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Some growth and ramification properties of certain integrals on algebraic manifolds

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1. Introduction and statement of results

We are going to work in complex n-space C^n with elements $x=(x_1,\ldots,x_n)$, etc., and shall consider a certain class $A(C^n)$ of analytic functions on C^n with special growth and ramification properties. To prepare the definition of $A(C^n)$ we make the following definition.

Definition. A path γ: $x=x(t)=(x_1(t),\ldots,x_n(t))$, $t_1\leqslant t\leqslant t_2$, in C^n is said to be of class $\mathcal{A}$ if it consists of a finite number of pieces where the components $x_i(t)$ are regular algebraic functions of t. If the number of pieces is not greater than n_1 and the algebraic functions all have degrees $\leqslant n_2$ (i.e. each of them may be defined by a polynomial of degree $\leqslant n_2$), then we shall say that γ has the rank (n_1,n_2). (It then also has the rank (n_1',n_2'), if $n_1\leqslant n_1'$ and $n_2\leqslant n_2'$).
We now define the class $A(C^n)$.

Definition. $A(C^n)$ consists of all functions f such that there exists an algebraic manifold V_f in C^n of the form $p(x)=0$, where $p(x)$ is a complex polynomial not identically zero, such that

(a) f is a regular analytic and in general many-valued function on the whole of (C^n-V_f)

(b) all the determinations of f in the neighbourhood of any point in (C^n-V_f) span a linear space over C of finite dimension

(c) there is a point $x^0\in(C^n-V_f)$, a real number a, a complex polynomial $R(x)$, $R(x)\neq0$ when $x\in(C^n-V_f)$, and for every determination f_0 of f at x^0 and every (n_1,n_2) a real number C such that

$$|f_{0\gamma}(x)|\leqslant C(|x|+1)^a|R(x)|^{-1}\quad(\forall x\in\gamma)$$

for all paths γ in C^n of rank (n_1,n_2) starting at x^0 and not meeting V_f. Here $f_{0\gamma}$ is the function on γ obtained by analytic continuation along γ out from f_0, and $|x|=(\sum_{i=1}^n|x_i|^2)^{\frac{1}{2}}$.

Remark. It may be proved that the condition (c) is not changed if we only permit pairs (n_1,n_2) of the form $(n_1,1)$.

Instead of $f\in A(C^n)$ we shall also say that $f(x)$ is of class A in x. If a function f is defined and regular analytic in a neighbourhood of a point $x^0\in C^n$ (or on any point set sufficiently dense to determine f uniquely), then saying that f is of class A naturally means that the function obtained from f by analytic continuation is of class A. Hence this means that there is a manifold V_f as above such that f may be continued analytically along any path in C^n which does not meet V_f and that the resulting analytic function on $(C^n - V_f)$ ($(C^n - V_f)$ being connected) satisfies (*b*) and (*c*). We may also observe that if $f \in A(C^n)$, then as the point x^0 in (*c*) we may choose any $x^0 \in (C^n - V_f)$, keeping the number a and the polynomial $R(x)$.

Example. An algebraic function f on C^n belongs to $A(C^n)$. For let f satisfy the equation

$$p_m(x) f(x)^m + \dots + p_0(x) = 0$$

where the $p_i(x)$ are complex polynomials and $p_m(x) \not\equiv 0$ and where the discriminant $D(x)$ is not identically zero. If we take V_f as the manifold $p_m(x) D(x) = 0$, then evidently f satisfies (*a*) and (*b*), and by an elementary estimate for all the roots of an algebraic equation we have

$$|f(x)| \leqslant \max_{0 \leqslant i \leqslant m} (|p_i(x)/p_m(x)|) + 1$$

for all determinations of f. Then we easily see that (*c*) holds, taking $R(x) = p_m(x)$.

In this example the number C in (*c*) is independent of the path of continuation γ, the estimate holding for all determinations of the function. A simple example where this is not the case is provided by $\log x_1$, which by later results (or by the remark of the definition of $A(C^n)$) may be seen to be of class A, and for which clearly no majoration may be independent of the special path of continuation. Another example of the same kind is x_1^α with α non-real.

Now let us consider an algebraic manifold $V(y) \subset C^n$, depending on a parameter $y \in C^m$, which is given by $(n-r)$ equations

$$p_1(y,x) = 0, \dots, p_{n-r}(y,x) = 0$$

where the $p_i(y,x)$ are complex polynomials in $y \in C^m$ and $x \in C^n$. We let $R(y)$ be the set of those points $x \in V(y)$ for which the gradients

$$\text{grad}_x \, p_1(y,x), \dots, \text{grad}_x \, p_{n-r}(y,x)$$

are linearly independent over C. $R(y)$ is then an analytic manifold of dimension r. When $R(y^0)$ is not empty, the same is true of $R(y)$ when y is sufficiently close to y^0. If $x^0 \in R(y^0)$, there is a subset $x'' = (x_{j_{r+1}}, \dots, x_{j_n})$ of the coordinates x_i in C^n such that $\text{grad}_{x''} \, p_1(y^0, x^0), \dots, \text{grad}_{x''} \, p_{n-r}(y^0, x^0)$ are linearly independent. If $x' = (x_{j_1}, \dots, x_{j_r})$ consists of the remaining coordinates x_i, we may in a neighbourhood (in C^n) of x^0 use x' as coordinates on $R(y)$ when y lies in a neighbourhood of y^0.

If $\omega_y(x)$ is a holomorphic differential p-form on $R(y)$, it may be written locally as

$$\omega_y(x) = \sum_{k_1 < \dots < k_p} f_{k_1 \dots k_p}(y, x') \, dx_{k_1} \wedge \dots \wedge dx_{k_p} \qquad (L)$$

where the k_i are among the numbers $j_1, \dots, j_r$.

Definition. A holomorphic differential p-form $\omega_y(x)$ on $R(y)$, defined for (y,x) in a neighbourhood of (y^0, x^0), $x^0 \in R(y^0)$, is said to be of class A at (y^0, x^0) if all the coefficients in a representation of $\omega_y(x)$ of the form (L) at (y^0, x^0) are of class A in (y, x').

Remark. If $\omega_y(x)$ is of class A at (y^0,x^0), then the coefficients in (L) are of class A for any permitted choice of x'. This will follow from Lemma 1 and the fact that two different sets (y,x') and $(y,\tilde{x}')$ in (L) are connected by an algebraic transformation.

Our main result is the following theorem.

Theorem 1. Let $\gamma(y)$ be a compact regular[1] p-cycle on $R(y)$ and $\omega_y(x)$ a holomorphic closed differential p-form on $R(y)$, where $\gamma(y)$ is defined for y in a neighbourhood of y^0 and $\omega_y(x)$ for (y,x) in a neighbourhood of $(y^0,\gamma(y^0))$. Further suppose that $\gamma(y)$ depends continuously on y and that $\omega_y(x)$ is of class A at any (y^0,x^0), $x^0 \in \gamma(y^0)$. Then the function

$$g(y)=\int_{\gamma(y)}\omega_y(x)$$

(hereby defined in a neighbourhood of y^0) is of class A.

In a particular case we shall show a sharper result:

Theorem 2. If in Theorem 1 we have $p=m=1$ and if the coefficients of $\omega_y(x)$ in any representation (L) are algebraic in (y,x'), then there exists a natural number N such that for $|y|$ sufficiently large

$$T^N g(y)=g(y)+h(y),\ T^N h(y)=h(y).$$

Here T denotes analytic continuation one turn in the positive sense along circles $|y|=\text{constant}$.

Theorem 2 is related to the theory of the variation of cycles on a variable plane section of an algebraic manifold, due to Lefschetz [1] and Picard and Simart [2]. In this theory, however, there are simplifying restrictions on the manifold.

The proof of Theorem 1 is based on lemma 2 where we prove that integration of a function of class A with respect to a single complex variable between bounds that are algebraic functions of the remaining variables yields a function of class A. In the proof of Lemma 2 the analytic continuation of the integral is performed by successive deformation of the path of integration, and the result is obtained by certain estimates for the path of integration and the integrand. By a triangulation of $\gamma(y)$ we then write $g(y)$ as a sum of functions, each of which has been obtained from a function of class A by repeated integrations with respect to a single complex variable between algebraic bounds. Theorem 1 then follows from Lemma 2. Theorem 2 is a consequence of Lemma 4, where, in a particular case, we prove a more precise result than in Lemma 2.

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2. Some properties of functions of class A

To prepare the proof of our theorems, we give some lemmas.

Lemma 1. $A(C^n)$ is a ring. Let $f \in A(C^n)$ and let $y \to x(y)$ be an algebraic mapping from C^m to C^n such that all its branches are regular analytic outside $W': q'(y)=0$, $q'(y) \not\equiv 0$, and that for some $y \in (C^m - W')$ all the determinations of $x(y)$ do not belong to V_f. Then the function $g(y)=f(x(y))$ is of class A.

[1] e.g. piecewise of class C^1.

Proof. It is easy to check that if $f_1, f_2 \in A(C^n)$, then $f_1 \pm f_2$ and $f_1 f_2$ also belong to $A(C^n)$. We may e.g. choose the corresponding manifold V_f as $V_{f_1} \cup V_{f_2}$ and the polynomial $R(x)$ in (c) as $R_1(x)R_2(x)$, if $R_1(x)$ and $R_2(x)$ correspond to f_1 and f_2. Now let $f \in A(C^n)$ and $y \to x(y)$ be a mapping as above. Then the condition '$x(y) \in V_f$ for some determination of $x(y)$' defines a manifold W'' in C^m of the form $q''(y)=0, q''(y) \not\equiv 0$, since for some $y \in (C^m - W')$ all the determinations of $x(y)$ are not on V_f. We then see that $g(y)$ has the properties (a) and (b) with respect to $V_g = W = W' \cup W''$. Let the polynomial $R(x,y)$ and the number k correspond to f in (c). Then $x(y)$ and $1/R(x(y),y)$ are algebraic, with all their branches regular outside W. Hence there exist polynomials $R_1(y)$ and $R_2(y)$, not vanishing on $(C^m - W)$, and real numbers k_1 and k_2 such that

$$|x(y)| \leqslant (|y|+1)^{k_1}/|R_1(y)|, \ (y \in (C^m - W)),$$

and

$$|1/R(x(y),y)| \leqslant (|y|+1)^{k_2}/|R_2(y)|, \ (y \in (C^m - W)),$$

for all determinations of $x(y)$. Hence

$$(|x(y)|+1)^k/|R(x(y),y)| \leqslant (|y|+1)^{k_3}/|R_3(y)|, (y \in (C^m - W))$$

for some polynomial $R_3(y)$, $R_3(y) \neq 0$ when $y \in (C^m - W)$, some real number k_3 and all determinations of $x(y)$. It then follows that $g(y)$ also satisfies (c) with respect to W, since it is clear that if γ is a path of class $\mathcal{A}$ in $(C^m - W)$, then $x(\gamma)$ is a path in C^n of class $\mathcal{A}$, of a rank depending only on the rank of γ and on the mapping $y \to x(y)$.

Our next lemma roughly states that integration with respect to one complex variable between algebraic bounds of a function of class A yields a function of class A.

Lemma 2. Let the function $f(y,x)$ be of class A in (y,x), where $y \in C^n$ and $x \in C$, with $V_f: p(y,x)=0$. Let $\eta_1(y)$ and $\eta_2(y)$ be algebraic functions on C^n such that $(y,\eta_1(y)), (y,\eta_2(y)) \bar{\in} V_f$ for some $y \in C^n$ and all determinations of $\eta_1(y)$ and $\eta_2(y)$. Then the function

$$g(y) = \int_{\eta_1(y)}^{\eta_2(y)} f(y,x)\,dx$$

is of class A. Here we take the integral along a path in the complex plane varying continuously with y and not passing through any point x such that $(y,x) \in V_f$. As integrand we have a branch of $f(y,x)$ (hence an analytic function of (y,x) along the path). As V_g we may take any manifold W of the form $q(y)=0$, $q(y) \neq 0$, such that outside W all the different branches of the algebraic function $x = x(y)$, defined by $p(y,x)=0$, are regular and non-intersecting, and such that outside W all the branches of $\eta_1(y)$ and $\eta_2(y)$ are regular and $p(y,\eta_1(y)) \neq 0$, $p(y,\eta_2(y)) \neq 0$.

For the proof we shall need the following lemma.

Lemma 3. Let l be a straight line segment in the complex plane, and let the function $\omega = f(z)$ be defined and analytic in a neighbourhood of l; assume that $f(z)$ is a branch of the algebraic function defined by $P(\omega,z)=0$. Then there is a number N, only depending of the degree of $P(\omega,z)$, such that either $\mathrm{Re}(f(z)) \equiv 0$ on l, or the number of zeros of $\mathrm{Re}(f(z))$ on l is smaller than N.

Proof. Without loss of generality we may assume that l is a piece of the real axis. Then in a neighbourhood of l, $w_1 = \overline{f(\bar{z})}$ is a branch of the algebraic function defined by $\overline{P}(\omega_1, z) = 0$. For $z \in l$ we have $\operatorname{Re}(f(z)) = 2^{-1}(f(z) + \overline{f(\bar{z})})$, and by well-known properties of algebraic functions $\omega_2 = \operatorname{Re}(f(z))$ is a branch of an algebraic function, $q(\omega_2, z) = 0$, whose degree may be majorized by a number only depending on the degree of $P(\omega, z)$. Hence no non-zero branch of the algebraic function $\omega_2(z)$ can have more zeros than the degree of $q(\omega, z)$, which proves the lemma.

We now return to the proof of Lemma 2. Consider $V_f : p(y,x) = 0$. If $p(y,x)$ actually contains x, then the equation $p(y,x) = 0$ defines x as an algebraic function of y, whose different branches are regular and non-intersecting outside some algebraic manifold $W_1 : q_1(y) = 0$, $q_1(y) \not\equiv 0$. In any case there is a finite (perhaps empty) set of points $\xi_1(y), \dots, \xi_r(y)$ which are all the different branches of the algebraic function $p(y,x) = 0$ and which keep apart and depend regularly on y for $y \in (C^n - W_1)$. There is an algebraic manifold $W_2 : q_2(y) = 0$, $q_2(y) \not\equiv 0$, such that all the branches of $\eta_1(y)$ and $\eta_2(y)$ are regular outside W_2, and also an algebraic manifold $W_3 : q_3(y) = 0$, $q_3(y) \not\equiv 0$, such that outside W_3 no branch of $\eta_1(y)$ or $\eta_2(y)$ intersects with any of the $\xi_i(y)$. In fact, by assumption, $(y, \eta_1(y))$, $(y, \eta_2(y)) \,\bar{\in}\, V_f$ for some y and all determinations of $\eta_1(y)$ and $\eta_2(y)$. Put $W : q(y) \equiv q_1(y) q_2(y) q_3(y) = 0$, i.e. $W = W_1 \cup W_2 \cup W_3$.

We are going to show that $g(y)$ is of class A with $V_g = W$. Let us first make a few observations and introduce some new notations.

1. For $y \in (C^n - W)$, let V_y be the subset of the complex plane consisting of the points $\xi_1(y), \dots, \xi_r(y)$ and $\eta_1(y)$ and $\eta_2(y)$ and their algebraic conjugates and let V'_y consist of $\xi_1(y), \dots, \xi_r(y)$ only. Put

$$M_1(y) = \max |x|, \; x \in V_y,$$

$$M_2(y) = \min |x_1 - x_2|, \; x_1 \in V_y, \; x_2 \in V'_y, \; x_1 \neq x_2.$$

One has

$$M_1(y) \leqslant (1 + |y|)^{k_1} / |r_1(y)| \tag{1}$$

$$1/M_2(y) \leqslant (1 + |y|)^{k_2} / |r_2(y)| \tag{2}$$

where k_1 and k_2 are real numbers and $r_1(y)$ and $r_2(y)$ are polynomials not vanishing outside W. This follows immediately from the fact that the functions to be estimated are the maxima of the absolute values of a finite number of algebraic functions having all their branches regular outside W.

2. Since $f(y,x)$ is of class A, we have, for $(x^0, y^0) \in (C^{n+1} - V_f)$, an estimate

$$|f_{0\gamma}(y,x)| \leqslant C(1 + |y| + |x|)^k / |R(y,x)| \tag{3}$$

where γ is a path of class $\mathcal{A}$ from (y^0, x^0) to (y, x), f_0 some branch of f at (y^0, x^0), and $C = C(y^0, x^0, f_0, \operatorname{rank} \gamma)$, but k and the polynomial $R(y,x)$ depend on f alone. Let $r(y)$ be the coefficient of the highest power of x in $R(y,x)$. Then

$$r(y) \neq 0 \text{ when } y \,\bar{\in}\, W.$$

In fact, put $U' : r(y) = 0$ and $U'' = W \cup U'$. Then for $y \,\bar{\in}\, U''$ we may factorize $R(y,x)$:

$$R(y,x) = r(y) \prod_{i=1}^{s} (x - x_i(y)). \tag{4}$$

We see that all the $x_i(y)$ are among the $\xi_i(y)$, for otherwise $R(y,x)$ would vanish at a point outside V_f. Let y^0 be arbitrary in $(C^n - W)$. Assume that $r(y^0) = 0$. Then we can make y approach y^0 outside U'', and since all the $\xi_i(y)$ are bounded in a neighbourhood of any point not in W, we get $R(y^0, x) \equiv 0$, which is a contradiction. Hence $r(y) \neq 0$ when $y \,\bar{\in}\, W$.

3. In this section of the proof we introduce some special paths in the complex plane from $\eta_1(y)$ to $\eta_2(y)$.

To each $\xi_i(y)$ we construct a triangle $T_i(y,\varrho)$ with corners

$$\xi_i(y) + \varrho\varepsilon_j, \quad \text{where} \quad \varepsilon_j^3 = 1, 0 < \varrho \leqslant M_2(y)/3. \tag{5}$$

It is clear that the triangles $T_i(y,\varrho)$ do not overlap and that $\eta_1(y)$ and $\eta_2(y)$ and all their algebraic conjugates stay outside them. Let $V_{y,\varrho}$ be the set of all the points (5) together with $\eta_1(y)$ and $\eta_2(y)$ and their algebraic conjugates. Let us denote the elements in $V_{y,\varrho}$ by $\zeta_1(y,\varrho), \ldots, \zeta_s(y,\varrho)$ in some order. For future use we notice that

$$|x| \leqslant (1 + |y|)^{k_3}/|r_3(y)| \quad \text{when} \quad x \in V_{y,\varrho}, \tag{6}$$

and
$$|1/R(y,x)| \leqslant (1 + |y|)^{k_4}/|r_4(y)| \quad \text{when} \quad x \bar{\in} \bigcup_{i=1}^{r} T_i(y, M_2(y)/3) \tag{7}$$

where k_3 and k_4 are real numbers and $r_3(y)$ and $r_4(y)$ polynomials, not vanishing outside W, and where k_3 and r_3 are independent of ϱ. This follows by (1), (2), (3), and (4).

It is clear that any path in $(C - V'_y)$ from $\eta_1(y)$ to $\eta_2(y)$ is homologous to a path $\pi(y,\varrho)$, consisting of a finite number of straight pieces with endpoints in $V_{y,\varrho}$ and not passing through the interior of any of the triangles $T_i(y,\varrho)$. We see that if ϱ depends continuously on $y, \varrho = \varrho(y)$, and satisfies the condition $0 < \varrho(y) < M_2(y)/3$, then $\pi(y,\varrho(y))$ (starting at a point y^0 with an arbitrary path $\pi_0 = \pi(y^0, \varrho(y^0))$ from $\eta_1(y^0)$ to $\eta_2(y^0)$) may be made to vary continuously with y. Obviously we can make the deformation so that we have to introduce a new corner in $\pi(y,\varrho(y))$ at most when some of the points $\zeta_i(y,\varrho(y))$ crosses the straight line connecting two others, i.e. at most when

$$\operatorname{Im}((\zeta_i - \zeta_k)/(\zeta_j - \zeta_k)) = 0 \tag{8}$$

for some i, j, and k, $i \neq k$. Let us say that $\operatorname{rank}_1\gamma = m$, if the path γ is of rank $(m,1)$, i.e. consists of at most m straight line segments. Similarly, we put $\operatorname{rank}_2\gamma = m$, if γ is of rank (m',m) for some m', i.e. $\gamma: x = x(t)$, $t_1 \leqslant t \leqslant t_2$, where $x(t)$ is piecewise regular algebraic of degree $\leqslant m$. We shall now prove that there is a number c such that if $\pi(y,\varrho)$ is any path in C of the type above, then for any $\varrho', 0 < \varrho' < M_2(y)/3$, there exists a path $\pi(y,\varrho')$ homologous to $\pi(y,\varrho)$ in $(C - V'_y)$ such that

$$\operatorname{rank}_1\pi(y,\varrho') \leqslant c \cdot \operatorname{rank}_1\pi(y,\varrho) \tag{9}$$

where c only depends on the number s of elements in $V_{y,\varrho}$ (the same for all (y,ϱ)). In fact, let us move the points in $V_{y,\varrho}$ simultaneously to the corresponding points in $V_{y,\varrho'}$, the motion being parametrized by, e.g., $V(y,t) = V_{y,t\varrho}$, $1 \leqslant t \leqslant \varrho'/\varrho$ (if $\varrho < \varrho'$). Correspondingly we deform the path continuously: $\pi = \pi(t) = \pi(y,\varrho(t))$. Then all the points of $V(y,t)$ are linear functions of t, and it follows that any straight line connecting two of the points of $V(y,t)$ will be crossed by each of the other points at most twice, when t goes from 1 to ϱ'/ϱ. At every crossing we shall at most have to multiply the number of straight pieces in the path by 2, and it follows that $\operatorname{rank}_1\pi(y,\varrho') \leqslant 2^{s^2(s-1)} \operatorname{rank}_1\pi(y,\varrho)$, proving (9).

4. Let $\gamma: y = y(t)$ be a path of class $\mathcal{A}$ outside W from $y^0 = y(t_0)$ to $y^1 = y(t_1)$. Consider the path

$$\pi(t,\varrho) = \pi(y(t),\varrho), 0 < \varrho < M_2(y(t))/3$$

obtained from $$\pi_0 = \pi(y(t_0), \varrho_0)$$

by continuous deformation (ϱ depending continuously on t). We shall see that

$$\operatorname{rank}_1 \pi(t,\varrho) \leqslant c \cdot \operatorname{rank}_1 \pi_0 \tag{10}$$

where c only depends on the rank of γ (and on V_f). In fact, (9) shows that it is sufficient to prove this when ϱ is constant along γ. We choose

$$\varrho = 3^{-1} \min_{y \in \gamma} M_2(y).$$

Then the functions $\zeta_i = \zeta_i(y,(t),\varrho)$ are algebraic in t with degrees that may be majorized by means of $\operatorname{rank}_2 \gamma$ only (and V_f). During the process of deformation of the path new corners will be introduced at most when (8) happens.

Hence, by Lemma 3, the number of new corners introduced in the process of deformation may be majorized by a number only depending on rank γ. Since the introduction of a new corner means that $\operatorname{rank}_1 \pi(t,\varrho)$ is multiplied by 2 at most, (10) is proved. The deformation of the path induces a continuation of $f(y,x)$, so that starting on $\pi(t_0,\varrho)$ with some branch f_0 of f and continuing f_0 with respect to (y,x) when t varies, we get a branch f_π of f on $\pi(t,\varrho)$. We see that f_π may be obtained from f_0 in two steps:

$$(y(t_0), \eta_1(y(t_0))) \to (y(t), \eta_1(y(t))) \to (y(t), x), x \in \pi(t,\varrho),$$

the first along $(y,x) = (y(u), \eta_1(y(u))), t_0 \leqslant u \leqslant t$, and the second along $\pi(t,\varrho)$ with y constant $= y(t)$. These two paths together make a path of class $\mathcal{A}$. In virtue of (10) it has a rank, only depending on the rank of γ and the rank of π_0 (and on V_f). We are now in a position to estimate $g_{0\gamma}(y), g_0$ being some branch at y^0 of the function g of Lemma 2 (hence determined by the choice of a special path π_0 in C from $\eta_1(y^0)$ to $\eta_2(y^0)$, not meeting V'_{y^0}, and a branch f_0 of f on π_0). It is clear that g_0 may be continued analytically along any path in $(C^n - W)$ since $f(y,x)$ is analytic outside V_f and since outside W the endpoints $\eta_1(y)$ and $\eta_2(y)$ are regular analytic functions of y. The function obtained from g_0 by continuation along γ (as above) is given by

$$g_{0\gamma}(y(t)) = \int_{\pi(t,\varrho)} f_\pi(y(t), x)\, dx.$$

For an arbitrary t taking $\varrho = M_2(y(t))/3$, we get by (3), (6), and (7)

$$|f_\pi(y(t),x)| \leqslant c(1 + |y(t)|)^{k_5} / |r_5(y(t))|, \ (x \in \pi(t,\varrho)) \tag{11}$$

where $r_5(y) \neq 0$ outside W, k_5 and r_5 only depend on the given function f, while $c = c(f_0, \pi_0, \operatorname{rank}\gamma)$. Moreover, (9) and (10) show that

$$\operatorname{rank}_1 \pi(t,\varrho) \leqslant c(y^0, g_0, \operatorname{rank}\gamma) \tag{12}$$

and that the maximal length of a straight piece in $\pi(t,\varrho)$ is majorized by $2(1 + |y(t)|)^{k_3} / |r_3(y(t))|$. Also using (11) and (12) it follows that

$$|g_{0\gamma}(y(t))| \leqslant c(1 + |y(t)|)^{k_6} / |r_6(y(t))|$$

where $r_6(y) \neq 0$ outside W, and where r_6 and k_6 only depend on the integrated function f, while $c = c(y^0, g_0, \operatorname{rank}\gamma)$. This proves that the function $g(y)$ of Lemma 2 satisfies the condition (c).

Finally, let us verify that $g(y)$ has the property (b). To do this we observe that $g(y)$ is a sum of functions

$$\int_{L_{ij}(y)} f(y,x)\,dx \tag{13}$$

where $L_{ij}(y)$ is a straight line segment between two points in $V_{y,\varrho}$. When y varies little, we can keep all the pieces $L_{ij}(y)$ (not introducing any new corners). Since the integrals (13) are finite in number and the possible determinations of f in them span a linear space of finite dimension, all linear combinations of them form a linear space of finite dimension. Hence all determinations of g span a finite dimensional linear space. This completes our proof of Lemma 2.

We shall now consider a special case of Lemma 2.

Lemma 4. If in Lemma 2 the function $f(y,x)$ is algebraic, and y is a single complex variable, then there exists a natural number m such that, in a neighbourhood of infinity, one has

$$T^m g(y) = g(y) + h(y), \; T^m h(y) = h(y).$$

Here T denotes analytical continuation one round in the positive sense along circles $|y| = \text{constant}$.

Proof. We shall perform the deformation of the path of integration in a special way, and we shall keep the notations of Lemma 2. All the points $\xi_i(y)$ are algebraic functions of y. Hence, in a neighbourhood of infinity, they may be expanded into Puiseux series of the type

$$\sum_{j=-\infty}^{j_0} a_j (y^R)^j$$

where R is a positive rational number. Let us divide the points $\xi_i(y)$ into rings, so that in the same ring we take points whose Puiseux expansions have the same exponent of the leading term and the same modulus of the leading coefficient. Consider such a ring, corresponding to the exponent α and modulus K of the coefficient. Then all the points of this ring will evidently keep in the annulus

$$(K-\varepsilon)|y|^\alpha \leqslant |x| \leqslant (K+\varepsilon)|y|^\alpha$$

for some $\varepsilon > 0$ and for y in a neighbourhood of infinity. We see that we may choose these annuli corresponding to the different rings so that for y in a neighbourhood of ∞ they are all disjoint. Now we divide each ring into 'groups', each group consisting of points $\xi_i(y)$ having not only the same modulus of the leading term but also the same value. These groups we divide into rings of second order, now according to the second coefficient in the Puiseux expansion but otherwise in the same way as for the rings of first order (above called rings only). The points in a ring of second order will stay in an annulus, the radii depending only on $|y|$, as y varies in a neighbourhood of ∞, just as points of the rings of first order, but now the centre of the annuli of second order corresponding to one and the same group is

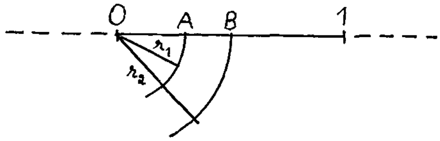

Fig. 1.

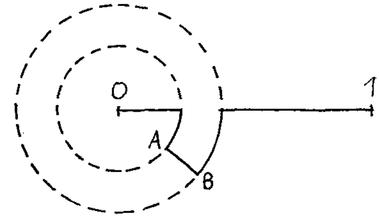

Fig. 2.

not zero but the point given by the (common) leading term of the Puiseux expansions. It is clear that all the annuli of second order of a "group" will stay in the corresponding annulus of first order, when y lies in a neighbourhood of infinity. Hence, if $|y|$ is sufficiently large, no point not belonging to a ring of second order will belong to the corresponding annulus of second order. We continue in the same way and divide the rings of second order into groups of second order, and these, according to the third coefficient in the Puiseux expansion, into rings of third order, and so on. In each step we get annuli analogous to those of rings of first and second order. Since there is only a finite number of points $\xi_i(y)$, this procedure is finished within a finite number of steps, and then any two of the points $\xi_i(y)$ have fallen into different rings or groups of some order.

As in Lemma 2, in a neighbourhood of a point $y_0 \bar{\in} W$, every branch of the function $g(y)$ is equal to the integral of some branch of $f(y,x)$ along some path in the complex plane from $\eta_1(y)$ to $\eta_2(y)$, not passing through any of the points $\xi_i(y)$. Let us first assume that $\eta_1(y)$ and $\eta_2(y)$ are constant, equal to 0 and 1, respectively, say. Further assume that y_0 is real and positive and that for y real $\geqslant y_0$, none of the points $\xi_i(y)$ will cross the real axis. Moreover, we suppose that the path of integration which we start with at y_0 is the straight piece $[0,1]$. Then, continuing $g(y)$ out from y_0 along the real axis in the positive sense, we need not deform the path of integration. We choose y_0 so large that for $|y| > y_0/2$ the points $\xi_i(y)$ behave as described above; we know that this is the case in some neighbourhood of ∞.

Now let us continue $g(y)$ out from y_0 along the circle $|y| = y_0$ in the positive sense. We shall investigate the deformation of the path of integration under this continuation. Let us first consider the deformation caused by the points $\xi_i(y)$ in one particular ring R of first order; assume that the radii of the corresponding annulus are $r_1(=r_1(|y|))$ and $r_2(=r_2(|y|))$, (see Fig. 1). If the straight piece AB does not divide any of the groups of R, we may, if necessary, deform the path as in Fig. 2, keeping AB between two groups, and in this case we have then fully treated the deformation necessary to avoid the points of R. If, however, AB does divide some group G, this is not enough. In this case we let AB follow the centre of G, i.e. the point corresponding to the common first term in the Puiseux expansions. We then move AB as in the former case, now letting AB pass through the centre of G. During this process AB will be crossed by the points of G but by no other points $\xi_i(y)$, and no point in G will cross the path elsewhere. So we shall have to deform AB with respect to the points of G. We have now, however, the same situation for this deformation as when we started, the relevant rings now being the rings of second order of points in G (the path of integration now lies on both sides of the centre, but that does not cause any essential complication). In this way we go on till after

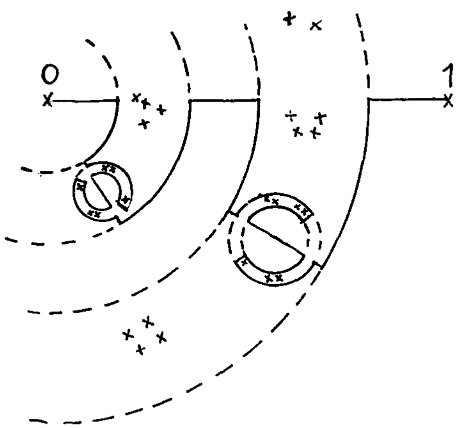

Fig. 3.

a finite number of steps the deformation is complete so that no point $\xi_i(y)$ crosses the path of integration during the process of deformation. For in any step the deformations with respect to the different rings are evidently independent of each other, because of the choice of rings. So we may perform them all simultaneously, and they will not affect each other (Fig. 3).

Let us by Σ denote the set of the straight pieces in all the deformations corresponding to AB in the special one considered. When we start with y at y_0, all the pieces in Σ lie on the positive real axis. For every piece σ in Σ there is a natural number N_σ such that when y has run through the circle $|y| = y_0$ N_σ times, then σ has returned to its original position. Then, when y has made $N' = \prod_{\sigma\in\Sigma} N_\sigma$ rounds, all the pieces of Σ lie on their original places on the positive real axis. This is also the case after any $\mu N'$ rounds (μ integer). Now let us consider the function elements of $f(y,x)$ that we have on the pieces of $\sum$. Since there is only a finite number of branches of f, it follow s that for every $\sigma\in\Sigma$ there is a positive integer μ_σ such that when y has made $\mu^\sigma N'$ rounds we have the same function element on σ as when we started. Then, after $N = \prod_{\sigma\in\Sigma}(\mu_\sigma N')$ rounds, we have on all the pieces of Σ the same function element of $f(y,x)$ as when we started. Then we see that

$$T^N g(y) = g(y) + h(y)$$

where $h(y)$ is a sum of integrals of $f(y,x)$ along circles. From the nature of the procedure we easily see that $T^{2N}g(y) = g(y) + 2h(y)$, and hence $T^N h(y) = h(y)$.

The lemma is then proved in the special case where $\eta_1(y)$ and $\eta_2(y)$ are constant, equal to $0,1$ respectively, and the real axis is not crossed by any of the $\xi_i(y)$ when y is large and positive, and the path of integration is the straight piece $[0,1]$. It is, however, not difficult to bring the general case back to this special case, For, there is a positive number c such that for $y \geqslant c$ none of the points $\xi_i(y), \eta_j(y)$ crosses (in the strict sense) the straight line between two others. Then we may introduce a finite number of points $\theta_1(y), \ldots, \theta_t(y)$, which are algebraic functions of y, such that the sum of the straight line segments $S_i(y) = [\theta_i(y), \theta_{i+1}(y)]$ is homologous (in $(C - \{\xi_i(y) \mid i = 1, \ldots, r\})$) to our path of integration and that none of the straight lines through the S_i is crossed by any of the $\xi_i(y)$ for $y>c$. By a linear mapping L_y ($L_y(z)=\alpha z+\beta$; α,β depending on y) we may then map S_i on $[0,1]$. Then the points $L_y(\xi_i(y))$ are algebraic functions of y and do not cross the segment $[0,1]$ for $y>c$. Then we may perform the deformation as in the special case considered, and applying L_y^{-1} to it we get a deformation of S_i with the properties used in the special case (since clearly $T^v L_y = L_y$ for some natural number v). Hence, arguing as above, we find that each function $\int_{S_i(y)} f(y,x)\,dx$ satisfies the property of the lemma, and this is then easily seen to be true also of $g(y)=\sum_{i=1}^t \int_{S_i} f(y,x)\,dx$. The lemma is proved.

We shall also need the following lemma, obtained by repeated use of lemma 2.

Lemma 5. Let E_z be the Euclidean simplex in R^p:

$$x_1 \geqslant z_1, \ldots, x_p \geqslant z_p, x_1 + \ldots + x_p \leqslant 1 - z_0$$

where $z_0, z_1, \ldots, z_p$ are small. Let $f(y,x)$ be a function of $y \in C^m$ and $x \in C^p$, which is analytic in a neighbourhood of (y^0, E_0) so that the function

$$g_z(y) = \int_{E_z} f(y,x)\,dx$$

is defined and regular analytic in (y,z) at $y=y^0, z=0$. Moreover, suppose that $f(y,x)$ is of class A. Then we may find a z^0, arbitrarily small, such that $g_{z^0}(y)$ is of class A in y.

Proof. Let us prove that $g_z(y)$ is of class A in (y,z) at $(y^0,0)$, from which the lemma immediately follows. We have that $g_z(y)$ is obtained from $f(y,x)$ by repeated integrations with respect to a single real variable:

$$f_k(y, z_0, \ldots, z_k, x_{k+1}, \ldots, x_p) = \int_{z_k}^{z_k+w_k} f_{k-1}(y, z_0, \ldots, z_{k-1}, x_k, \ldots, x_p) \cdot dx_k$$

where $f_0 = f$ and $f_p = g$, and where

$$w_k = 1 - z_0 - z_1 - \ldots - z_k - x_{k+1} - \ldots - x_p \ (0 \leqslant k < p)$$

and

$$w_p = 1 - z_0 - z_1 - \ldots - z_p.$$

Assume now that f_{k-1} is of class A with

$$V_{f_{k-1}}: q_{k-1}(y, z_0, \ldots, z_{k-1}, x_k, \ldots, x_p) = 0$$

and that q_{k-1} is not divisible by w_{k-1}. Put

$$q_k(y, z_0, \ldots, z_k, x_{k+1}, \ldots, x_p) = p_{k-1}(y, z_0, \ldots, z_{k-1}, x_{k+1}, \ldots, x_p) \cdot q_{k-1}(y, z_0, \ldots, z_k, x_{k+1}, \ldots, x_p) \cdot$$
$$q_{k-1}(y, z_0, \ldots, z_{k-1}, z_k + w_k, x_{k+1}, \ldots, x_p) \tag{14}$$

where p_{k-1} is the product of the leading coefficient and the discriminant of q'_{k-1} with respect to x_k, and q'_{k-1} is obtained from q'_{k-1} by taking away multiple factors (considering q_{k-1} as a polynomial in x_k). Then q_k is not divisible by w_k. In fact, the first factor of (14) cannot be divisible by w_k, since it does not contain z_k. If the product of the two other factors in (14) were divisible by w_k, it would follow that

$$w_k = 0 \Rightarrow q_{k-1}(y, z_0, \ldots, z_k, x_{k+1}, \ldots, x_p) = 0, \tag{15}$$

i.e. changing z_k to x_k

$$w_{k-1} = 0 \Rightarrow q_{k-1}(y, z_0, \ldots, z_{k-1}, x_k, \ldots, x_p) = 0.$$

But this is a contradiction, since we have supposed that q_{k-1} is not divisible by w_{k-1}. In particular, $q_k \not\equiv 0$, and, since the choice of q_k evidently corresponds to the description of V_{f_k} in Lemma 2, it follows from Lemma 2 that f_k is of class A with V_{f_k}: $q_k(y, z_0, \ldots, z_k, x_{k+1}, \ldots, x_p) = 0$. Now, f_0 is of class A and since q_0 does not depend on z_0, it follows that q_0 is not divisible by w_0. Hence we get by induction that $g_z(y) = f_p(y, z_0, \ldots, z_p)$ is of class A in (y, z), proving the lemma.

3. Proof of the main theorem

Let us recall the situation of Theorem 1. We were given an algebraic manifold in C^n:

$$V(y): p_1(y,x) = 0, \ldots, p_{n-r}(y,x) = 0$$

depending on $y \in C^m$. By $R(y)$ we denoted the subset of $V(y)$ of points x such that $\operatorname{grad}_x p_1(y,x), \ldots, \operatorname{grad}_x p_{n-r}(y,x)$ are linearly independent, $R(y)$ hence becoming an analytic manifold in C^n of dimension r. We had also, in the neighbourhood of some $y^0 \in C^m$, a p-cycle $\gamma(y)$ on $R(y)$ varying continuously with y. Further $\omega_y(x)$ was a closed holomorphic differential p-form on $R(y)$, defined for (y,x) in a neighbourhood of $(y^0, \gamma(y^0))$ and of class A at any $(y^0, x^0), x^0 \in \gamma(y^0)$. Then the function g was defined: $g(y) = \int_{\gamma(y)} \omega_y(x)$. We have to prove that it is analytic in a neighbourhood of y^0 and of class A. First, for y in a neighbourhood of y^0, we shall define a mapping $T(y, \cdot)$: $R(y^0) \to R(y)$. For this purpose, let us at every $x^0 \in R(y^0)$ consider the manifold

$$N(x^0): x = x^0 + t_1 \overline{\operatorname{grad}_x p_1(y^0, x^0)} + \ldots + t_{n-r} \overline{\operatorname{grad}_x p_{n-r}(y^0, x^0)}$$

where the t_i are complex parameters. Then, for y in a neighbourhood of $y^0, N(x^0)$ intersects $R(y)$ in exactly one near-lying point $\xi = \xi(y, x^0)$. Putting $\xi = T(y, x^0)$, then given any compact subset K of $R(y^0)$, in a sufficiently small neighbourhood of y^0 $T(y, \cdot)$ is a bijective and bicontinuous mapping from K to $T(y, K)$. Further, for any $x^0 \in K$, $T(y, x^0)$ is a regular analytic function of y when y lies in a neighbourhood of y^0, which may be taken the same for all $x^0 \in K$. These statements follow from general theorems on implicit functions and the fact that the gradients $\operatorname{grad}_x p_i(y^0, x^0)$ are linearly independent for every $x^0 \in R(y^0)$. Since our differential form $\omega_y(x)$ is closed and $T(y, x^0)$ depends continuously on (y, x^0), we have

$$g(y) = \int_{\gamma(y)} \omega_y(x) = \int_{T(y, \gamma_0)} \omega_y(x)$$

where $\gamma_0 = \gamma(y^0)$. By the mapping $T(y, \cdot)$ we may then write $g(y)$ in the form

$$g(y)=\int_{\gamma_0}\tilde{\omega}_y(x)$$

where $\tilde{\omega}_y(x)$ is a closed differential form on $R(y^0)$ which is regular analytic in y in one and the same neighbourhood of y^0 for all $x\in\gamma_0$. It follows that $g(y)$ is regular analytic in a neighbourhood of y^0.

In the sequel we shall regard $V(y)$ as a $2r$-dimensional real algebraic manifold in R^{2n}, then denoting it by $\check{V}(y)$ and similarly writing $\check{R}(y)$ and $\check{\gamma}(y)$. Instead of $T(y,x)$ we write $\check{T}(y,\xi)$, where we now consider $\check{T}(y,\xi)$ as an element in R^{2n} and where $\xi=(\mathrm{Re}(x),\mathrm{Im}(x))$. We suppose that $y^0=0$, which is no essential restriction. When y is real, $\check{V}(y)$ is then given by $(2n-2r)$ real equations

$$q_1(y,\xi)=0,\dots,q_{(2n-2r)}(y,\xi)=0.$$

It is clear that $\check{T}(y,\xi)$ depends algebraically on (y,ξ). Let $x^0(\leftrightarrow\xi^0)$ be an arbitrary point on γ_0 and choose a set $x'=(x_{j_1},\dots,x_{j_r})$ of local coordinates on $R(y)$ at (y^0,x^0) (as in (L)). If we put $\xi'=(\mathrm{Re}(x'),\mathrm{Im}(x'))=(\xi_{l_1},\dots,\xi_{l_{2r}})$ we have then, in a real neighbourhood of (y^0,ξ^0):

$$\omega_y(x)=\sum_{k_1<\dots<k_p}f_{k_1\dots k_p}(y,\xi')\,d\xi_{k_1}\wedge\dots\wedge d\xi_{k_p},$$

where $k_1,\dots,k_p\in\{l_1,\dots,l_{2r}\}$, and analogously for $\tilde{\omega}_y(x)$ with coefficients $\tilde{f}_{k_1\dots k_p}$. From our assumption that $\omega_y(x)$ is of class A at every $(y^0,x^0),x^0\in\gamma_0$, it follows that the functions $f_{k_1\dots k_p}(y,\xi')$ are of class A in (y,ξ') at (y^0,ξ'^0). Every $\tilde{f}_{k_1\dots k_p}(y,\xi')$ is a sum of terms of the form $h(y,\xi')f(y,\check{T}(y,\xi)')$, where h is algebraic, f is a coefficient of $\omega_y(x)$, and ξ is the point on $\check{R}(y^0)$ with coordinates ξ' in the local system, and, conversely, $\check{T}(y,\xi)'$ stands for the local coordinates of $\check{T}(y,\xi)\in\check{R}(y)$.

It is clear that the range of the mapping $(y,\xi')\rightarrow(y,\check{T}(y,\xi)')$ contains a full neighbourhood of $(y^0,\xi^0_{l_1},\dots,\xi^0_{l_{2r}})$, and so it cannot be contained in V_f, and it follows by Lemma 1 that $\tilde{f}_{k_1\dots k_p}$ is of class A. Hence we have reduced the proof to a case where only the differential form but not the manifold depends on y. Now it is possible to find a finite set $\{S_i\}_{i=1}^{s}$ of Euclidean simplexes (e.g. with their vertices on $\check{R}(y^0)$) such that projection P along the manifolds $N(x)$ is a bijective regular algebraic mapping from $\sum\check{S}_i$ to $\sum\check{P}(S_i)$ and such that $\sum P(S_i)$ is homologous to γ_0 on $R(y^0)$ and lies so close to γ_0 that $g(y)=\int_{\gamma_0}\tilde{\omega}_y(x)=\sum\int_{P(S_i)}\tilde{\omega}_y(x)$ for y in a neighbourhood of y^0 and that $\tilde{\omega}_y(x)$ is of class A at every (y^0,x^0), $x^0\in\sum P(S_i)$. Moreover, we may suppose that these properties remain valid when the vertices of the S_i (each of which is common to at least two of the S_i) vary little around their original positions. Denote by ζ all the coordinates of all the vertices of the simplexes S_i, taken as points in R^{2n} and let ζ^0 correspond to the original position. We may map each of the simplexes S_i onto the unit simplex E_0 in R^p by an affine mapping (depending algebraically on ζ), and this mapping is uniquely determined if we prescribe the order of the vertices. By the projection P and this affine mapping we may write $g(y)$ as a sum:

$$g(y)=\int_{\gamma(y)}\omega_y(x)=\int_{\gamma_0}\tilde{\omega}_y(x)=\sum\int_{P(S_i)}\tilde{\omega}_y(x)=\sum\int_{E_0}f_i(y,\zeta,v)\,dv.$$

Here the functions $f_i(y, \zeta, v)$ are sums of terms $h(\zeta, v)f(y, \xi'(\zeta, v))$, where h is algebraic, ξ' local coordinates on $R(y^0)$, f a coefficient of $\tilde{\omega}_y(x)$ in these coordinates and

$$(\zeta, v) \to \xi'(\zeta, v)$$

an algebraic mapping such that its range contains a full neighbourhood of a point in R^{2r}. Hence it follows by Lemma 1 that all the functions f_i are of class A. Then, by Lemma 5 there is a Euclidean simplex E_{z^0} (with z^0 small) such that

$$h_1'(y, \zeta) = \int_{E_{z^0}} f^1(y, \zeta, v)\, dv$$

is of class A. But by the construction of our mappings we see that $h_1(y, \zeta) = h_1'(y, U(\zeta))$, where we have put $h_i(y, \zeta) = \int_{E_0} f_i(y, \zeta, v)\, dv$, and where U is an algebraic mapping whose range contains a full neighbourhood of ζ^0, and so by Lemma 1 it follows that $h_1(y, \zeta)$ is of class A. The same statement is true for all the other functions $h_i(y, \zeta)$, and so $g(y) = \sum h_i(y, \zeta)$ is of class A in (y, ζ) and hence also in y. This proves Theorem 1.

Theorem 2 also follows since by lemma 4 all the functions $h_i(\cdot, \zeta)$ have the property of Theorem 2 for any ζ.

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