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Analytic properties of expansions, and some variants of Parseval-Plancherel formulas

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We shall be concerned with the following question. Consider an expansion in Legendre (or other) functions given by

$$F(z) = \sum a(n) P_n(z). \tag{0.1}$$

Here, as indicated, the coefficients arise from a function $a(s)$ which is in fact assumed to be analytic in an appropriate region of the s plane. The problem that occurs is of characterizing the analytic properties of F in terms of those of a. More particularly: How do we characterize those F which arise when a is an entire function of suitable kind, or when a may have poles, etc.? One may also ask if there is an identity of the Parseval–Plancherel type which relates a quadratic class of coefficient functions analytic in a right-half plane, with a quadratic class of F's so that the corresponding mapping is unitary.

These questions have some intrinsic merit, but their answers have additional interest for the following two reasons:

(1) Series of type (0.1), where $a(s)$ is meromorphic have recently attracted considerable attention in some problems of physics, see e.g. Regge [10], and Khuri [7]. The classical setting of this problem in connection with physics goes back to Poincaré [9]; see also Watson [14] and Sommerfeld [11], p. 282.

(2) In studying such series one may put on a solid analytic footing and make quite precise the following idea hitherto expressed only heuristically: that the completeness of the expansion of spherical functions for $SO(3)$ (the completeness of the Legendre polynomials on $(-1, 1)$) should lead by analytic continuation to the completeness of the corresponding expansion on $SL(2, R)$.

Let us describe these things in more detail. It can be expected that there should be an analogy between the behavior of the series (0, 1), and the corresponding power series.

$$\mathcal{J}(z) = \sum_{n=0}^{\infty} a(n) z^n. \tag{0.2}$$

In the case of power series it is well-known that under appropriate assumption on the function $a(s)$ a variety of conclusions about $\mathcal{J}(z)$ follow; see e.g. Lindelöf [8], Bieberbach [3]. The simplest of these, which unlike most is both necessary and sufficient is the well-known theorem of Carlson and Wigert.

Theorem A. *A necessary and sufficient condition that, up to an additive constant, $\mathcal{F}(z)$ be representable in the form (0.2) where $a(s)$ is entire and of zero exponential type is that $\mathcal{F}(z)$ be an entire function of $1/1-z$.*

There is also a modification of this result when finitely many poles are allowed to occur, i.e. when $a(s)$ is replaced by $a(s)+R(s)$, with R rational. The modification is, however, rather drastic because the resulting $\mathcal{F}(z)$ must then necessarily be multivalued. For simplicity let us assume that the poles of $R(s)$ are distinct and non-integral.

Theorem A′. *The class of $\mathcal{F}$'s which arise as (0.2) from such a's coincides with the class $\mathcal{F}$'s which may be continued into the complex plane slit along the positive real axis from 1 to ∞, and so that in this slit plane, when $|z|>1$,*

$$\mathcal{F}(z)=\varphi(z)+\sum c_n z^{\alpha_n}, \tag{0.3}$$

where $\varphi(z)$ is analytic and single-valued in $|z|>1$ (including ∞); α_n are the poles of $a(s)$.

While this theorem does not seem to be stated in the literature, it is in fact an easy consequence of the reasoning given in Lindelöf [8], section 61, combined with Theorem A.

Our first results, combined in section 2, are the analogues of Theorems A and A′ for Legendre series (0.1). The analogue of Theorem A is that in effect $F(z)$ is representable as (0.1) with $a(s)$ of zero exponential type if and only if $\sqrt{1-z}\,F(z)$ is an entire function of $1/1-z$. The matter is put in better perspective by considering, at this stage, the more general ultra-spherical expansions, which contain as the special case $\lambda=\frac{1}{2}$ the Legendre polynomials. An analogue of Theorem A′ is also found (see Theorem 2). Here the expression (0.3) is replaced by a similar one with the Legendre functions of the second kind $Q_{-1-\alpha_n}(-z)$ instead of z^{α_n}.

We next consider the case where the coefficient function is analytic in a right half-plane. It is natural, in this case, to look for an identity between a Hilbert space of such analytic functions and a corresponding Hilbert space of F's analytic in the complex plane, slit along the positive axis from 1 to ∞. The result obtained here is new even in the case of power series although its proof is quite simple. More particularly, the Hilbert space of coefficient functions will be $H^2(Rs>-\frac{1}{2})$, those a which are analytic when $R(s)>-\frac{1}{2}$, with norm $\|a\|=\sup_{\sigma>-\frac{1}{2}}(\int_{-\infty}^{\infty}|a(\sigma+it)|^2dt)^{\frac{1}{2}}$. The class of $\mathcal{F}$'s will be denoted by $H^2(S)$, ($S=$ the slit plane); this class will consist of the $\mathcal{F}$ analytic in S, and which are simultaneously in H^2 in the upper and lower half-planes. Equivalently, it is those $\mathcal{F}$ analytic in S, with $\lim_{y\to\infty}|\mathcal{F}(x+iy)|\to 0$, and where the jump $\lim_{y\to 0,\, y>0}$ $\mathcal{F}(x+iy)-\mathcal{F}(x-iy)$ exists in L^2 norm. Our result (Theorem 3, section 2) is then that $\mathcal{F}\in H^2(S)$ if and only if $a(s)\in H^2(Rs>-\frac{1}{2})$, with an identity of corresponding norms. Now the usual Parseval relation for power series identifies a quadratic norm of the coefficients $((\sum|a(n)|^2)^{\frac{1}{2}})$ with a quadratic norm of $\mathcal{F}$, taken on the unit circle. Theorem 3 might be considered as a variant of that, identifying another quadratic norm of the a's, with a quadratic norm of the values of $\mathcal{F}$ on $(1,\infty)$.

To describe the analogue of this for Legendre series it is convenient to change the definition of $a(s)$ by setting instead of (0.1).

$$F(z)=\sum(2n+1)\,a(n)\,P_n(z). \tag{0.4}$$

The space of a is modified to consist of those a which belong to $H^2(Rs>-\frac{1}{2})$ and for which the norm

$$\|a\|_*=\left(\int_{-\infty}^{\infty}|a(-\tfrac{1}{2}+it)-a(-\tfrac{1}{2}-it)|^2\,\frac{t}{\tanh \pi t}\,dt\right)^{\frac{1}{2}}$$

is finite. This class will be referred to as $H^2_*(Rs>-\frac{1}{2})$. We prove then that F is in $H^2(S)$ if and only if it can be represented by (0.4) with $a\in H^2_*(Rs>-\frac{1}{2})$, and with an identity of norms.

The proof requires the Legendre analogue of the Mellin transform, which takes the form $\int_1^\infty Q_s(x)\,\varphi(x)\,dx$; by considering it, together with the class $H^2(S)$, one can pass by analytic continuation from the completeness of spherical functions an $SO(3)$ to the corresponding completeness for $SL(2, R)$.

Section 1. The case when $a(s)$ is meromorphic in the entire plane

We begin by making some remarks to clarify the results given below.

Let us recall that an entire function $a(s)$ is said to be of zero exponential type if $a(s)=O(e^{\varepsilon|s|})$, as $|s|\to\infty$, for every $\varepsilon>0$. Such functions have a useful representation: a is in this class if and only if there exists a $\varphi(w)$, which is an entire function of $1/w$, so that

$$a(s)=\frac{1}{2\pi i}\int_C e^{sw}\,\varphi(w)\,dw.$$

and with C an arbitrarily small circle centered at the origin. It should also be recalled that such functions are determined by their values at the positive integers; see [4], Chapters 5 and 9. We should next point out that the series (0.1) may in general never converge. It will, therefore, be necessary to interpret the sum as $\lim_{r\to 1}\sum a(n)\,r^n P_n(z)$. But in making this interpretation we must keep in mind that this limit may be identically zero without a being so. A simple exemple of this arises for the "δ-function" $\sum(2n+1)\,P_n(x)$. Finally, in proving our theorem it will be instructive to consider the more general class of ultra-spherical polynomials $\{P_n^\lambda(x)\}$ given by the generating relation $(1-2xr+r^2)^{-\lambda}=\sum_{n=0}^\infty r^n P_n^\lambda(x)$, $(\lambda>0)$. When $\lambda=\frac{1}{2}$, we have the Legendre case, $P_n^{\frac{1}{2}}(x)=P_n(x)$; while $P_n^1(x)=(\sin(n+1)\,\theta)/\sin\theta$, $x=\cos\theta$. The consine polynomials, $\cos n\theta$, may be obtained as a limiting case, $\lambda=0$. See [12, Chapter 4]. Of course, the spherical functions for $SO(n)$ arise in the case $\lambda=n-2/2$.

Theorem 1. *Let $\lambda>0$. Suppose $a(s)$ is an entire function of zero exponential type. If* $-1\leqslant x<1$

$$F(x)=\lim_{r\to 1}\sum_{n=0}^{\infty}a(n+\lambda)\,r^n\,P_n^\lambda(x)$$

exists, and $(1-x)^{\lambda}F(x)$ has an extension into the complex plane as an entire function of $1/1-z$. Conversely, every entire function of $1/1-z$ arises this way. Finally F vanishes identically if and only if $a(s)$ is an odd function of s.

The proof is elementary.

We begin with the generating relation $(1-2xr+r^2)^{-\lambda}=\sum_{n=0}^{\infty} r^n P_n^{\lambda}(x)$, which obviously extends to complex w of absolute value less than one, i.e.

$$(1-2wx+w^2)^{-\lambda}=\sum_{n=0}^{\infty} w^n P_n^{\lambda}(x), \quad -1\leqslant x\leqslant 1, \ |w|<1. \tag{1.1}$$

The series on the right converges uniformly for w in closed subsets lying in the interior of the unit disc, because the $P_n^{\lambda}(x)$ are of at most polynomial growth in n. Notice also that the expression $1-2xw+w^2$ is not zero for $-1\leqslant x\leqslant 1$, $|w|<1$. Next, since $a(z)$ is an entire function of zero exponential type, we know that it can be represented in the form

$$a(s)=\frac{1}{2\pi i}\int_C e^{sw}\varphi(w)\,dw,$$

where $\varphi(1/w)$ is intire in w, and C is any circle about the origin. From this we get that

$$F_{\varepsilon}(x)=\sum_{n=0}^{\infty} a(n+\lambda)\,e^{-\varepsilon n}P_n^{\lambda}(x)=\frac{1}{2\pi i}\int_C e^{\lambda w}(1-2\,e^{w-\varepsilon}x+e^{2w-2\varepsilon})^{-\lambda}\varphi(w)\,dw.$$

Here, and from this point on, C will be a circle of sufficiently small radius about the origin, and x will be assumed to lie away from 1. From the above we get immediately

$$e^{-\varepsilon\lambda}(1-x)^{\lambda}F_{\varepsilon}(x)=\frac{2^{-\lambda}}{2\pi i}\int_C\left(\frac{1-\cosh(w-\varepsilon)}{x-1}-1\right)^{-\lambda}\varphi(w)\,dw.$$

However $(t-1)^{-\lambda}=\sum_{n=0}^{\infty}B_n^{\lambda}t^n$, where $B_n^{\lambda}\cong n^{\lambda}$ and the series converges uniformly inside the unit disc. Thus by taking ε sufficiently small, and C sufficiently near the origin we get

$$e^{-\varepsilon\lambda}(1-x)^{\lambda}F_{\varepsilon}(x)=2^{-\lambda}\sum_{n=0}^{\infty}B_n^{\lambda}(1-x)^{-n}C(n,\varepsilon),$$

where
$$C(n,\varepsilon)=\frac{1}{2\pi i}\int_C(\cosh(w-\varepsilon)-1)^n\varphi(w)\,dw.$$

Thus
$$|C(n,\varepsilon)|\leqslant A_{\varepsilon}(\varepsilon')^{2n}, \text{ for each } \varepsilon'>\varepsilon.$$

Passing to the limit $\varepsilon\to 0$, for $1\leqslant x<1$, we get

$$(1-x)^{\lambda}F(x)=2^{-\lambda}\sum B_n^{\lambda}(1-x)^{-n}C(n)$$

with $C(n)=C(n,0)$.

What we have said about the growth of the binomial coefficients B_n^λ and the constants $C(n,\varepsilon)$, makes it clear that $(1-x)^\lambda F(x)$ has an extension to an entire function of $1/1-z$ given by

$$\cdots (1-z)^\lambda F(z) = 2^{-\lambda} \sum_{n=0}^{\infty} B_n^\lambda (1-z)^{-n} C(n). \tag{1.2}$$

Now suppose that $a(s)$ is an odd function, $a(s)=\sum d_k s^{2k+1}$. Then

$$\varphi(w) = \sum_{k=0}^{\infty} (d_k(2k+1)!\, w^{-2k} \quad \text{is even, and} \quad C(n) = \frac{1}{2\pi i}\int_C \varphi(w)(\cosh w-1)^n\, dw = 0,$$

since the integration involves only even (positive and negative) powers of w. Thus $(1-z)^\lambda F(z)\equiv 0$. Conversely suppose $(1-z)^\lambda F(z)\equiv 0$. Then since $B_n^\lambda \neq 0$ for every n, $\int_C \varphi(w)(\cosh w-1)^n dw = 0$ all n,

For each integral k, $\cosh kw$ is a polynomial in $\cosh w$, and hence is a polynomial in $\cosh w-1$. Therefore because for all n, $\int_C \varphi(w)(\cosh w-1)^m dw=0$, it follows that $\int_C \varphi(w)\cosh kw\, dw = 0$. Hence $(a(k)+a(-k))/2=0$. But $a(s)+a(-s)$ is of zero exponential type; its vanishing at the integers implies its vanishing identically. So $a(s)$ is odd.

What remains to be shown is that every entire function of $1/1-z$ can be obtained this way (that is a (1.2)). Since $B_n^\lambda \sim n^\lambda$, it suffices to see that when ever C_n are the Taylor coefficients of an entire function, there exist a $\varphi(w)$, which is an entire function of $1/w$ so that

$$C_n = \frac{1}{2\pi i}\int_C \varphi(w)(\cosh w-1)^n\, dw.$$

Now make the change of variables $(\cosh w-1)^{\frac{1}{2}} = z$, which is a regular mapping of a neighborhood of $w=0$ to a neighborhood of $z=0$. Then we get

$$C_n = \frac{1}{2\pi i}\int_{C'} \psi(z) z^{2n-1}\, dz, \quad \text{where} \quad \psi(z) = 2\,\frac{(\cosh w-1)}{\sinh w}\,\varphi(w).$$

Now there certainly exists a $\psi(z)$ analytic of $z\neq 0$, so that $C_n = 1/2\pi i \int \psi(z) z^{2n-1} dz$. (Take $\psi(z)=\sum_{n=0}^{\infty} C_n z^{-2n}$.) Let

$$\varphi_0(w) = \frac{1}{2}\left(\frac{\sinh w}{\cosh w-1}\right)\psi(z);$$

then $\varphi_0(w)$ is analytic in small disc punctured at $w=0$, and $C_n = 1/2\pi i \int_c \varphi_0(w)$ $(\cosh w-1)^n dw$. Finally let $\varphi(w)=\varphi_0(w)-\varphi_1(w)$, where $\varphi_1(w)$ is that part of the Laurent development of $\varphi_0(w)$ which involves positive powers of w. Then

$$\frac{1}{2\pi i}\int_C \varphi_1(w)(\cosh w-1)^n\, dw = 0,$$

and our conclusion is achieved.

We wish now to indicate the extension of the above theorem when the coefficient function is no longer assumed to be of *zero* exponential type. If we assume therefore that a is of exponential type, its growth in various directions in the complex s-plane is best described in terms of a convex set D, the indicator diagram of $a(s)$; see [4, Chapter 5]. In the case of zero exponential type D is in fact the origin. It is important to assume that the width of the indicator diagram along the imaginary axis is less than 2π; from this it follows that a is completely determined by its restriction to the positive integers. A more elaborate argument than the one given for Theorem 1 shows that under these assumptions on a the series (0.1) leads to a function analytic outside the set $S=\{z \mid z=\cosh w,\, w \in \bar{D}\}$, ($\bar{D}=$ complex conjugate of D) including ∞, and single-valued there; the converse also holds. We shall not prove this here, but the reader may consult [3] for the analogue for power series.

We now consider the extension of the theorem when the coefficient function $a(s)$ is allowed to have poles. Thus we shall assume that $a(s)=a_0(s)+R(s)$, where $a_0(s)$ is entire of zero exponential type and R is rational. For the sake of simplicity the poles α_n of R will be assumed to be simple, non-integral, and non-half-integral. These limitations can be dropped by passing to the limit in the argument given below. From here on we shall consider only the Legendre case, $\lambda=\frac{1}{2}$. There is an extension to the general ultraspherical case; but the extra complication of details might tend to obscure the main ideas, and so we limit ourselves to the Legendre case.

Our theorem requires the consideration of the Legendre functions of the cecond kind, $Q_s(z)$. This function is jointly annalytic in z and s, when z lies in the complex plane cut along the real axis from $-\infty$ to 1, and when s is not a negative integer. This function is related to the Legendre function $P_s(z)$ (which is the continuation of $P_n(z)$) by

$$Q_s(z)-Q_{-1-s}(z)=\pi \cot \pi s\, P_s(z); \text{ see [2] p. 140.} \tag{1.5}$$

Theorem 2. *A necessary and sufficient condition that $F(x)$ defined on $-1<x<1$ be given as*

$$F(x)=\lim_{r\to 1} \sum a(n)\, r^n P_n(x), \tag{1.6}$$

is that $F(x)$ be analytically continuable into the complex plane slit from 1 to $+\infty$ along the real axis; and that in this slit plane, when $|z|>1$

$$F(z)=(z-1)^{-\frac{1}{2}}\varphi(z)+\sum c_n Q_{-1-\alpha_n}(-z), \tag{1.7}$$

where $\varphi(z)$ is analytic in $|z|>1$ (including ∞) and single-valued there.

The following clarifying remarks are in order here. In the previous treatment of related problems using the Watson–Sommerfold transform, see e.g. [11], p. 282 et seq. and [7], the contributions due to poles were given in terms of the function $P_{\alpha_n}(-z)$ instead of $Q_{-1-\alpha_n}(-z)$. The latter form is compatible with the right asymptotic behavior at ∞, only when $R(\alpha_n)\geqslant -\frac{1}{2}$, since

$$Q_{\alpha_n}(-z)\cong |z|^{-R(\alpha_n)-1}, \text{ as } |z|\to\infty,$$

and the relation (1.5) holds. Moreover, the form (1.7) has the advantage of separating the "single-valued" part $\varphi(z)$ due to the regular part of $a(s)$, and the multiple valued part $\sum c_n Q_{1-\alpha_n}(-z)$ due to the poles of $a(s)$. It should be noticed that (1.7) could equally well be written as $F(z)=z^{-\frac{1}{2}}\psi(z)+\sum c_n Q_{1-\alpha_n}(-z)$, where $\psi(z)$ is analytic and single-valued for $|z|>1$ (including ∞). However the form (1.7) makes the compatibility with theorem 1 evident.

Proof. Suppose first $-1 \leqslant x < 1$. Then

$$F(x)=\lim_{r\to 1}\frac{i}{2}\int_C \frac{a(s)\,r_s}{\sin \pi s}P_s(-x)\,ds=\frac{i}{2}\int_C \frac{a(s)}{\sin \pi s}P_s(-x)\,ds,$$

where C is an infinite loop surrounding the positive x-axis in the negative direction (i.e., beginning below the axis at $+\infty$, and ending above the axis at $+\infty$), containing none of the poles of $a(s)$, and all the non-negative poles of $1/\sin \pi s$. $P_s(-x)$ is the analytic extension of $P_n(-x)$ given by, e.g.

$$P_s(\cos\theta)=\int_0^\theta (\cos r-\cos\theta)^{-\frac{1}{2}}\cos\left[(s+\tfrac{1}{2})\,r\right]dr,\quad x=\cos\theta$$

which is clearly the only extension of $P_n(x)$ of exponential type $<\pi$ in s.

In view of the growth of $P_s(-x)$ and the assumed growth of $a(s)$ we can write

$$F(x)=\frac{i}{2}\int_{C_1}\frac{a(s)}{\sin \pi s}P_s(x)\,ds-\pi\sum_{R(\alpha_j)\geqslant -\frac{1}{2}}\frac{\beta_i}{\sin \pi\alpha_j}P_{\alpha_j}(-x),$$

where β_j are the residues at the poles $s=\alpha_j$ of $a(s)$, and C_1 is the axis $R(s)=-\frac{1}{2}$, taken with increasing $t, (s=\sigma+it)$, except for an indentation so that all poles α_n of $a(s)$, for which $R(a_n)\geqslant-\frac{1}{2}$, are to the right of *it*. Now the formula

$$P_s(z)=\frac{2^{-s}}{\Gamma(-s)\,\Gamma(s+1)}\int_0^\infty (z+\cos t)^{-s-1}(\sinh t)^{2s+1}\,dt,$$

[2, p. 155] shows that $P_s(-x)$ is analytically continuable in the slit plane when, $-1<R(s)<0$, and for fixed z is of exponential type π, in the strip $-1<R(s)<0$. This of course shows that $F(z)$ is also so analytically continuable, and then using identity (1.5) we can write

$$F(x)=\frac{1}{2\pi}\int_{C_2}\frac{a(s)}{\cos \pi s}\varphi_s(-z)\,ds-\frac{i}{2\pi}\int_{C_2}\frac{a(s)}{\cos \pi s},$$

$$\times\, Q_{-1-s}(-z)\,ds-\pi\sum_{\alpha_j}\frac{\beta_j}{\sin \pi\alpha_j}P_{\alpha_j}(-z)$$

where the contour C_2 is like C_1, except that now all poles α_j, and $s=-\frac{1}{2}$, lie to the right of C_2. Let us consider the integral involving $Q_s(-z)$ first. Deforming the contour C_2 back to C again we get

$$\frac{i}{2\pi}\int_{C_2}\frac{a(s)}{\cos \pi s}\,Q_s(-z)\,ds=\frac{i}{2\pi}\int_{C}\frac{a(s)}{\cos \pi s}\,Q_s(-z)\,ds+\sum_{\alpha_j}\frac{\beta_j}{\cos \pi\alpha_j}\,Q_{\alpha_j}(-z).$$

We have used the fact that $Q_s(-z)$ behaves favorably in the half-plane $R(s)\geqslant -\frac{1}{2}$, which can be read of the identity

$$Q_s(z)=2^{-s-1}\int_0^{\pi}(z+\cos t)^{-s-1}(\sin t)^{2s+1}\,dt, \tag{1.8}$$

[2, p. 155]. However,

$$\frac{i}{2\pi}\int_{C}\frac{a(s)}{\cos \pi s}\,Q_s(-z)\,ds=\sum_{n=0}^{\infty}(-1)^{n+1}a(n-\tfrac{1}{2})\,Q_{n-\frac{1}{2}}(-z).$$

The integral involving Q_{-1-s} is deformed into a loop similar to C, but surrounding the negative axis from $-\infty$ to -1. In this case there is no contribution due to the poles of $a(s)$, and we get

$$\frac{i}{2\pi}\int_{C_2}\frac{a(s)}{\cos \pi s}\,Q_{-1-s}(-z)\,ds=-\sum_{n=1}^{\infty}(-1)^{-n+1}a(-n-\tfrac{1}{2})\,Q_{-n-\frac{1}{2}})(-z).$$

However, $Q_{n-\frac{1}{2}}(-z)=Q_{-n-\frac{1}{2}}(-z)$ (n integral, see (1.5)).

Combining the above we get

$$F(z)=a(-\tfrac{1}{2})\,Q_{-\frac{1}{2}}(-z)+\sum_{n=1}^{\infty}[a(n-\tfrac{1}{2})+a(-n-\tfrac{1}{2})]\,Q_{n-\frac{1}{2}}(-z)$$
$$-\sum\frac{\beta_j}{\cos \pi\alpha_j}\,Q_{-1-\alpha_j}(-z). \tag{1.9}$$

Finally by (1.8) and the fact that

$$\left|\frac{1-u^2}{z+u}\right|\leqslant\frac{2}{|z|},\quad \text{if}\quad |z|\geqslant 1,\quad -1\leqslant u\leqslant 1,$$

we get that

$$|Q_{n-\frac{1}{2}}(-z)|\leqslant A\,|z|^{-n-\frac{1}{2}},\; n=0,1,2,\ldots,\; |z|\geqslant 1.$$

This with the fact that $a(+n-\frac{1}{2})=O(e^{\varepsilon n})$, for every $\varepsilon>0$, $n>0$ assures the convergence of the series in (1.9) uniformly in $|z|\geqslant 1+\delta$, $\delta>0$. Since as is easily seen from (1.8) $(z-1)^{\frac{1}{2}}\,Q_{n-\frac{1}{2}}(-z)$ is also single valued in $|z|>1$, our representation (1.7) is proved.

The converse can now be proved as follows. Suppose we are given an $F(z)$ in the slit plane, which for $|z|>1$ has the representation (1.7). Let

$$F_0(z)=\lim_{r\to 1}\sum a_0(n)\,r^n\,P_n(z),\quad \text{where}\quad a_0(s)=-\sum\frac{c_j\cos \pi\alpha_j}{s-a_j}.$$

Then by what we have just proved, when $|z|>1$

$$F_0(z) = (z-1)^{-\frac{1}{2}} \varphi_0(z) + \sum c_n Q_{-1-\alpha_n}(-z).$$

Hence
$$F(z) - F_0(z) = (z-1)^{-\frac{1}{2}} [\varphi(z) - \varphi_0(z)].$$

This shows that $(1-z)^{\frac{1}{2}} \{F(z) - F_0(z)\}$ has a single valued analytic continuation into the extended complex plane with only singularity $z=1$; i.e., $(1-z)^{\frac{1}{2}} \{F(z) - F_0(z)\}$ is an entire function of $1/1-z$. Thus by Theorem 1,

$$F(x) - F_0(x) = \lim_{r \to 1} \sum a_1(n) r^n P_n(x), \quad -1 \leqslant x < 1,$$

where $a_1(s)$ is entire of zero exponential type. Now we need only take $a(s) = a_1(s) + a_0(s)$.

2. Case when $a(s)$ is given in a half-plane; variants of Parseval-Plancherel formula

As before, we shall denote by S the complex plane slit along the positive real axis from 1 to ∞. $H^2(S)$ will denote the class of functions $F(z)$, analytic in S, for which $\sup_{y \neq 0} \int_{-\infty}^{\infty} |F(x+iy)|^2 dx < \infty$. Then as is well-known, $F_{\pm}(x) = \lim_{y \to 0,\, y>0} F(x \pm iy)$ exists in L^2 norm. We then define the norm by

$$\|F\|^2 = \int_{-\infty}^{\infty} |F_+(x)|^2 dx + \int_{-\infty}^{\infty} |F_-(x)|^2 dx.$$

Of course, this makes $H^2(S)$ into a Hilbert space. These functions may be characterized in another way. While the boundary values $F_+(x)$ and $F_-(x)$ exist (and agree for $-\infty < x < 1$), they do not agree on the cut. However, the jump across the cut, $F_+(x) - F_-(x)$ completely determines this function. In fact, if $f(x) = F_+(x) - F_-(x)$, then

$$F(z) = \frac{1}{2\pi i} \int_1^{\infty} \frac{f(u)\, du}{u-z}, \tag{2.1}$$

and
$$\|F\|^2 = \int_1^{\infty} |f(u)|^2 du.$$

Conversely, if $f(u)$ is an arbitrary element of $L^2(1, \infty)$, the Cauchy integral (2.1) is in $H^2(S)$, and $F_+(x) - F_-(x) = f(x)$. This follows from the fact that if $y \geqslant 0$, $F(x+iy) \mp F(x-iy)$ are essentially the Poisson and conjugate Poisson integrals of f; and that as $y \to 0$ they converge respectively to f and $i\tilde{f}$ with $\tilde{f}$ the Hilbert transform of f. The mapping $f \to \tilde{f}$ is a unitary mapping on $L^2(-\infty, \infty)$. (For all these facts concerning the Hilbert transform see [13], Chapter 5.) Thus the Cauchy integral (2.1) gives a unitary equivalence between $H^2(S)$ and $L^2(1, \infty)$.

For the coefficient function $a(s)$ we will take the space $H^2(Rs > -\frac{1}{2})$, i.e., the $a(s)$ will be assumed to be analytic in

$$R(s) > -\tfrac{1}{2}, \quad \text{and} \quad \sup_{\sigma > -\frac{1}{2}} \int_{-\infty}^{\infty} |a(\sigma+it)|^2\,dt = \int_{-\infty}^{\infty} |a(-\tfrac{1}{2}+it)|^2\,dt = \|a\|^2 < \infty.$$

It should be remarked that if $a(s) \in H^2(Rs > -\frac{1}{2})$, then it is automatically bounded in any half plane $R(s) \geqslant -\frac{1}{2}+\delta$, $\delta > 0$, (see (2.2) below), and hence is uniquely determined by its values on the positive integers. Our theorem for power series can be formulated as follows:

Theorem 3. *Suppose* $F(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then* $F \in H^2(S)$ *if and only if* $a_n = a(n)$, *where* $a \in H^2(Rs > -\frac{1}{2})$. *Moreover,* $\|F\|^2 = 2\pi \|a\|^2$.

Proof. Suppose, to begin with, $a(s) \in H^2(Rs > -\frac{1}{2})$. Then according to the Paley–Wiener representation in the context of Mellin transforms

$$a(s) = \int_1^{\infty} u^{-s-1} \varphi(u)\,du, \tag{2.2}$$

where $\qquad \varphi \in L^2(1, \infty), \text{ and } \|a\|^2 = 2\pi \int_1^{\infty} |\varphi(u)|^2\,du.$

Now if $|z| < 1$,

$$F(z) = \sum a(n) z^n = \sum_{n=0}^{\infty} \int_1^{\infty} u^{-n-1} z^n \varphi(u)\,du = \int_1^{\infty} \frac{\varphi(u)}{u-z}\,du.$$

Since if $u \geqslant 1$, and $|z| < 1$, z fixed, the series converges, its partial sums bounded in absolute value by A/u. Thus $F(z) = \int_1^{\infty} \varphi(u)/(u-z)\,du$ in the slit plane and $F \in H^2(S)$. Finally, as pointed out above, $\|F\|^2 = 4\pi^2 \int_1^{\infty} |\varphi(u)|^2\,du = 2\pi \|a\|^2$.

Conversely, suppose $F(z) \in H^2(S)$. Then

$$F(z) = \frac{1}{2\pi i} \int_1^{\infty} \frac{f(u)}{u-z}\,du, \quad \text{with} \quad \|F\|^2 = \int_1^{\infty} |f(u)|^2\,du.$$

Expanding $1/(u-z)$ again as $\sum z^n u^{-n-1}$, we get with

$$|z| < 1, \ u \geqslant 1, \ F(z) = \sum_{n=0}^{\infty} a(n) z^n, \quad \text{where} \quad a(s) = \frac{1}{2\pi i} \int_1^{\infty} u^{-s-1} \varphi(u)\,du$$

and $$\|a\|^2 = 2\pi \int_1^{\infty} |\varphi|^2\,du = \frac{1}{2\pi} \|F\|^2.$$

For the analogue of Theorem 3 for Legendre expansions we need to consider a modification of the space $H^2(Rs > -\frac{1}{2})$ of coefficient functions. For this purpose we define $H^2_*(Rs > -\frac{1}{2})$ the space of functions $a(s)$ which are analytic on $Rs > \frac{1}{2}$, belong to $H^2(Rs > -\frac{1}{2})$ and for which in addition, the norm squared $\|a\|^2_* = \int_{-\infty}^{\infty} |a(-\frac{1}{2}+it) - a(-\frac{1}{2}-it)|^2\,(t\,dt/\tanh \pi t)$ is finite.

Let us make a few simple remarks about this space. Since

$$\frac{t}{\tanh \pi t} \geqslant c > 0 \quad \text{then} \quad \|a\|_*^2 \geqslant c \int_{-\infty}^{\infty} |a(-\tfrac{1}{2}+it) - a(-\tfrac{1}{2}-it)|^2 \, dt$$

$$= c \int_{-\infty}^{\infty} |a(-\tfrac{1}{2}+it) + (a-\tfrac{1}{2}-it)|^2 \, dt.$$

The latter fact follows since for functions in $H^2(Rs > -\frac{1}{2})$, $a(-\frac{1}{2}+it)+a(-\frac{1}{2}-it)$ is (except for a factor of i) the Hilbert transform of $a(-\frac{1}{2}+it)-a(-\frac{1}{2}-it)$. Hence $\|a\|_*^2 \geqslant 2c\|a\|^2$. Thus the norm of $H_*^2(Rs > -\frac{1}{2})$ essentially dominates the norm of $H^2(Rs > -\frac{1}{2})$, and this domination shows easily that $H_*^2(Rs > -\frac{1}{2})$ is then a complete (Hilbert) space. We shall also prove below that if $a \in H_*^2$ $(Rs > -\frac{1}{2})$ then $\sum_{n=0}^{\infty}(2n+1)|a(n)|^2 < \infty$, and so the series

$$F(x) = \sum (2n+1)\, a(n)\, P_n(x) \quad -1 < x < 1 \tag{2.3}$$

converges in the $L^2(-1,1)$ norm.

Our result is as follows:

Theorem 4. *$F(z)$ is in $H^2(S)$ if and only if for $-1 \leqslant x \leqslant 1$ we have the development* (2.3), *where $a(s) \in H_*^2(Rs > -\frac{1}{2})$. Moreover, $\|F\|^2 = 2\pi\|a\|_*^2$.*

Proof. We begin by showing that under our hypotheses on $a(s)$, $\sum(2n+1)|a(n)|^2 < \infty$. In fact, since $a(s)$ belongs to $H^2(Rs > -\frac{1}{2})$ it can be represented either in terms of the Poisson integral or the conjugate Poisson integral of its boundary values. Because of the oddness of the conjugate Poisson kernel the latter form is more convenient here. Thus

$$a(\sigma) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{a(-\frac{1}{2}+it)\,t}{(\sigma+\frac{1}{2})^2+t^2}\,dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(a(-\frac{1}{2}+it)-a(-\frac{1}{2}-it))\,t\,dt}{(\sigma+\frac{1}{2})^2+t^2}$$

hence

$$|a(n)| \leqslant A\left\{n^{-1}\int_{|t|\leqslant n} |a(-\tfrac{1}{2}+it)-a(-\tfrac{1}{2}-it)|\,dt + \int_{|t|\geqslant n} \frac{|a(-\frac{1}{2}+it)-a(-\frac{1}{2}-it)|}{|t|}\,dt\right\}, \quad n \geqslant 1.$$

The result then follows by well-known arguments, see ([5], Chapter 9.)

The main part of the proof of the theorem is based on an analogue of the Mellin representation (2.2).

Lemma. *Suppose $a(s) \in H_*^2(Rs > -\frac{1}{2})$, then*

$$a(s) = \frac{1}{2\pi i}\int_1^{\infty} Q_s(x)\,\varphi(x)\,dx, \quad R(s) > -\tfrac{1}{2}, \tag{2.4}$$

where $2\pi^2 \|a\|_*^2 = \int_1^\infty |\varphi(u)|^2 du$; *such a* φ *is unique. Conversely, if* $\varphi \in L^2(1, \infty)$, *then* $a(s)$ *given by* (2.4) *belongs to* $H_*^2(Rs > -\frac{1}{2})$. *Thus* (2.4) *gives a unitary correspondence between* $L^2(1, \infty)$ *and* $H_*^2(Rs > -\frac{1}{2})$.

Proof of the lemma. Let us first make a remark about the space $H_*^2(Rs > -\frac{1}{2})$. By its definition, if $a(s) \in H_*^2(Rs > -\frac{1}{2})$, then the function $f(t) = a(-\frac{1}{2} + it) - a(-\frac{1}{2} - it)$ belongs to $L^2(-\infty, \infty, (t\,dt)/\tanh \pi t)$. We claim that, conversely, every odd function $f(t)$ which belongs to $L^2(-\infty, \infty, (t\,dt)/\tanh \pi t)$ arises in this way. In fact such a function belongs automatically to $L^2(-\infty, \infty)\,dt)$. We let $u(s)$ and $v(s)$ denote respectively the Poisson integral and conjugate Poisson integral of $f(t)$. Thus $\alpha(s) = u(s) + iv(s)$ belongs to $H^2(Rs > -\frac{1}{2})$, and $u(s)$ and $v(s)$ are respectively odd and even in t. Hence $\alpha(-\frac{1}{2} + it) - \alpha(-\frac{1}{2} - it) = 2f(t)$, and therefore $\alpha(s) \in H_*^2(Rs > -\frac{1}{2})$. In this way the space $H_*^2(Rs > -\frac{1}{2})$ is unitarily equivalent with the space of odd functions in $L^2((-\infty, \infty,)\,(t\,dt)/\tanh \pi t)$.

Next, let us notice that formula (1.8) shows that $|Q_s(x)| \leqslant Ax^{-\sigma-1}$ $s = \sigma + it$, $x > 1$. This shows that whenever $\varphi \in L^2(1, \infty)$, $a(s)$ given by (2.4) is analytic in $R(s) > -\frac{1}{2}$; if in addition $\varphi \in L^1(1, \infty)$, then $a(s)$ is analytic and bounded in $R(s) \geqslant -1$, and vanishes as $R(s) \to +\infty$.

At this stage we invoke the Plancherel formula for the Legendre functions on the interval $(1, \infty)$. (See Bargmann [1], Harish Chandra [6]; also Bateman [2] p. 175.) As was pointed out, this is the analogue of the discrete Legendre expansion on $(-1, 1)$; except that here we are dealing with the Poincare upper half-space, and the group $SL(2, R)$, instead of the surface of the sphere and the group $SO(3)$. For our purposes this result may be stated as follows: Whenever

$$\varphi \in L^1(1, \infty) \cap L^2(1, \infty) \quad \text{and} \quad \Phi(t) = \int_1^\infty P_{-\frac{1}{2}+it}(x)\,\varphi(x)\,dx,$$

then
$$\frac{1}{2}\int_{-\infty}^{\infty} |\Phi(t)|^2\, t \tanh \pi t\, dt = \int_1^\infty |\varphi(x)|^2\, dx. \tag{2.5}$$

Moreover, the $\Phi(t)$ which arise this way are a dense subspace among the even functions of t which belong to $L^2((-\infty, \infty), t \tanh \pi t\, dt)$. This, of course, is another way of saying that the transform $\varphi \to \Phi$ extends to a unitary mapping from $L^2(1, \infty)$ to the even functions on $L^2(-\infty, \infty, t \tanh \pi t\, dt)$.

Finally we should recall the relation (1.5).

In proving the lemma, let us begin with a given $\varphi \in L^1(1, \infty) \cap L^2(1, \infty)$. If $a(s)$ is defined as in (2.4) then as pointed out before, a is analytic and bounded in $R(s) \geqslant -1$. Moreover,

$$a(-\tfrac{1}{2} + it) - a(-\tfrac{1}{2} - it) = \frac{\tanh \pi t}{2\pi i}\, \Phi(t).$$

Thus according to (2.5)

$$\frac{4\pi^2}{2}\int_{-\infty}^{\infty} |a(-\tfrac{1}{2} + it) - a(-\tfrac{1}{2} - it)|^2 \frac{t}{\tanh \pi t}\, dt = \int_1^\infty |\varphi(x)|^2\, dx.$$

In order to conclude that $a \in H_*^2(Rs > -\frac{1}{2})$, we must therefore show only that

$a \in H^2(Rs > -\frac{1}{2})$. However, the function $a(\sigma+it)-a(\sigma-it)$ is harmonic and bounded in $\sigma \geqslant -\frac{1}{2}$. It is therefore the Poisson integral of its boundary value, which by assumption certainly belongs to $L^2(-\infty,\infty)$. Hence the conjugate harmonic function which equals $(a(\sigma+it)+a(\sigma-it\varphi))/i+c$, has the same property, if it is normalized to vanish as $\sigma\to\infty$.

This arises when $c=0$, since $a(s)$ itself vanishes as $\sigma\to\infty$. Thus $a(s)$ is the Poisson integral of an L^2 function and hence belongs to $H^2(Rs > -\frac{1}{2})$. Summarizing all the above, we see that whenever $\varphi \in L^1(1,\infty)\cap L^2(1,\infty)$, then $a(s)$ given by (2.4) belongs to $H^2_*(Rs > -\frac{1}{2})$, $2\pi^2\|a\|^2_* = \int_1^\infty |\varphi(x)|^2\,dx$, and the $a(s)$ so represented are a dense subset of $H^2_*(Rs > -\frac{1}{2})$. From this we see that the mapping (2.4) gives a unique (abstract) extension to all of $L^2(1,\infty)$ onto $H^2_*(Rs > -\frac{1}{2})$ so that $2\pi\|a\|^2_* = \int_1^\infty |\varphi(x)|^2\,dx$. However, an arbitrary element $\varphi\in L^2(1,\infty)$ could be approximated in the L^2 norm by a sequence

$$\varphi_n \in L^1(1,\infty)\cap L^2(1,\infty).$$

If
$$a_n(s) = \frac{1}{2\pi i}\int_1^\infty Q_s(x)\,\varphi_n(x)\,dx,$$

the sequence $a_n(s)$ converges uniformly in compact sub-domains of $R(s) > = \frac{1}{2}$; thus

$$a(s) = \frac{1}{2\pi i}\int_1^\infty Q_s(x)\,\varphi(x)\,dx, \quad R(s) > -\tfrac{1}{2}.$$

This concludes the proof of the lemma.

We now conclude the proof of the theorem. Consider the class of $\varphi(x)\in L^2(1,\infty)$, and which vanish near $x=1$ and outside a bounded interval. This class is clearly dense in $L^2(1,\infty)$. For each such φ, define its Cauchy integral, i.e.,

$$F(z) = \frac{1}{2\pi i}\int_1^\infty \frac{\varphi(u)}{u-z}\,du.$$

Then according to what was said earlier, the F so defined belong to $H^2(S)$, $\|F\|^2 = \int_1^\infty |\varphi(u)|^2\,du$, and these F form a dense subspace of $H^2(S)$. Each such F belongs, of course, to $L^2(-1,1;dx)$. It therefore has a Legendre polynomial development, $F(x)\sim\sum_{n=0}^\infty (2n+1)\,a_n P_n(x)$. What is it?

In order to find out we shall use the expansion of the Cauchy kernel in Legendre polynomials, i.e.,

$$\frac{1}{u-x} = \sum (2n+1)\,P_n(x)\,Q_n(u). \tag{2.6}$$

As is known (see [12] p. 244) the series converges uniformly in $1\leqslant x\leqslant 1$, as long as u is restricted to lie in a proper subinterval of $(1,\infty)$. This gives

$$F(x) = \sum (2n+1)\,a(n)\,P_n(x), \quad \text{where} \quad a(s) = \frac{1}{2\pi i}\int_1^\infty Q_s(u)\,\varphi(u)\,du,$$

the series for F converging uniformly. However, by the lemma

$$a(s) \in H^2_*(Rs > -\tfrac{1}{2}), \quad \text{and} \quad 2\pi \|a\|^2_* = \int_1^\infty |\varphi(u)|^2 du = \|F\|^2.$$

Moreover, the class of such a (corresponding to φ whose support is restricted to lie in a proper sub-interval of $(1, \infty)$) forms a dense set in $H^2_*(Rs > -\tfrac{1}{2})$. This gives the required identification of a dense subspace of $H^2(S)$ with the corresponding dense subspace of $H^2_*(Rs > -\tfrac{1}{2})$. The extension of the identification to the full spaces is now a straight-forward matter.

As a final point, we shall explain how the "completeness relation" in the non-compact case can be deduced from the completeness relation of the Legendre polynomials over $(-1, 1)$. To make matters precise, and stripped of any technical difficulties, we shall assume that $\varphi \in L^1(1, \infty) \cap L^2(1, \infty)$, and its transform vanishes, that is

$$\Phi(t) = \int_1^\infty P_{-\frac{1}{2}+it}(x)\,\varphi(x)\,dx \equiv 0.$$

We want to show that $\varphi \equiv 0$, without using anything like the lemma. We shall however make use of the Legendre series expansion of the Cauchy kernel (2.6), which, of course, is an expansion arising from the compact case. Now let $a(s) = (1/2\pi i) \int_1^\infty Q_s(x)\,\varphi(x)\,dx$. As was pointed out earlier (and this depends only on the asymptotic behavior of the Q's) $a(s)$ is bounded and continuous in the right half-plane $R(s) \geqslant -\tfrac{1}{2}$, and tends to zero as $\sigma \to \infty$. However, $a(-\tfrac{1}{2} + it) - a(-\tfrac{1}{2} - it) = -\tfrac{1}{2} \tanh \pi t\, \Phi(t)$, since $Q_s - Q_{-1-s} = \pi \cot \pi s\, P_s$. Thus $a(s) - a(-1-s)$ vanishes when $R(s) = -\tfrac{1}{2}$, and hence $a(s)$ is entire and $a(s) = a(-1-s)$. Thus a is bounded, hence a constant, and finally $a(s) \equiv 0$, since $a(\sigma) \to 0$, $\sigma \to \infty$. Now let $F(z)$ be the Cauchy integral of φ. Then restricted to $[-1, 1]$ $F(x)$ is in $L^2(-1, 1)$, and because of the development (2.6) as we have seen, $F(x) \sim \sum (2n+1)\,a(n)\,P_n(x)$. Thus $F(x) = 0$, in $-1 < x < 1$, and therefore everywhere. But $\varphi(x) = \lim_{y \to 0,\, y>0} F(x+iy) - F(x-iy)$. This shows $\varphi = 0$.

The whole matter can be restated and summarized in another way. Let $\varphi(x)$ be an arbitrary L^2 function on $(1, \infty)$. Then there exists a unique function $F(z)$ analytic in the complex plane slit along the interval $(1, \infty)$, which vanishes at ∞ (in an appropriate sense) and whose jump across the cut in the L^2 sense is $\varphi(x)$. This F is then automatically in L^2 on $[-1, 1]$. Let $F(x) \sim \sum (2n+1) \cdot a_n P_n(x)$, be its Legendre series development there. Then again there exists a unique function $a(s)$ which is analytic in $Rs > -\tfrac{1}{2}$ and bounded at ∞ so that $a_n = a(n)$. Finally

$$a(-\tfrac{1}{2} + it) - a(-\tfrac{1}{2} - it) = -\tfrac{1}{2} \tanh \pi t \int_1^\infty P_{-\frac{1}{2}+it}(x)\,\varphi(x)\,dx.$$

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BIBLIOGRAPHY

1. BARGMANN, V., Representations of the Lorentz group, Annals of Math. *48*, 568–640 (1947).
2. Bateman manuscript project, Higher transcendental functions Volume I. 1953.
3. BIEBERBACH, L., Analytische Fortsetzung, Berlin 1955.
4. BOAS, R. P., Entire Functions, New York 1954.
5. HARDY, G. H., LITTLEWOOD, J. E., and POLYA, G., Inequalities, Cambridge 1934.
6. Harish Chandra, Plancherel formula for the 2×2 real unimodular group, Proc. Nat. Acad. Sci. *38*, 337–342 (1952).
7. KHURI, N. N., Regge Poles, Power Series, and a Crossing-Symmetric Watson–Sommerfeld Transformation, Phys. Rev. *132*, 914–926 (1963).
8. LINDELÖF, L. L., Le calcul des résidus ..., Paris, 1905.
9. POINCARÉ, H., Œuvres, Vol. 10, pp. 94–203; especially section 19.
10. REGGE, R., Introduction to Complex Orbital Momenta, Il Nuovo Cimento *14*, 951–976 (1959).
11. SOMMERFELD, A., Partial Differential Equations in Physics, New York, 1949.
12. SZEGO, G., Orthogonal Polynomials, 1939.
13. TITCHMARSH, E. C., Introduction to the theory of Fourier integrals, Oxford, 1937.
14. WATSON, G. N., The Diffraction of Electric Waves by the Earth, Proc. Roy. Soc. London *95*, 83–99 (1918).

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