\sum_{l=1}^N (d_{jk}^l - \bar{e}_{kj}^l) p_l = \sum_{l=1}^N (e_{jk}^l - \bar{d}_{kj}^l) \bar{p}_l,$$

so that

$$\sum_{l=1}^N d_{jk}^l p_l \varphi = \sum_{l=1}^N \bar{e}_{kj}^l p_l \varphi$$

if (3) is valid. In the following definition no ambiguity is therefore caused by the indeterminacy of the coefficients d^l_{jk} and e^l_{jk}, and the form (4) introduced there is hermitian symmetric.

Definition. *A real valued function $\varphi \in C^2(\Omega)$ is said to be convex with respect to a system* (1) *satisfying* (i) *and* (iii), *provided that at every point in Ω where* (3) *is fulfilled we have*

$$\operatorname{Re} \sum_{j,k=1}^{N} (p_j \bar{p}_k \varphi + \sum_{l=1}^{N} e^l_{jk} \bar{p}_l \varphi) f_j \bar{f}_k \geq c \sum_{j=1}^{N} |f_j|^2 \quad \text{if} \quad \sum_{j=1}^{N} f_j p_j \varphi = 0. \tag{4}$$

Here c shall have a positive lower bound on every compact subset of Ω.

Example 1. If $P_j = \partial/\partial x_j$, $j = 1, \ldots, n$, the convexity condition means that the only critical points of φ are non-degenerate minimum points. (Theorem 1 below therefore gives an existence theorem for this system in Ω if and only if every component of Ω is homeomorphic to an open ball.)

Example 2. If $n = 2\nu$ and $\mathbf{R}^n$ is identified with $\mathbf{C}^\nu$, the operators P_j can be chosen as $\partial/\partial \bar{z}_j$ where z_j are the complex coordinates. Then the convexity means that φ is strictly plurisubharmonic at the critical points and that the level surfaces of φ are strictly pseudoconvex elsewhere. (Theorem 1 below therefore gives an existence theorem for this system in Ω if and only if Ω is a domain of holomorphy.)

To state our global existence theorem we need a final hypothesis:

(iv) *In a neighborhood of any point in the complement with respect to Ω of the closure of the set where* (3) *is valid one can choose functions $b_1, \ldots, b_N \in C^1$ so that $\sum_{j=1}^N b_j p_j$ is real and $\sum_{j=1}^N b_j p_j \varphi \neq 0$ in the whole neighborhood.*

Note that (iv) is valid if the dimension of the space of $(b_1, \ldots, b_N) \in \mathbf{C}^N$ such that $\sum_1^N b_j p_j$ has real coefficients at the point $x \in \Omega$ is independent of x.

Theorem 1. *Let the hypotheses* (i), (ii), (iii) *be fulfilled, and assume that there is a real valued function $\varphi \in C^2(\Omega)$ satisfying* (iv) *such that φ is convex with respect to the system* (1) *and*

$$\Omega_\gamma = \{x;\ x \in \Omega,\ \varphi(x) < \gamma\}$$

is relatively compact in Ω for every $\gamma < \sup_\Omega \varphi$. For all $f_j \in L^2_{\mathrm{loc}}(\Omega)$, $j = 1, \ldots, N$, which satisfy the compatibility conditions (2), *the equations* (1) *then have a solution $u \in L^2_{\mathrm{loc}}(\Omega)$.*

The hypotheses concerning φ can be stated in a seemingly stronger form.

Lemma 1. *The function φ in Theorem 1 and the coefficients in* (iii) *can be modified so that the inequality* (4) *is valid for all f at all points in Ω, with a positive continuous function c, and the set Ω_γ is relatively compact in Ω for every real number γ.*

Proof. First note that a function φ which is convex with respect to (1) cannot have a local maximum in Ω. Hence $\varphi(x) < \Gamma = \sup_\Omega \varphi$ for every $x \in \Omega$. Now replace φ by $\chi(\varphi)$ where χ is a convex, increasing function $\in C^2(-\infty, \Gamma)$. Then the form (4) is replaced by

$$\chi'(\varphi) \sum_{j,k=1}^{N} (p_j \bar{p}_k \varphi + \sum_{l=1}^{N} e_{jk}^l \bar{p}_l \varphi) f_j \bar{f}_k + \chi''(\varphi) \left| \sum_{j=1}^{N} f_j p_j \varphi \right|^2.$$

From this and an obvious compactness argument we conclude that if $\chi''(\varphi)/\chi'(\varphi)$ is larger than a certain increasing function of φ on $[\min \varphi, \Gamma)$, then (4) holds for all f at all points in a neighborhood ω of the closure in Ω of the set where (3) is valid, if φ is replaced by $\chi(\varphi)$. We can choose χ so that $\chi(\varphi) \to \infty$ when $\varphi \to \Gamma$. Now we can use a partition of unity to construct functions $b_1, \ldots, b_N \in C^1(\Omega)$ such that $\sum_1^N b_j p_j$ has real coefficients and $\sum_1^N b_j p_j \varphi \geqslant 0$ with strict inequality in $\Omega \cap \complement \omega$. In fact, the local existence of such functions $b_1, \ldots, b_N$ is guaranteed by condition (iv). We can add to d_{jk}^l and e_{jk} matrices $\lambda b_l \delta_{jk}$ and $\lambda \bar{b}_l \delta_{jk}$ respectively, where δ_{jk} is the Kronecker δ and $0 \leqslant \lambda \in C^1$. If λ is chosen larger than some continuous function of φ and φ is replaced by $\chi(\varphi)$, the inequality (4) will be valid without restriction.

Remark. It may seem that we could as well have required from the beginning that (4) shall hold in Ω for all f. However, such an assumption would not have been independent of the choice of e_{jk}^l and would not have been invariant under non-singular linear transformations of the operators $p_1, \ldots, p_N$ whereas the assumptions we have made are invariant under such operations. This is of course particularly important if the theorem is extended to bundles as indicated above.

The proof of Theorem 1 will at the same time give a local existence theorem.

Theorem 2. *Let* (i), (ii), (iii) *be satisfied in a neigborhood of* 0 *and assume that the vectors*

$$(a_j^1(0), \ldots, a_j^n(0)), \; j = 1, \ldots, N,$$

are linearly independent, that is, that the operators p_j, $j = 1, \ldots, N$, are linearly independent at 0. *If Ω is the ball $\{x; |x| < \delta\}$ and δ is sufficiently small, the equations* (1) *have a solution $u \in L^2_{\text{loc}}(\Omega)$ for all $f_j \in L^2_{\text{loc}}(\Omega)$ satisfying* (2).

Lemma 2. *The hypotheses of Theorem* 2 *imply that for small δ there is a function $\varphi \in C^2(\Omega)$ with the properties listed in Lemma* 1.

Proof. The form (4) with $\varphi(x) = |x|^2$ reduces to

$$2 \sum_{\nu=1}^{n} \sum_{j,k=1}^{N} a_j^\nu \overline{a_k^\nu} f_j \bar{f}_k = 2 \sum_{\nu=1}^{n} \left| \sum_{j=1}^{N} a_j^\nu f_j \right|^2 \quad \text{at } x = 0,$$

which is positive definite since by hypothesis $f = 0$ if $\sum_{j=1}^N a_j^\nu f_j = 0$, $\nu = 1, \ldots, n$, at $x = 0$. If δ is sufficiently small, the form (4) is therefore uniformly positive definite in Ω when $\varphi(x) = |x|^2$. Hence the function $\varphi(x) = 1/(\delta^2 - |x|^2)$ has all required properties, for $\varphi(t) = (\delta^2 - t)^{-1}$ is convex and increasing when $t < \delta^2$, and $\varphi(t) \to \infty$ when $t \to \delta^2$.

In proving both Theorem 1 and Theorem 2 we now have available a function φ with the properties listed in Lemma 1, and it remains to prove that the existence of such a function together with conditions (i), (ii), (iii) leads to the existence theorems we want. In the proof we use the following elementary fact from functional analysis.

Lemma 3. *Let T be a linear closed densely defined operator from one Hilbert space H_1 to another H_2, and let F be a closed subspace of H_2 containing the range R_T of T. Then $F = R_T$ if and only if*

$$\|f\|_{H_2} \leqslant C\,\|T^* f\|_{H_1}, \quad f \in F \cap D_{T^*}. \tag{5}$$

We shall apply this lemma with $H_1 = L^2(\Omega, \sigma_1)$, the space of functions which are square integrable in Ω with respect to the measure $\sigma_1\,dx$, where dx denotes the Lebesgue measure and the density σ_1 will be chosen later. We take H_2 as the direct sum of N copies of $L^2(\Omega, \sigma_2)$ and set

$$Tu = (P_1 u, \ldots, P_N u).$$

Thus $u \in D_T$ if $u \in L^2(\Omega, \sigma_1)$ and $P_j u$ defined in the distribution sense belongs to $L^2(\Omega, \sigma_2)$ for $j = 1, \ldots, N$. For suitable densities σ_1 and σ_2 we shall prove that R_T consists of all $f \in H_2$ such that (2) is fulfilled. We therefore introduce a third Hilbert space H_3 which is the direct sum of $\binom{N}{2}$ copies of $L^2(\Omega, \sigma_3)$ and define an operator S from H_2 to H_3 by

$$Sf = \{P_j f_k - P_k f_j - \sum_{l=1}^{N} c_{jk}^l f_l\}_{1 \leqslant j < k \leqslant N},$$

the precise definition being analogous to that of T. Then we have $ST = 0$ so R_T is contained in the null space N_S of S. We wish to prove that $R_T = N_S$, that is, that (5) is valid with $F = N_S$. To do so it is sufficient to prove that

$$\|f\|_{H_2}^2 \leqslant C(\|T^* f\|_{H_1}^2 + \|Sf\|_{H_3}^2), \quad f \in D_S \cap D_{T^*}, \tag{6}$$

and this we shall do when the densities are conveniently chosen. Moreover, we shall prove that the densities can be chosen so that (6) is valid and any given f with components in $L^2_{\text{loc}}(\Omega)$ belongs to H_2. This will prove the theorems.

The operators T and S have been defined as maximal differential operators, so $D_S \cap D_{T^*}$ is the intersection of the minimal domain of one differential operator with the maximal domain of another. However, we prefer to prove estimates involving only elements $f \in H_2$ with components in $C_0^\infty(\Omega)$. This will suffice if $\sigma_1, \sigma_2, \sigma_3$ are chosen so that the distinction between minimal and maximal differential operators disappears.

Let us therefore choose a sequence $\eta_\nu \in C_0^\infty(\Omega)$ so that $0 \leqslant \eta_\nu \leqslant 1$ for every ν and $\eta_\nu = 1$ on any compact subset of Ω when ν is large. We can choose $\sigma_j \in C^\infty(\Omega)$ so that $\sigma_j > 0$ and

$$\sum_{k=1}^{N} \sigma_{j+1} |p_k \eta_\nu|^2 \leqslant \sigma_j \; (j = 1, 2;\; \nu = 1, 2, \ldots). \tag{7}$$

In fact, on any compact subset of Ω this means only a finite number of positive bounds for the quotient σ_{j+1}/σ_j. Since

$$T(\eta_\nu u) - \eta_\nu Tu = (p_1 \eta_\nu, \ldots, p_N \eta_\nu)\, u,$$

we have

$$\eta_\nu T^* f - T^*(\eta_\nu f) = \sigma_2 \sigma_1^{-1} \sum_1^N f_k \bar{p}_k \eta_\nu,$$

and Cauchy–Schwarz' inequality gives in view of (7)

$$|\eta_\nu T^* f - T^*(\eta_\nu f)|^2 \sigma_1 \leqslant \sigma_2 \sum_1^N |f_k|^2.$$

Hence we obtain by dominated convergence that

$$\|\eta_\nu T^* f - T^*(\eta_\nu f)\|_{H_1} \to 0 \quad \text{when} \quad \nu \to \infty. \tag{8}$$

Similarly we can prove that

$$\|S(\eta_\nu f) - \eta_\nu Sf\|_{H_3} \to 0 \quad \text{when} \quad \nu \to \infty. \tag{9}$$

If $f \in D_{T^*} \cap D_S$, it follows that $\eta_\nu f \to f$, $S(\eta_\nu f) \to Sf$ and $T^*(\eta_\nu f) \to T^* f$ when $\nu \to \infty$, that is, $\eta_\nu f \to f$ in the graph norm. But Friedrichs' lemma (see e.g. Hörmander [2]) shows that $\eta_\nu f$ can be approximated in the graph norm by elements with components in $C_0^\infty(\Omega)$. Hence we have proved

Lemma 4. *If* (7) *holds, it follows that elements in H_2 with components belonging to $C_0^\infty(\Omega)$ are dense in $D_{T^*} \cap D_S$ in the graph norm*

$$f \to \|f\|_{H_2} + \|T^* f\|_{H_1} + \|Sf\|_{H_3}.$$

Proof of Theorems 1 and 2. Let σ_j^0, $j = 1, 2, 3$, be some fixed positive C^∞ functions satisfying (7), and let $\varphi \in C^2(\Omega)$ have the properties listed in Lemma 1. If we set

$$\sigma_j = \sigma_j^0 e^{-\chi(\varphi)},$$

where χ is a convex increasing function in $C^2(\mathbf{R})$, it is clear that (7) is fulfilled, and we shall show that (6) is valid if χ is sufficiently rapidly increasing. We may assume that $\sigma_1^0 \sigma_3^0 = (\sigma_2^0)^2$, which implies that $\sigma_1 \sigma_3 = \sigma_2^2$. The adjoint of P_j with respect to the usual L^2 norm can be written $-\bar{p}_j + c_j$ where c_j is a continuous function. If $f_j \in C_0^\infty(\Omega)$, which we assume from now on, we obtain

$$T^* f = \sigma_1^{-1} \sum_{j=1}^N (-\bar{p}_j + c_j)(f_j \sigma_2).$$

If we introduce the definition of σ_j and write $\psi = \chi(\varphi)$, it follows that

$$A_1 = \int \left| \sum_1^N \bar{p}_j (f_j e^{-\psi}) \right|^2 e^\psi \sigma_3^0 \, dx \leqslant 2\, \|T^* f\|_{H_1}^2 + N(f),$$

where the left-hand side is a definition. Here and in what follows $N(f)$ denotes an error term for which there is an estimate of the form

$$N(f) \leqslant \int |f|^2 C(x)\, e^{-\psi}\, dx,$$

where C is continuous in Ω and independent of f and of ψ. It is also clear that

$$A_2 = \sum_{1\leqslant j<k\leqslant N} \int |p_j f_k - p_k f_j|^2 e^{-\psi} \sigma_3^0\, dx \leqslant 2\, \|Sf\|_{H_3}^2 + N(f).$$

With the notation

$$B_1 = \sum_{j,k=1}^{N} \int |p_j f_k|^2 e^{-\psi} \sigma_3^0\, dx$$

we have

$$A_1 + A_2 = B_1 + \sum_{j,k=1}^{N} \left\{ \int \bar{p}_k (f_k e^{-\psi}) \overline{\bar{p}_j (f_j e^{-\psi})}\, e^{\psi} \sigma_3^0\, dx - \int p_j f_k \overline{p_k f_j}\, e^{-\psi} \sigma_3^0\, dx \right\}.$$

We shall integrate by parts here, moving all differentiations to the left. This gives if we use obvious estimates for all but the main terms

$$\operatorname{Re} \int \sum_{j,k=1}^{N} \{\bar{p}_k (e^{-\psi} p_j f_k) - e^{-\psi} p_j (e^{\psi} \bar{p}_k (f_k e^{-\psi}))\} \bar{f}_j \sigma_3^0\, dx + B_1$$
$$\leqslant A_1 + A_2 + (A_1 N(f))^{\frac{1}{2}} + (B_1 N(f))^{\frac{1}{2}}.$$

The paranthesis on the left can be simplified to

$$e^{-\psi} (\bar{p}_k p_j f_k - p_j \bar{p}_k f_k + (p_j \bar{p}_k \psi) f_k)$$
$$= e^{-\psi} \left(\sum_{l=1}^{N} e_{jk}^l \bar{p}_l f_k - \sum_{l=1}^{N} d_{jk}^l p_l f_k + (p_j \bar{p}_k \psi) f_k \right).$$

By another integration by parts in the terms involving $\bar{p}_l f_k$, it follows that

$$\operatorname{Re} \int \sum_{j,k=1}^{N} \left(p_j \bar{p}_k \psi + \sum_{l=1}^{N} e_{jk}^l \bar{p}_l \psi \right) f_k \bar{f}_j e^{-\psi} \sigma_3^0\, dx + B_1 \leqslant A_1 + A_2 + (A_1 N(f))^{\frac{1}{2}} + (B_1 N(f))^{\frac{1}{2}}.$$

If we introduce that $\psi = \chi(\varphi)$, use the fact that χ is convex, and make obvious estimates like $(B_1 N(f))^{\frac{1}{2}} \leqslant B_1 + N(f)$, we obtain

$$\int \chi'(\varphi) \operatorname{Re} \sum_{j,k=1}^{N} \left(p_j \bar{p}_k \varphi + \sum_{1}^{N} e_{jk}^l \bar{p}_l \varphi \right) f_k \bar{f}_j e^{-\chi(\varphi)} \sigma_3^0\, dx$$
$$\leqslant 3 \left(\|T^* f\|_{H_1}^2 + \|Sf\|_{H_3}^2 + \int |f|^2 C(x)\, e^{-\chi(\varphi)}\, dx \right)$$

in view of the estimates initially given for A_1 and A_2. But the quadratic form in the integrand can be estimated from below by $c|f|^2$ where c is positive and continuous in Ω. If we choose χ so rapidly increasing that

$$\chi'(\varphi)\, c\, \sigma_3^0 \geq 3(\sigma_2^0 + C), \tag{10}$$

which is possible since Ω_γ is relatively compact in Ω for every γ, it follows that

$$\|f\|_{H_2}^2 \leq \|T^* f\|_{H_1}^2 + \|Sf\|_{H_3}^2, \quad f \in D_{T^*} \cap D_S.$$

Here we have also used Lemma 4. Since any given f with components in $L^2_{\text{loc}}(\Omega)$ belongs to H_2 for some choice of $\chi(\varphi)$ satisfying (10), the proof of the theorem is complete.

Remark. The smoothness assumptions made above are slightly more restrictive than necessary. In fact, it is everywhere sufficient to require measurability and local boundedness instead of continuity, and local Lipschitz continuity instead of continuous differentiability. In the definition one must only strengthen the requirement then by assuming (4) in a neighborhood of the closure of the set of points where (3) is valid.

The convexity conditions considered here agree in the case $N=1$ with the first order case of those introduced in Hörmander [1], Chapter VIII. At the International Colloquium on Differential Analysis in Bombay, January 1964, J. Kohn has also indicated some convexity conditions for systems which seem close to the one used here.

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