Estimates of harmonic measures

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Introduction

Estimates of harmonic measures in terms of Euclidean quantities are useful in many situations. In the two-dimensional case one can apply methods of conformal mapping and extremal lengths, and many sharp results are well known. Different means of harmonic measures can be studied in the \( n \)-dimensional case. This paper is intended to provide a survey of methods available to estimate harmonic measures.

The two-dimensional case is treated in Chapter I. The second paragraph contains well-known distortion inequalities from the theory of conformal mapping, and § 3 contains well-known results from the theory of extremal lengths. In § 4 we prove two symmetrization theorems with the aid of a result from § 3. In § 5 we apply results from § 3 to comb domains.

Chapter II gives \( n \)-dimensional methods. In § 6 a method of Carleman [6.1] is applied to harmonic measures. The derivation of Carleman’s method in Theorem 6.1 follows that of Dinghas [6.3]. The estimates of harmonic measures in Theorems 6.2 and 6.3 are new in the case \( n > 2 \). In § 7 we treat Nevanlinna’s mean value in a special case. In § 8 we prove some symmetrization results with probabilistic methods.

The main problem is to provide upper bounds for harmonic measures. Lower bounds are discussed in § 2 and § 7.

Bearing in mind the possibility of exhausting a given domain with more regular domains we have not aimed at generality in assumptions about the domains considered.

The subject of this paper was suggested by Professor L. Carleson, to whom I am deeply grateful for all his advice.

1. Definitions

\( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space, \( n \geq 2 \), with points \( z = (x_1, y_1, \ldots, y_{n-1}) = (x, y) \). In Chapter I we treat the case \( n = 2 \) and prefer to write \( z = x + iy \). The following definitions are then to be understood with \( \text{Re} \ z \) instead of \( x \).

\( D \) denotes a domain (open connected set) and \( \partial D \) the boundary of \( D \).

\( \Theta_x = \{ z \mid x_1 = x, z \in D \} \).

\( D_x \) is the subdomain of \( \{ z \mid x_1 < x, z \in D \} \) that contains a given point \( z_0 \).

\( \theta_x = \{ z \mid z \in \Theta_x, z \in \partial D_x \} \).

Without \( D \) being specified \( D_0 \) denotes a domain in \( \{ z \mid x_1 < \xi \} \) with part of its boundary on \( \{ z \mid x_1 = \xi \} \) and \( \theta_\xi \) then denotes the interior of \( \{ z \mid x_1 = \xi, z \in \partial D_\xi \} \).
Given $D_2$, $\vartheta_z = \{z | x_1 = x, z \in D_1\}$, $x < \xi$.

$\vartheta^i$, $i = 1, 2, \ldots, n(x)$, are the components or unions of components of $\Theta_z(\vartheta_z)$ that separate two given points or surfaces. (Cf. § 2.)

$\Theta(x)$, $\theta(x)$, $\vartheta(x)$, $\vartheta(x)$ are the measures of the respective sets.

$\Theta_z$ will be used in preference to $\vartheta_z$ and $\theta_z$ if $D$ is such that all the $\Theta_z$ are connected. $\vartheta_z$ can also be used to denote a set in the $(n - 1)$-dimensional $y$-space.

$\omega(z; z; D)$ denotes the harmonic measure at the point $z$ of $z \in \partial D$ with respect to $D$. $c$ may denote various constants.

Symmetrization of an $n$-dimensional open set $A$ with respect to an $(n-1)$-dimensional hyperplane $p$ (Steiner symmetrization) [1.1, p. 5, pp. 151–152] means the following: $A$ is transformed into $A^*$ so that any straight line perpendicular to $p$ that intersects $A$ also intersects $A^*$. Both intersections have the same measure (length) and the intersection with $A^*$ is a single line-segment symmetric with respect to $p$.

When $n = 2$ this reduces to the definition of symmetrization with respect to a straight line.

A continuous function $f$ is symmetrized with respect to a hyperplane $p$ by symmetrizing the sets $\{z | f(z) > a\}, \inf f(z) \leq a < \sup f(z)$, in the manner described above.

Symmetrization of an $n$-dimensional open set, $n > 2$, with respect to a straight line $l$ (Schwarz symmetrization) [1.1, pp. 151–152] means the following: $A$ is transformed into $A^*$ so that any $(n - 1)$-dimensional hyperplane perpendicular to $l$ that intersects $A$ also intersects $A^*$. Both intersections have the same measure and the intersection with $A^*$ is a sphere with its centre on $l$.

Chapter I. The two-dimensional case

2. Distortion theorems in the theory of conformal mapping

Problems of distortion in the theory of conformal mapping have been widely studied. We shall refer to the survey given by Lelong–Ferrand [2.2, Ch. VI, in particular pp. 185–202, pp. 216–217].

Let $D$ be a simply connected domain in the $z$-plane not containing the point at infinity. Let $A$ and $B$ be two accessible boundary points. We limit the discussion to the following situation: $A$ and $B$ are the only boundary points of $D$ at infinity and $\Re z \rightarrow -\infty$, when $z \rightarrow A, z \in D$, and $\Re z \rightarrow +\infty$, when $z \rightarrow B, z \in D$. Let $L$ be a Jordan arc in $D$ joining $A$ and $B$. $\Theta^1$, $i = 1, 2, \ldots, n(x)$, are the segments of $\Theta_z$ that separate $A$ and $B$. $\Theta^1$ is the first of the $\Theta^i$ that is met when moving along $L$ from $A$ to $B$. $\Theta^1$ can also be defined as that segment among the $\Theta^i$ that separates the largest subdomain of $D$ from $A$. Cf. Fig. 2.1. $\Theta_1(x)$, the length of $\Theta_1$, is lower semicontinuous. A detailed discussion of the definition of $\Theta_1$ is given by Lelong–Ferrand [2.2, pp. 185–186].

$D$ is mapped conformally onto $G$ in the $w$-plane, $w = u + iv$. $G = \{w | \frac{|v|}{\sqrt{u^2 + v^2}} < \frac{1}{2\pi} \}$ and $A$ corresponds to $u = -\infty$ and $B$ to $u = +\infty$. $\gamma_x$ is the image of $\Theta^1, u_1$ and $u_2$ are defined by

$$u_1(x) = \inf_{w \in \gamma_x} u, \quad u_2(x) = \sup_{w \in \gamma_x} u.$$

Let $D^1$ be the subdomain of $D$ separated from $B$ by $\Theta^1$. $G_a = \{w | u < a, |v| < \frac{1}{2\pi}\}$ and $l_a = \{w | u = a, |v| < \frac{1}{2\pi}\}$. We use conformal invariance of harmonic measures, the
extension principle, and explicit harmonic measures in $G_\xi$ to establish the following relations:

$$\max_{z \in \Omega_\xi^-} \omega(z; \theta_\xi^1; D_\xi^1) \leq \omega(u_2(x); l_{u_1(\xi)}; G_{u_1(\xi)})$$

$$-\frac{4}{\pi} \arctg \exp (- u_2(\xi) + u_2(x)), u_2(x) < u_1(\xi);$$

$$\max_{z \in \Omega_\xi^-} \omega(z; \theta_\xi^1; D_\xi^1) \geq \omega(u_1(x); l_{u_1(\xi)}; G_{u_1(\xi)}) = \frac{4}{\pi} \arctg \exp (- u_2(\xi) + u_1(x)).$$

Ahlfors' first distortion inequality [2.1, pp. 7-12], [2.2, pp. 187-190], [2.3, pp. 93-100], states that

$$u_1(\xi) - u_2(x) > \pi \int_\xi^z \frac{dt}{\theta_1(t)} - 4\pi, \quad \text{when} \quad \int_\xi^z \frac{dt}{\theta_1(t)} > 2.$$ (2.3)

By (2.1) this yields an upper bound for $\omega(z; \theta_\xi^1; D_\xi^1)$. (Also cf. [2.3, pp. 76-78].) However, a more general result is proved in Theorem 3.2 with the method of extremal lengths. In connection with estimates of harmonic measures, distortion inequalities in the other direction are more useful, since there are few other methods for finding lower bounds of harmonic measures.

Distortion inequalities in the other direction require various restrictive assumptions about $D$. Ahlfors' original second inequality [2.1, pp. 12-17] is contained (with a different constant term) in an inequality by Lelong-Ferrand [2.2, pp. 194-198]. Another variant was proved by Warschawski [2.4, pp. 291-296], [2.2, p. 202]. With (2.2) this yields the following theorem.

**Theorem 2.1.** Let $D$ be bounded by the curves $y = q_2(x)$ and $y = q_1(x)$, $q_2(x) > q_1(x)$, $-\infty < x < \infty$. Let $q_1$ and $q_2$ have bounded derivatives; $|q_1(x)| < m$, $|q_2(x)| < m$, $-\infty < x < \infty$. $\psi(x) - \frac{1}{2} (q_1(x) + q_2(x))$, $\Theta(x) = q_2(x) - q_1(x)$. Then
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\[ \max_{z \in \partial \Omega} \omega(z; \Theta_\Omega; D_\Omega) \geq c \exp \left( -\pi \int_{\Omega}^z \frac{dt}{\Theta(t)} - \pi \int_{\Omega}^z \left( \frac{\Theta''(t)}{\Theta(t)} + \frac{\Theta'(t)^2}{12\Theta(t)} \right) dt \right), \]

where \( c = \exp \left( -8\pi \left(1 + \frac{1}{\delta} m^2 \right) \right) \).

Another method for determining lower bounds of harmonic measures will be discussed in § 7. Lower bounds of the form \( c \exp \left( -\pi \int_{\Omega}^z \frac{dt}{\Theta(t)} \right) \) can not be established if \( \Theta \) oscillates too much. An example illustrating this is given in § 5.

3. Relations between extremal lengths and harmonic measures

This subject dates back to Beurling’s thesis [3.2], and appears to have been well known to many mathematicians before an account of it was published by Hersch [3.3].

Let \( \Gamma \) be a family of locally rectifiable curves (denoted \( \gamma \)) in a domain \( D \) (i.e. each compact subcurve of \( a \gamma \) is rectifiable).

Consider non-negative functions \( \varrho \) in \( D \) for which

\[ L_\varrho = L_\varrho(\Gamma) = \inf_{\gamma} \int_{\gamma} \varrho \, |dz| \]

\[ A_\varrho = A_\varrho(D) = \int_{D} \varrho^2 \, dxdy \]

are defined and are not both 0 or both \( \infty \). (For a locally rectifiable \( \gamma \) \( \int_{\gamma} \varrho \, |dz| \) is defined as the supremum of the integrals over subcurves of \( \gamma \).) The extremal length \( \lambda(\Gamma) \) is defined by

\[ \lambda(\Gamma) = \sup_{\varrho} \frac{L_\varrho^2}{A_\varrho}. \]

The functions \( \varrho \) define a conformal metric by \( d\sigma = \varrho(z) \, |dz| \).

The definition of extremal length is due to Ahlfors and Beurling [3.1, p. 114]. The extremal length is a conformal invariant. Thus, if relations between some extremal lengths and harmonic measures are known in, for instance, the circle, these relations can be used to find estimates of harmonic measures.

In the following, when discussing a family of curves we shall assume that they are locally rectifiable. Let \( D \) be simply connected and let all points of \( \partial D \) be accessible. Let four points be picked on \( \partial D \), so as to divide \( \partial D \) into four parts, \( \alpha_1, \beta_1, \alpha_2, \beta_2 \), in this order. \( D \) is then called a quadrangle. Let \( \Gamma \) be the family of curves joining \( \alpha_1 \) to \( \alpha_2 \) within \( D \). Then

\[ \lambda(\Gamma) = \lambda_\varrho(\alpha_1, \alpha_2) = \lambda(\alpha_1, \alpha_2) \]

is called the extremal distance between \( \alpha_1 \) and \( \alpha_2 \) in \( D \).

We now list a few well-known properties of extremal lengths and distances [3.1, p. 115], [3.3, pp. 305–308]. We only discuss extremal lengths different from zero and infinity. Considering those \( \varrho \) for which \( \int_{\gamma} \varrho \, |dz| \geq 1 \), for all \( \gamma \in \Gamma \), \( \lambda(\Gamma) \) is defined by

\[ \lambda(\Gamma) = \sup_{\varrho} \frac{1}{A_\varrho}. \]

Such a \( \varrho \) will be called admissible with respect to \( \Gamma \).
Lemma 3.1. \( \lambda(\Gamma) \) is a conformal invariant.

Lemma 3.2. If \( \Gamma_1 \subset \Gamma_2 \), then \( \lambda(\Gamma_1) > \lambda(\Gamma_2) \).

The first two lemmata follow immediately from the definition.

Lemma 3.3. Let the \( \Gamma_k \) be in disjoint domains \( D_k, k = 1, 2, ..., n \). Let \( \Gamma = \{ \gamma \} \) be such that each \( \gamma \) contains at least one \( \gamma_k \in \Gamma_k \) for each \( k, k = 1, 2, ..., n \). Then

\[
\hat{\lambda}(\Gamma) \geq \sum_{k=1}^{n} \lambda(\Gamma_k).
\]

Proof. Let \( \varphi_k \) be admissible with respect to \( \Gamma_k, k = 1, 2, ..., n \). Let \( t_k, k = 1, 2, ..., n \), be positive numbers with \( \sum_{k=1}^{n} t_k = 1 \). Then \( \varphi = \sum_{k=1}^{n} t_k \varphi_k \) is admissible with respect to \( \Gamma \). The lemma is proved by choosing \( t_k = \lambda(\Gamma_k) \cdot (\sum_{k=1}^{n} \lambda(\Gamma_k))^{-1}, k = 1, 2, ..., n \).

Lemma 3.4. Let \( D \) be a rectangle with sides \( \alpha_1 \) and \( \alpha_2 \) of length \( a \), and sides \( \beta_1 \) and \( \beta_2 \) of length \( b \). Then

\[
\lambda(\alpha_1, \alpha_2) = ba^{-1}.
\]

Proof. Let \( D \) be \( \{ z \mid 0 < \text{Re} \, z < a, \, 0 < \text{Im} \, z < b \} \). Then, by the Schwarz inequality,

\[
L_{\alpha}^2 \leq \left( \int_0^b \varrho \, dy \right)^2 \leq b \int_0^b \varrho^2 \, dy.
\]

Integrating over \( x \) we obtain

\[
a L_{\alpha}^2 \leq b A_{\alpha}
\]

and hence

\[
\lambda(\alpha_1, \alpha_2) = ba^{-1}.
\]

Equality in (3.1) holds for \( \varrho = b^{-1} \). This proves the lemma.

Lemma 3.5. Let \( D \) be a quadrangle with \( \partial D \) divided into \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) in this order. \( \beta_1 \) and \( \beta_2 \) are assumed to be analytic arcs and \( \alpha_1 \) and \( \alpha_2 \) simple arcs. Let \( u \) be harmonic in \( D \) with boundary values 1 on \( \alpha_2 \), 0 on \( \alpha_1 \), and \( \partial u/\partial n = 0 \) on \( \beta_1 \) and \( \beta_2 \). Then

\[
\lambda^{-1}(\alpha_1, \alpha_2) = \iint_D |\text{grad} \, u|^2 \, dx \, dy.
\]

Proof. By Lemma 3.1 it is sufficient to prove this in the case of a rectangle. Let \( \alpha_1 \) in Lemma 3.4 be on the real axis. Then \( u = yb^{-1} \) and the lemma is correct.

Lemma 3.6. Let \( - \) denote reflection in the real axis. Let \( D \) be symmetric with respect to the real axis and \( \Gamma \) such that \( \gamma \in \Gamma \Rightarrow \bar{\gamma} \in \Gamma \). Then it is sufficient to consider \( \varphi \)'s symmetric with respect to the real axis to determine \( \lambda(\Gamma) \).

Proof. Let \( \varphi \) be admissible with respect to \( \Gamma \). Then \( \bar{\varphi} \) and \( \frac{1}{2}(\varphi + \bar{\varphi}) \) are also admissible. Furthermore
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\[ A_{\partial \Omega, 2} = \frac{1}{4} \int_{C} (q + \overline{q})^2 \, dy \leq \frac{1}{2} \int_{C} (q^2 + \overline{q}^2) \, dy = A_q. \]

This proves the lemma.

Next we collect the information that we shall need about elliptic integrals. We assume that \( 0 < k < 1 \). Define

\[ K(k) = \int_0^1 (1 - x^2)^{-1/2} (1 - k^2 x^2)^{-1} \, dx, \quad K'(k) = K((1 - k^2)^1), \quad (3.2) \]

and

\[ t(k) = \frac{K'(k)}{K(k)}, \quad (3.3) \]

Then

\[ t(k) \leq \frac{2}{\pi} \log \frac{4}{k}, \quad (3.4) \]

\[ t(k) - 2 \log \frac{4}{k} = A(k) k^2, \quad |A(k)| \leq C_\theta, \quad \text{when} \quad 0 \leq k \leq k_0 < 1. \quad (3.5) \]

(3.4) and (3.5) follow, for instance, from [3.4, p. 54] and are used by Hersch [3.3, pp. 316–319].

The following theorem giving an explicit relation between a harmonic measure and an extremal length was proved by Hersch [3.3, pp. 319–320].

**Theorem 3.1.** Let \( D \) be simply connected and let all points of \( \partial D \) be accessible. \( \partial D \) is divided into two connected parts \( \alpha \) and \( \beta \). \( z_0 \) is a fixed point in \( D \) and \( \omega = \omega(z_0; \alpha; D) \). Let \( \Gamma \) be the family of curves in \( D \) joining points on \( \alpha \) and separating \( z_0 \) from \( \beta \). \( t \) is defined by (3.2) and (3.3). Then

\[ \lambda(\Gamma) = 2t \left( \sin \frac{\pi \omega}{2} \right). \quad (3.6) \]

**Proof.** \( D \) is mapped conformally onto \( G \), the interior of the unit circle in the \( w \)-plane, so that \( z = z_0 \) corresponds to \( w = 0 \) and \( z = z_0 \) to \( \alpha_1 = \{ w \} \) \( -\pi \omega < \arg w < \pi \omega, \quad |w| = 1 \). \( \beta_1 = \partial G - \alpha_1 \) and \( \eta_1 = \{ w \} \) \( -1 < \Re w \leq 0, \Im w = 0 \). \( \Gamma_1 \) is the family of curves \( \gamma_1 \) in \( G \) joining points on \( \alpha_1 \) and separating \( w = 0 \) from \( \beta_1 \). \( C \) is the family of curves \( c \) joining \( \eta_1 \) and \( x_1 \) in \( G_1 = G - \eta_1 \).

By Lemma 3.6 it is sufficient to consider \( \phi \) symmetric with respect to the real axis to determine \( \lambda(\Gamma_1) \) and \( \lambda(C) \). A curve \( \gamma_1 \in \Gamma_1 \) contains two curves \( c' \) and \( c'' \) in \( C \). Let \( \overline{\phi} \) denote reflection in the real axis. Then, for a symmetric \( \phi \),

\[ \int_{\gamma_1} \overline{\phi} \, |dw| \geq \int_{c'} \overline{\phi} \, |dw| + \int_{c''} \overline{\phi} \, |dw| = \int_{c''} \overline{\phi} \, |dw| + \int_{c'} \overline{\phi} \, |dw| \]

\[ \geq \min \left( \int_{c''} \overline{\phi} \, |dw|, \int_{c'} \overline{\phi} \, |dw| \right). \]

Hence

\[ L_{\phi}(\Gamma_1) = 2 L_{\phi}(C) \]

and

\[ \lambda(\Gamma_1) = 4 \lambda(C). \quad (3.7) \]
\( \lambda(C) - \lambda_{\gamma_1}(\gamma_1, \alpha_1) \) is now determined by a conformal mapping of the quadrangle \( G_1 \) onto a rectangle. By virtue of conformal invariance the theorem follows.

The following corollary will be useful in the next paragraph.

**Corollary 3.1.** Let \( D_\xi \) (containing \( z_0 \)) satisfy the assumptions of Theorem 3.1 with \( \theta_\xi = \alpha \) and let \( \partial D_\xi \) be piecewise smooth. \( D_\xi \) is mapped conformally onto \( G = \{ w \mid |w| < 1 \} \) so that \( z = z_0 \) corresponds to \( w = 0 \) and \( \theta_\xi \) to \( \{ w \mid -\pi \omega < \arg w < \pi \omega, \; |w| = 1 \} \), where \( \omega = \omega(z_0, \theta_\xi; D_\xi) \). \( \alpha_1 \) denotes \( \{ w \mid 0 \leq \text{Re} w < 1, \; \text{Im} w = 0 \} \) and \( z \) is the image of \( \alpha_1 \) in \( D_\xi \). Let \( u \) be harmonic in \( D_\xi - \alpha \) with boundary values 1 on \( \alpha \), 0 on \( \partial D_\xi - \theta_\xi \), and \( \partial u/\partial n = 0 \) on \( \theta_\xi \) (except at the endpoint of \( \alpha \)). Then

\[
2t \left( \sin \frac{\pi \omega}{2} \right) = \int \int_{D_\xi - \alpha} |\nabla u|^2 \, dx \, dy.
\]

**Proof.** We consider \( G \). Then by (3.6) and (3.7)

\[
\lambda_{\gamma_1}(\alpha_1, \beta_1) = \frac{1}{2} t \left( \sin \frac{\pi (1 - \omega)}{2} \right) = \frac{1}{2} t \left( \sin \frac{\pi \omega}{2} \right).
\]

The corollary now follows by Lemma 3.1 and Lemma 3.5.

**Corollary 3.2.** With the notation of Theorem 3.1

\[
\omega(z_0; x; D) \leq 4 \exp \left( -\frac{\pi}{4} \lambda(\Gamma) \right).
\]

**Proof.** This follows from (3.6) and (3.4).

**Theorem 3.2.** Let \( D \) be simply connected and let all points of \( \partial D \) be accessible. Assume that \( D \) has no boundary point at infinity with finite \( \text{Re} z \). Let \( D_\xi \) (containing \( z_0 = x_0 + iy_0 \)) be such that \( \theta_\xi \) consists of one segment. Let \( \theta_\xi, i = 1, 2, ..., n(x) \), separate \( z_0 \) and \( \theta_\xi \). Assume that \( n(x) = 0, \; x < x_0 \). Then
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\[ \omega(z_0; \theta; D_{\xi}) \leq 4 \exp \left( -\pi \int_{x_0}^{x} \left( \sum_{i=1}^{n(x)} \frac{1}{\theta_i(x)} \right) dx \right). \]

**Proof.** The \( \theta_i \) cover the shaded area in Fig. 3.2. Let \( \Gamma \) be the family of curves in \( D_{\xi} \) joining points on \( \theta_i \) and separating \( z_0 \) from \( \partial D_{\xi} - \theta_i \). We use Corollary 3.2 with \( x = \theta \) and \( D = D_{\xi} \). Thus

\[ \omega(z_0; \theta; D_{\xi}) \leq 4 \exp \left( -\pi \frac{1}{4} \lambda(\Gamma) \right). \]

A simple estimate of \( \lambda(\Gamma) \) is obtained by the following choice of \( \varphi \):

\[ \varphi(z) = \begin{cases} 1, & z \in \theta_i, \; i = 1, 2, \ldots , n(x), \; x_0 < x < \xi, \\ 0, & \text{otherwise}. \end{cases} \]

We note that each \( \theta_i(x) \) is lower semicontinuous. Now

\[ \lambda(\Gamma) \geq L^0_{\varphi} A_{-1} \geq 4 \int_{x_0}^{\xi} \left( \sum_{i=1}^{n(x)} \frac{1}{\theta_i(x)} \right) dx, \]

and the theorem is thus proved.

**Remark 1.** The possibility of such a choice of metric (in the case of \( n(x) = 1 \)) was noted by Hersch [3.3, pp. 325–326].

**Remark 2.** Let the \( D \) in Theorem 3.2 have a boundary point \( B \) such that \( \text{Re} \; z \to +\infty \), \( z \to B \), \( z \in D \). Let \( \theta_i \), \( i = 1, 2, \ldots , n(x) \), separate \( z_0 \) and \( B \). \( D_{\xi} \) is the subdomain of \( D \) separated from \( B \) by \( \theta_i \). We drop the assumption that \( n(x) = 0, x < x_0 \). Let \( I \) be the interval generated by the \( \theta_i \) separating \( z_0 \) and a fixed \( \theta_i \). Then

\[ \omega(z_0; \theta_i; D_{\xi}) \leq 4 \exp \left( -\pi \int_{I} \left( \sum_{i=1}^{n(x)} \frac{1}{\theta_i(x)} \right) dx \right), \]

where ' means that the sum is to be taken over the \( \theta_i \) separating \( z_0 \) and \( \theta_i \).

**Remark 3.** The estimate of \( \omega(z_0; \theta_i; D_{\xi}) \) in Theorem 3.2 in terms of the lengths of the \( \theta_i \) separating \( z_0 \) and \( \theta_i \) can be generalized to higher dimensions, cf. Theorem 6.2.

**Remark 4.** It does not appear possible to extend the method of § 3 to higher dimensions. In Theorem 3.1 \( \lambda(\Gamma) \) differs little from \( 4\lambda_D(\alpha), \alpha \), when \( D \) is a quadrangle with two opposite boundary arcs \( \alpha \) and \( \alpha \), such that the distance between \( \alpha \) and \( \alpha \) is large in comparison with the length of \( \alpha \) and \( z_0 \) is near \( \alpha \). Now let \( G \) be a vertical right cylinder of height \( b \) and let the area of the two horizontal sides \( \alpha \) and \( \alpha \) be \( A \). Then, in analogy with Lemma 3.4, \( \lambda_C(\alpha, \alpha) = bA^{-1} \). However, the harmonic measure of \( \alpha \) with respect to \( G \) depends not only on the size but also on the shape of a cross-section of the cylinder.
4. Symmetrization results for harmonic measures

Symmetrization with respect to a straight line is defined in §1. The following two theorems are proved with the aid of Corollary 3.1. Another method to prove symmetrization results is discussed in §8.

**Theorem 4.1.** Let $D_\zeta$ (containing $z_0$) be bounded by a piecewise smooth simple closed curve. Let $\theta_\zeta$ consist of one segment. * denotes symmetrization with respect to the real axis. Then

$$\max_{\text{Re } z_0 = x_0} \omega(z_0; \theta_\zeta; D_\zeta) \leq \omega(x_0; \theta_\zeta^*; D_\zeta^*).$$

Equality holds if and only if $D_\zeta$ is a translate of $D_\zeta^*$.

**Proof.** We use the same notation as in Corollary 3.1. Let $\kappa$ be the image of $\kappa_1$ in $D_\zeta$. The slit $\kappa' = \{z | x_0 < \text{Re } z < x, \text{Im } z = 0\}$ is the image of $\kappa_1$ in $D_\zeta^*$. Let $u$ be harmonic in $D_\zeta - \kappa$ with boundary values 1 on $\kappa$, 0 on $\partial D_\zeta - \theta_\zeta$, and $\partial u/\partial n = 0$ on $\theta_\zeta$ (except at the endpoint of $\kappa$). Let $v$ be harmonic in $D_\zeta^* - \kappa'$ with boundary values 1 on $\kappa'$, 0 on $\partial D_\zeta^* - \theta_\zeta^*$, and $\partial v/\partial n = 0$ on $\theta_\zeta^*$ (except at the endpoint of $\kappa'$). Then by (3.8),

$$2t \left( \sin \frac{\pi \omega^*}{2} \right) = \int \int_{D_\zeta - \kappa} |\text{grad } u|^2 \, dx \, dy,$$

$$2t \left( \sin \frac{\pi \omega}{2} \right) = \int \int_{D_\zeta^* - \kappa'} |\text{grad } v|^2 \, dx \, dy,$$

where $\omega = \omega(z_0; \theta_\zeta; D_\zeta)$ and $\omega^* = \omega(x_0; \theta_\zeta^*; D_\zeta^*)$.

Let $u$ be symmetrized with respect to the real axis. The symmetrized function is $u^*$. $\kappa$ corresponds to $\kappa^*$ on the real axis so that $u^* = 1$ on $\kappa^*$. It is possible that $\kappa^*$ extends to the left of $x_0$. We now reflect $D_\zeta$ and $D_\zeta^*$ in $l = \{z | \text{Re } z = x\}$ and denote the reflected domains by $D_\zeta$ and $D_\zeta^*$. Set $G = D_\zeta \cup \theta_\zeta \cup D_\zeta^*$ and $G^* = D_\zeta^* \cup \theta_\zeta^* \cup D_\zeta$. $\kappa$, $\kappa'$, and $\kappa^*$ (including endpoints) are also reflected in $l$, and $s$, $s'$, $s^*$ denote the unions of the given slits and their reflections. By reflection in $l$ $u$ is defined to be harmonic in $G - s$ and $v$ is defined to be harmonic in $G^* - s'$. The domain of $u^*$ is also extended in this way. According to a result of Pólya and Szegő [4.4, p. 186–187]

$$\int \int_{G - s} |\text{grad } u|^2 \, dx \, dy \geq \int \int_{G^* - s^*} |\text{grad } u^*|^2 \, dx \, dy. \quad (4.1)$$

When $s^* - s' = \phi \partial v/\partial n = 0$ on $s^* - s'$. By Dirichlet’s principle (with free boundary values)

$$\int \int_{G^* - s'} |\text{grad } u^*|^2 \, dx \, dy \geq \int \int_{G^* - s'} |\text{grad } v|^2 \, dx \, dy$$

and hence

$$\int \int_{G - s} |\text{grad } u|^2 \, dx \, dy \geq \int \int_{G^* - s'} |\text{grad } v|^2 \, dx \, dy. \quad (4.2)$$

From this we obtain
Since \( t(k) \) is strictly increasing, \( 0 < k < 1 \), we obtain the result \( \omega \leq \omega^* \).

The Dirichlet integrals in (4.2) are the inverted values of the modules of the doubly connected domains \( G - s \) and \( G^* - s' \). These modules are (without restrictions in the boundary assumptions) equal if and only if \( G - s \) is a translate of \( G^* - s' \). This follows from a result by Jenkins [4.2, p. 106, p. 115]. Theorem 4.1 is now proved.

**Remark 1.** We mention another possibility of discussing equality in Theorem 4.1. This is to make a detailed examination of a proof of (4.1) along the lines of a proof given in [4.1, pp. 416–419]. Such an investigation was made by Ohtsuka [4.3, pp. 202–205] in the case of circular symmetrization (cf. the following remark). By the result of Jenkins above the case of equality can be settled for more general domains.

**Remark 2.** Theorem 4.1 can also be formulated for circular symmetrization with respect to the positive real axis. To define circular symmetrization, in the definition of symmetrization in § 1 straight lines are replaced by circles with their centres at the origin [4.4, pp. 193–195].

**Theorem 4.2.** Let \( D_\xi \) (containing \( z_0 = x_0 \)) be bounded by a piecewise smooth closed curve. Let \( \theta_\xi \) consist of one segment. \( D_\xi \) is assumed to be symmetric with respect to the real axis. \( D_\xi \) is reflected in \( l: \text{Re } z = \xi \). The reflected domain is \( D^*_\xi \) and \( G = D_\xi \cup \theta_\xi \cup D_\xi^* \). \( G \) is symmetrized with respect to \( l \), the symmetrized domain being \( G^*; \theta_\xi^* = \{ z \mid z \in G^*, \text{Re } z = \xi \} \) and \( D_\xi^* = \{ z \mid z \in G^*, \text{Re } z < \xi \} \). Then

\[
\omega(x_0; \theta_\xi; D_\xi) \leq \omega(x_0; \theta_\xi^*; D_\xi^*)
\]

with equality if and only if \( D_\xi = D_\xi^* \).

**Proof.** We use Corollary 3.1 and its notation again. Let \( \kappa \) be the image of \( \kappa \) in \( D_\xi \). Let \( u \) be harmonic in \( D_\xi - \kappa \) with boundary values 1 on \( \kappa \), 0 on \( \partial D_\xi - \theta_\xi \), and \( \partial u / \partial n = 0 \) on \( \theta_\xi \) (except at \( z = \xi \)). Set \( \kappa^* = \{ z \mid |\text{Re } z - \xi| \leq \xi - x_0, \text{Im } z = 0 \} \). By reflection in \( l \), \( u \) is defined to be harmonic in \( G - \kappa^* \). Let \( v \) be harmonic in \( G^* - \kappa^* \) with boundary values 1 on \( \kappa^* \) and 0 on \( \partial G^* \). Then by (3.8)
\[ 4t \left( \sin \frac{\pi x}{2} \right) = \int \int_{\Gamma_{\infty}^*} |\text{grad } u|^2 \, dx \, dy, \]
\[ 4t \left( \sin \frac{\pi x^*}{2} \right) = \int \int_{\Gamma_{\infty}^*} |\text{grad } v|^2 \, dx \, dy, \]

where \( \omega = \omega(x_0; \theta_1; D_1) \) and \( \omega^* = \omega(x_0^*; \theta_1^*; D_1^*). \)

Let \( u \) be symmetrized with respect to \( l \). The symmetrized function is \( u^* \). We then obtain in the same way as in the proof of Theorem 4.1
\[ \int \int_{\Gamma_{\infty}^*} |\text{grad } u|^2 \, dx \, dy \geq \int \int_{\Gamma_{\infty}^*} |\text{grad } v|^2 \, dx \, dy. \]

The proof can now be completed in the same way as in the proof of Theorem 4.1.

5. An application to comb domains

Let a simply connected domain \( D \) satisfy the following conditions. \( \Theta_2 = \emptyset \) outside \( A < x < B \) (\( -\infty < A < B < \infty \)). \( \Theta_2 = \{z \mid \Re z = x, \, |\Im z| < \infty \} \) for all \( x \) in \( A < x < B \) except \( x = x_m \). The number of points \( x_m \) in a finite interval is finite. Each \( \Theta_{x_m} \) consists of one bounded line-segment. We then call \( D \) a comb domain.

First we mention an explicit example illustrating § 2. Let \( D \) be bounded by the straight lines \( \{z \mid \Re z = x_m = -2ma, \, |\Im z| > b\}, \, m = 1, 2, ..., (a > 0) \) and the imaginary axis, where \( \mathcal{Z} = \{z \mid \Re z = 0, \, |\Im z| < b\} \). \( \omega(z; x; D) \) can be determined explicitly when \( z = x_m \). We write \( \exp(-\pi ba^{-1}) = k \) and use the notation in (3.2) and (3.3).

By a conformal mapping of \( \{z \mid -2a < \Re z < 0\} \) onto a rectangle \( \{w \mid |\Re w| < kK, \, 0 < \Im w < kK'\} \) in the \( w \)-plane and by analytic continuation, we obtain a conformal mapping of \( D \) onto a strip \( \{w \mid |\Re w| < kK', \, 0 < \Im w < kK'\} \). Hence
\[ \omega(x_m; \alpha; D) = \frac{4}{\pi} \arctg \exp \frac{\pi x_m}{\alpha(k)}. \]

When \( a \to 0 \), the term on the right tends to \( 4\pi^{-1} \arctg \exp (\pi x_m/2b) \), by (3.5).

Now let \( G \) be a domain such that \( \Theta_2 = \{z \mid \Re z = x, \, |\Im z| < \Theta(x)/2\}, \, -\infty < x < 0 \), and \( \Theta_2 = \emptyset, \, x \geq 0 \). Let \( \inf \Theta(x) = 2b \) be attained at the points \( x_m, \, m = 1, 2, ... \). Then \( \omega(x_m; x; G) < \omega(x_m; x; D) \). For small values of \( a \), the above estimate for \( \omega(x_m; x; G) \) can, for suitably chosen \( \Theta(x) \), be considerably smaller than \( c \exp (-\pi \int_{x_m}^\infty dx/\Theta(x)) \).

Theorem 5.1. Let the comb domain \( D \) be bounded by the lines \( \{z \mid \Re z = x_m, \, |\Im z| > b_m\}, \, m = 1, 2, ..., 0 > x_1 > x_2 > ..., \) and the imaginary axis. Set \( z = \{z \mid \Re z = 0, \, |\Im z| < b_0\}, \, \Theta_m = \{z \mid \Re z = x_m, \, |\Im z| < b_m\}, \) and \( b_m = \max (b_m, b_{m-1}), \, m = 1, 2, ... \). Given \( x \), let \( n \) be such that \( x_{n+1} < x < x_n \). Assume that
\[ \sum_{m=1}^n \left( \frac{x_{m-1} - x_m}{b_m} \right)^2 < M. \]

Then there is a constant \( c \) such that
\[ \omega(x; \alpha; D) \leq c \exp \left( -\pi \sum_{m=1}^n \frac{x_{m-1} - x_m}{2b_m} \right). \]
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Proof. \( \omega = \omega(x; z; D) \). Let \( \Gamma \) be the family of curves in \( D \) joining points on \( z \) and separating the point \( z = x \) from \( \partial D - z \). By Corollary 3.2

\[
\omega \leq 4 \exp \left( -\frac{\pi}{4} \lambda(\Gamma) \right).
\]

Let \( C \) be the family of curves in \( \{z \in D \mid \Re z > x_n\} \) joining \( \Theta_n \) and \( x \). By the same reasoning as in the proof of (3.7),

\[
\lambda(\Gamma) \geq 4 \lambda(C).
\]

We now estimate \( \lambda(C) \). Set \( x_0 = 0 \) and \( \Theta_0 = x \). Let \( D_m \) denote \( \{z \mid x_m < \Re z < x_{m-1}\} \), \( m = 1, 2, \ldots \). By Lemma 3.3

\[
\lambda(C) \geq \sum_{m=1}^{n} \lambda_{D_m}(\Theta_m, \Theta_{m-1}).
\]

Set \( \Theta'_m = \{z \mid \Re z = x_m, \Im z < b'_m\} \) and \( \Theta'_{m-1} = \{z \mid \Re z = x_{m-1}, \Im z < b'_m\} \), where \( b'_m = \max(b_m, b'_{m-1}), m = 1, 2, \ldots, n \). By Lemma 3.2

\[
\lambda_{D_m}(\Theta'_m, \Theta'_{m-1}) \geq \lambda_{D_m}(\Theta'_m, \Theta'_{m-1}).
\]

\( \lambda(\Theta'_m, \Theta'_{m-1}) \) is determined explicitly by a conformal mapping of \( D_m \) onto a rectangle. We write

\[
k_m = \exp \left( -\frac{2\pi b'_m}{x_{m-1} - x_m} \right)
\]

and define \( t_m \) by (3.2) and (3.3). Then

\[
\lambda_{D_m}(\Theta'_m, \Theta'_{m-1}) = 2t_m^{-1}
\]

and hence

\[
\omega \leq 4 \exp \left( -\frac{2\pi}{\sum_{m=1}^{n} t_m^{-1}} \right).
\]

By (3.5)

\[
2t_m^{-1} = \frac{x_{m-1} - x_m}{2b'_m} + \left( \frac{x_{m-1} - x_m}{2b'_m} \right)^2 B_m,
\]

where \( |B_m| \) is less than a constant depending only on \( M \).

Remark. If, in Theorem 5.1, \( 2b_m = \Theta(x_m) \) for a suitable continuous \( \Theta(x) \), we can, by a limiting process, obtain a special case of Theorem 3.2.

Chapter II. The general case

6. Carleman's method

This method was first used by Carleman in 1933 in a proof of Denjoy's conjecture concerning the number of finite asymptotic values of an integral function of finite order [6.1]. Denjoy's conjecture had been proved earlier by Ahlfors who used his distortion inequality (2.3). An account of Carleman's method in two dimensions
is given in two text-books [6.4, pp. 219-224] and [6.6, pp. 121-126]. The growth of harmonic and subharmonic functions of \( n \) variables has been investigated by several authors with Carleman’s method. Besides the references mentioned here in the text see [6.9]-[6.32] in the bibliography. In these investigations the main problem has been to study the growth of a given function; constants appearing in the estimates are allowed to depend on the function under consideration. However, a direct application to harmonic measures was given by Tsuji [6.8, pp. 112-117] in polar coordinates in the plane, cf. Lemma 6.7.

Carleman’s method consists in establishing a differential inequality for the Carleman mean (6.1) of a harmonic function. This can also be interpreted as a differential inequality for a certain Dirichlet integral. In Theorem 6.1 Carleman’s method is applied to harmonic measures. The proof of Theorem 6.1 follows—with some alterations—a proof given by Dinghas [6.3, pp. 3-9]. We use Theorem 6.1 and some lemmata to establish the estimates of harmonic measures in Theorem 6.2 and Theorem 6.3. In the case \( n>2 \) these estimates are new.

We assume that \( D \) satisfies the condition \( A \) below.

\textbf{A.} \( D \) is such that \( \Theta_x = \phi \) for \( x < 0 \) and \( \Theta_x = \Phi \) for \( x > 0 \). \( D \) is bounded by a finite number of piecewise smooth surfaces. \( D \) has no boundary point at infinity for which \( x_1 \) is finite. \( \Theta(x) \) is bounded and \( \leq M \) for all \( x > 0 \). Set \( \theta_0 = \{ z | x_1 = 0, z \in \partial D \} \). The measure of \( \theta_0 \) is positive.

To begin with, we collect information about the principal eigenvalues in \( B \).

\textbf{B.} Let the domain \( G \) in \( \mathbb{R}^n \) be bounded by a finite number of piecewise smooth surfaces. Let \( V \) be the class of functions \( f \) such that

1. \( f \) is continuous in \( G \cup \partial G \) and piecewise continuously differentiable in \( G \),
2. \( f(z) = 0, z \in \partial G \),
3. \( f(z) \neq 0 \).

Consider the variational problem of minimizing in \( V \) the Rayleigh quotient

\[ R(f) = \frac{\int_G |\text{grad } f|^2 dz}{\int_G f^2 dz} \]

Let \( v \) be the first (normed) eigenfunction of \( \Delta v + \lambda v = 0 \) in \( G, v = 0 \) on \( \partial G \), and \( \lambda \) the principal eigenvalue. It is well known that the Rayleigh quotient is minimized in \( V \) by \( v \) and that the minimum is \( \lambda \) [6.2, p. 399]. \( \lambda \) decreases when \( G \) increases [6.2, p. 409]. \( \lambda \) varies continuously with \( G \) [6.2, p. 423].

Furthermore \( \lambda \) decreases when \( G \) is symmetrized with respect to an \((n-1)\) dimensional hyperplane (cf. §1) [6.5, p. 419]. We also mention the Faber–Krahn inequality. Let \( A \) be the volume of \( G \). Let \( V_n \) be the volume and \( \Lambda_n \) the principal eigenvalue of the \( n \)-dimensional unit sphere. Then [6.5, p. 413]

\[ A^{2/n} \lambda \geq \Lambda_n V_n^{2/n}. \]

For a domain \( G \) with a less regular boundary, we define \( \lambda \) as \( \inf_\Omega \lambda \), where \( \Omega \) is of the type considered above and \( \Omega \subset G \).
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Let \( z_0 = (x_0, y_0) \) be a fixed point in \( D \). We write

\[ u(z) = \omega(z; \theta_0; D). \]

We shall establish upper bounds for \( u(z) \) at \( z = z_0 \) by studying the Carleman mean \( q \), defined by

\[ q(x) = \int_{\theta_x} u^2(x, y) \, dy, \quad 0 < x < \xi. \tag{6.1} \]

The definition of \( q \) is completed by setting \( q(0) = 0 \) and \( q(\xi) = \theta(\xi) \).

C. We define \( \lambda(x) \) in the following way. Consider a decreasing sequence, \( \{\varepsilon\} \), of positive numbers with limit zero. We choose the \( \varepsilon \) so that \( \text{grad } u = 0 \) on the surfaces \( u = \varepsilon \) in \( D_\varepsilon \). (The number of values \( a \) for which \( \text{grad } u \) vanishes at points of the equipotential set \( u = a \) in \( D_\varepsilon \) is enumerable [6.7, p. 276].) We write \( u_\varepsilon = \max (u - \varepsilon, 0) \) and set \( \theta_\varepsilon = \{x \mid x_1 = x, u_\varepsilon(z) > 0\} \). \( \theta_\varepsilon \) consists of a finite number of components \( \theta^i \). To each \( \theta^i \) we define \( \lambda_i^\varepsilon(x) \) according to B. Finally we define

\[ \lambda_\varepsilon(x) = \min_i \lambda_i^\varepsilon(x), \quad \lambda(x) = \lim_{\varepsilon \to 0} \lambda_\varepsilon(x). \]

The existence of \( \lambda(x) \) will be established in the proof of Theorem 6.1.

Let \( \lambda \) be defined by C. Then we define \( \psi \) by

\[ \psi(x) = \int_0^\pi \exp \left( 2 \int_0^x \lambda^1(u) \, du \right) \, dt. \tag{6.2} \]

We can now state

**Theorem 6.1.** Let \( D \) satisfy the assumptions A. Let \( \varphi \) and \( \psi \) be defined by (6.1) and (6.2). Then \( \varphi \) is a convex function of \( \psi \) for \( 0 \leq x \leq \xi \) and

\[ \varphi(x) \leq \psi(x) \theta(\xi) \psi^{-1}(\xi). \tag{6.3} \]

To prove Theorem 6.1 we consider

\[ \varphi_\varepsilon(x) = \int_{\theta_\varepsilon} u_\varepsilon^2(x, y) \, dy. \]

In the following lemmata 6.1–6.4 we drop the index \( \varepsilon \). This means that we work under the assumptions that \( u = 0 \) on \( \partial D \) and that \( u \) is harmonic in a neighbourhood of each point of \( \partial D \) for which \( x_1 < \xi \). Actually \( \vartheta_{x, \varepsilon} = \phi \) for \( x < x_\varepsilon \), \( \vartheta_{x, \varepsilon} = \theta \) for \( x > x_\varepsilon > 0 \), but when dropping the index \( \varepsilon \) we also write 0 instead of \( x_\varepsilon \).

**Lemma 6.1.** \( \theta(x) \) is continuous and \( \lambda(x) \) is upper semicontinuous, \( 0 < x < \xi \).

**Proof.** For simplicity we treat only the case \( n = 3 \). Suppose that \( \theta(x) \) is discontinuous at \( x = \sigma \). Then \( S = \{x \mid u(x) = 0\} \) contains a surface element of positive measure in the plane \( x_1 = \sigma \). This is impossible, since an analytic surface has at most a finite number of points in common with any straight line (not contained in the surface). Thus \( \theta(x) \) is continuous.
When the plane \( x_1 = a \) is not tangent to \( S \), each component \( \partial^i_\sigma \) of \( \partial_\sigma \) is bounded by a finite number of analytic curves, and \( \lambda_i(\sigma) \) is the principal eigenvalue of \( \partial^i_\sigma \). If the plane \( x_1 = a \) is tangent to \( S \), \( \partial^i_\sigma \) may for instance contain isolated points. In such a case \( \lambda_i(\sigma) \) is taken to be \( \inf \Omega \lambda_\Omega \), where \( \Omega \) is contained in \( \partial^i_\sigma \) and \( \partial \Omega \) consists of a finite number of analytic curves. Then \( \lambda_i(\sigma) = \inf R(f) \) for \( f \in V \) in \( \partial^i_\sigma \).

Given a number \( A > \lambda(\sigma) = \lambda_i(\sigma) \), we can choose \( \Omega \) so that \( A \) is the principal eigenvalue of \( \Omega \) and \( \Omega \) is strictly contained in \( \partial^i_\sigma \). Then \( \Omega \) is contained in some component of \( \partial_\sigma \) for all \( x \) sufficiently near \( \sigma \). Thus \( A > \lambda(x) \) for all \( x \) sufficiently near \( \sigma \) and \( \lambda \) is upper semicontinuous.

**Lemma 6.2.** \( q'(x) = 2 \int_{\partial_x} uu dy, \ 0 < x < \xi. \)

**Proof.** Let \( \partial^i_x \setminus \partial_x \) be the set of points \( y \) belonging to \( \partial_x \) but not to \( \partial_x \). Then

\[
q(x + h) - q(x) = \int_{\partial_x + h} u^2(x + h, y) dy - \int_{\partial_x} u^2(x, y) dy = \int_{\partial_x} (u^2(x + h, y) - u^2(x, y)) dy
\]

\[
+ \int_{\partial_x + h \setminus \partial_x} u^2(x + h, y) dy - \int_{\partial_x \setminus \partial_x + h} u^2(x + h, y) dy = I_1 + I_2 + I_3.
\]

Dividing \( I_1 \) by \( h \) and letting \( h \) tend to zero we obtain \( 2 \int_{\partial_x} uu dy \). Now consider \( I_2 \) for small \( h \). To each point \( (x + h, y) \), \( y \in \partial_x + h \setminus \partial_x \), belongs some point \( (s, y) \), \( x + s \leq x + h \), such that \( u(s, y) = 0 \). As the measure of \( \partial_x + h \setminus \partial_x \) is \( O(1) \), \( I_2 \) is \( O(h^2) \). In the same way \( I_3 \) is \( O(h^2) \). This proves the lemma.

**Lemma 6.3.** \( q''(x) = 2 \int_{\partial_x} |\text{grad } u|^2 dy, \ 0 < x < \xi. \)

**Proof.** By Lemma 6.2 and Green’s formula

\[
q'(x) = 2 \int_0^x \left( \int_{\partial_x} |\text{grad } u|^2 dy \right) dx.
\]

The integrand \( \int_{\partial_x} |\text{grad } u|^2 dy \) being continuous by Lemma 6.1, we obtain our lemma.

**Lemma 6.4.** (Carleman’s differential inequality.)

\[
q''(x) \geq 2q'(x) \lambda^4(x), \ 0 < x < \xi.
\]

**Proof.** By Lemma 6.3

\[
q''(x) = 2 \int_{\partial_x} u^2 dy + 2 \int_{\partial_x} |\text{grad } u|^2 dy, \ 0 < x < \xi.
\]

The first integral is estimated by applying the Schwarz inequality to \( q'(x) \) in Lemma 6.2. Thus

\[
\int_{\partial_x} u^2 dy \geq \frac{q'^2}{4q}.
\]
The second integral is estimated by $B$.

$$\int_{\Theta} \|\nabla_x u\|^2 dy \geq \lambda_i(x) \int_{\Theta} u^2 dy \geq \lambda(x) \int_{\Theta} u^2 dy.$$ 

Summing over $i$ we obtain

$$\int_{\Theta} \|\nabla_x u\|^2 dy \geq \lambda(x) \Phi(x).$$

Hence

$$\frac{2\Phi''}{\Phi} \geq \frac{\Phi'^2}{\Phi^2} + 4\lambda, \quad 0 < x < \xi.$$  

(6.5)

By the proof of Lemma 6.3 $\Phi'$ is a Dirichlet integral and thus positive for $0 < x < \xi$. By the inequality $(\Phi''/\Phi' - \Phi'/\Phi)^2 \geq 0$ and taking the square root the lemma now follows from (6.6).

**Proof of Theorem 6.1.** We now use the index $\varepsilon$ again. By $B$ and Lemma 6.1 $\{\lambda_i^\varepsilon\}$ is a decreasing sequence of integrable functions. $\lambda_i^\varepsilon = \lim \lambda_i^\varepsilon$ is integrable over $0 < x < \xi$. (Integrability at $x=0$ is guaranteed by the assumption that the measure of $\partial \Sigma$ is positive.) By Lemma 6.4

$$\Phi_i^\varepsilon(x) \geq 2\Phi_i^\varepsilon(x) \lambda_i^\varepsilon(x) \geq 2\Phi_i^\varepsilon(x) \lambda_i^\varepsilon(x), \quad \varepsilon < x < \xi.$$ 

For $\Phi$ defined by (6.2)

$$\Phi''(x) = 2\lambda_i^\varepsilon(x) \Phi'(x) \text{ a.e.}$$

Hence

$$\frac{d}{dx} (\log \Phi' - \log \Phi') \geq 0 \text{ a.e.}$$

Since $\log \Phi_i^\varepsilon$ and $\log \Phi_i$ are absolutely continuous on an interval $\alpha < x < \beta$, $\varepsilon < \alpha < \beta < \xi$, $\Phi_i^\varepsilon$ is thus a convex function of $\Phi$ on any such interval. By continuity at $x=0$ and $x=\xi$, $\Phi = \lim \Phi_i^\varepsilon$ is a convex function of $\Phi$ for $0 \leq x \leq \xi$. Thus (6.3) is true and Theorem 6.1 is proved.

**Remark.** (Asymptotic equality in Carleman’s differential inequality.)

Equality in (6.4) holds if and only if $u=f\nu$, where $f$ depends only on $x$. When $\partial \Sigma$ is connected and $\partial \Sigma$ smooth, equality holds in (6.5) if and only if $u=g\nu$, where $g$ depends only on $x$ and $\nu$ is the first eigenfunction of $\Delta \nu + \lambda \nu = 0$ in $\partial \Sigma$, $\nu=0$ on $\partial \Sigma$.

Now let $D_\delta$ be a right cylinder, $D_\delta = \Sigma | \Sigma \times \Sigma, y \in \Theta$, where $\Theta$ is simply connected and has a smooth boundary. Let $\{\lambda_j^\delta\}_{j=1}^\infty$ be the eigenvalues ($\lambda_j^\delta = \lambda$), $\{v_j^\delta\}_{j=1}^\infty$ the corresponding eigenfunctions of $\Delta \nu + \lambda \nu = 0$ in $\Theta$, $\nu=0$ on $\partial \Theta$. Then, by the method of separation of variables (the $c_n$ denoting constants)

$$u(x, y) \sim \sum_{n=1}^\infty c_n \exp (\lambda_i^\delta(x-\xi)) v_n(y).$$

Thus, for large negative values of $x-\xi$,

$$u(x, y) \sim c_1 \exp (\lambda_i^\delta(x-\xi)) v_1(y),$$

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\[ u_z(x, y) \sim c_1 \lambda^k \exp \left( \lambda^1(x - \xi) \right) v_1(y). \]

\( \varphi \), as well as \( \varphi_x \), is now twice differentiable, and asymptotic equality in (6.4) and (6.5) holds for \( \varphi \).

Instead of defining \( \varphi \) as an integral over \( \partial_x \) as in (6.1) we now define \( \varphi \) as an integral over \( \theta_x \). In order to estimate \( \phi \) in this case we introduce the following additional assumptions about \( D \).

**D.** \( \Theta_z \) is connected, \( x < x_0 \). \( \partial \) is smooth. In any finite interval \( \partial D \) has a finite number of tangent hyperplanes \( x_1 = c \).

**E.** \( \lambda(x) \) is defined as in \( C \), but with respect to \( \theta_x \) instead of \( \theta_x \). (For \( x < x_0 \) \( \theta_x \) is understood to be equal to \( \Theta_x \)).

**Lemma 6.5.** Let \( D \) satisfy A and D. Then (6.3) in Theorem 6.1 is true for \( x-x_0 \) with \( \varphi \), \( \lambda \), and \( \psi \) defined with respect to \( \theta_x \) instead of \( \theta_x \).

**Proof.** The choice of \( \Theta_x \) is due to Tsuji [6.8, p. 112]. In the case of simply connected \( \Theta_x \) and \( D_x \) he takes into account only the first of the \( \Theta_x \) separating \( \Theta_x \) from \( \Theta_x \), cf. Ahlfors’ distortion inequality (2.3). However, some irrelevant components of \( \Theta_x \) not separating \( \Theta_x \) from \( \Theta_x \) also enter the discussion. In Fig. 6.1 the \( \Theta_x \) cover the shaded area. Let \( x_1 = a \) be those hyperplanes tangent to \( \partial D \) for which \( D_x \) is a proper subdomain of \( D_x \).

Let the \( a \) belonging to the interval \( 0 < x < \xi \) be \( a_1 < a_2 < \ldots < a_m \). Then

\[ \varphi_x(a_1) \leq \varphi_x(a_1 + 0), \quad \varphi_x(a_1 - 0) \leq \varphi_x(a_1 + 0), \quad \mu = 1, 2, \ldots, m. \]

Thus \( \varphi_x \) and \( d\varphi_x/d\psi \) have positive jumps at \( x - a_\mu, \mu = 1, 2, \ldots, m \). According to the proof of Theorem 6.1 \( \varphi_x \) is a convex function of \( \psi \) on intervals not containing any of the \( \sigma \). Thus, for \( x_1 < x < x_0 < \xi \)

\[ \varphi_x(x) \leq \varphi_x(x_0) \frac{\psi(x) - \psi(x_0)}{\psi(\xi) - \psi(x_0)}. \]

The lemma now follows by letting \( \epsilon \) tend to zero.

**Lemma 6.6.** \( u(z_0) \leq c_0 \psi^1(x_0), \) where \( z_0 = (x_0, y_0) \) and \( c \) depends only on the geometry of \( D \) near \( z_0 \).

**Proof.** This lemma is an immediate consequence of Harnack’s inequality for positive harmonic functions. If a closed sphere \( S(r; z_0) \) with radius \( r \) and centre at \( z_0 \) is contained in \( D \), then for \( z \in S(r/2; z_0) \)

\[ u(z) \geq \left( \frac{3}{2} \right)^{n-2} u(z_0), \]

and hence

\[ \varphi(x) \geq c_n r^{n-1} u^2(z_0), \]

where \( c_n \) only depends on \( n \). This proves the lemma.

We now discuss estimating \( \varphi \) from

\[ \varphi(x) \leq \varphi(x) \theta(\xi) \psi^{-1}(\xi). \quad (6.3) \]
Writing \( \mu \) instead of \( 2\lambda^1 \), we have instead of (6.2)

\[
\phi(x) = \int_0^x \exp \left( \int_0^t \mu(u) \, du \right) \, dt.
\]

Since \( \theta(x) \leq M \), it follows by the Faber-Krahn inequality that \( \mu(x) \geq m > 0 \). Hence

\[
\phi(x) = \exp \left( \int_0^x \mu(u) \, du \right) \int_0^x \exp \left( - \int_0^t \mu(u) \, du \right) \, dt \leq m^{-1} \exp \left( \int_0^x \mu(u) \, du \right).
\]

(6.7)

In the same way, assuming that \( \mu(x) \leq m_1 \), we obtain for \( \xi > m^{-1} \log 2 \)

\[
\phi(\xi) \geq (2m_1)^{-1} \exp \left( \int_0^\xi \mu(u) \, du \right)
\]

and hence

\[
\phi(x) \leq c \exp \left( \int_x^\xi \mu(u) \, du \right).
\]

(6.8)

It is, of course, unsatisfactory to require that \( \mu \) be bounded. In general the estimate (6.8) does not follow from (6.3). For instance take \( \theta(x) \equiv \theta(x) \equiv 1 \) and \( \mu(x) = 2x, x > 0 \). Then, for large \( \xi \)

\[
\phi(\xi) \sim (2\xi)^{-1} \exp \xi^2.
\]

A general result is given in the following lemma.

**Lemma 6.7.** Let \( D \) satisfy \( A \) and \( \lambda \) be defined by \( C \). Then for \( 0 < l < \xi - x_0 \), \( z_0 = (x_0, y_0) \),

\[
u(z_0) \leq cl^{-\xi} \exp \left( - \int_{z_0}^{\xi-1} \lambda^1(x) \, dx \right),
\]

where the constant \( c \) depends only on \( A \) and the geometry of \( D \) near \( z_0 \). If \( D \) also satisfies \( D \) this is true with \( \lambda \) defined by \( E \).

**Proof.** We use a trivial estimate of \( \phi(\xi) \). Writing \( \mu = 2\lambda^1 \), we have for \( l > 0 \)

\[
\phi(\xi) \geq \exp \left( \int_0^{\xi-l} \mu(u) \, du \right) \int_{\xi-l}^\xi \exp \left( \int_{\xi-l}^t \mu(u) \, du \right) \, dt \geq l \exp \left( \int_0^{\xi-l} \mu(u) \, du \right).
\]
Hence, by (6.3) and (6.7)

\[ \varphi(x_0) \leq e^{l^{-1}} \exp \left( - \int_{z_n}^{z_0} \mu(u) \, du \right). \]

Our lemma now follows from Lemma 6.5 and Lemma 6.6.

Remark. Tsuji proved this result in polar coordinates in the plane [6.8, pp. 112–117], but with a more complicated derivation of the estimate of \( \varphi \).

In Theorem 6.3 we shall estimate \( \varphi \) from (6.3) in some special cases. First, however, we shall make the same choice among the components of \( \varphi \) as in Theorem 3.2 with Carleman's method. By Lemma 6.2 and Green's formula \( \varphi /2 \) is a Dirichlet integral. By working in terms of convexity in the proof of Theorem 6.1 difficulties due to infinite Dirichlet integrals are avoided. Now we have to introduce auxiliary functions possessing finite Dirichlet integrals. We shall assume that \( D \) satisfies \( D \) and \( D \) \( F \).

F, \( \theta \xi \) is connected, and the diameter of \( D \xi - D \xi \) is less than a fixed constant.

G. Given \( \xi \), let \( \theta \xi \), \( i=1, 2, \ldots, n(x) \), denote the components or unions of components of \( \vartheta \xi \) separating \( \theta \xi \) from \( \theta \xi \). \( \lambda \xi \) is now defined with respect to \( \theta \xi \) in \( \mathcal{C} \). We write

\[ \lambda \xi = \sum_{i=1}^{n(x)} \lambda \xi_i(x). \]

Lemma 6.8. Let \( D \) satisfy \( A \) and \( D \) and let \( D \xi \) satisfy \( F \). Let \( \lambda \xi \) be defined by G. Then for \( z_0 = (x_0, y_0), \xi > x_0 + 1, \)

\[ u(z_0) \leq c \exp \left( - \int_{z_n}^{z_0} \lambda \xi(x) \, dx \right), \]  \[(6.9)\]

where the constant \( c \) depends only on \( A \), \( F \), \( x_0 \), and the geometry of \( D \) near \( z_0 \).

Proof. We first modify \( D \xi \). Since, by \( F \), the diameter of \( D \xi - D \xi \) is bounded, each \( \vartheta \xi \) with \( \xi - 1 \leq x \leq \xi \) is contained in an \((n - 1)\)-dimensional sphere of fixed radius \( R \) and centre \( Y \). Set \( G \xi = \{ |x| - 1 < x < \xi, \, |y - Y| < R \} \). We consider \( D \xi \cup G \xi \) instead of \( D \xi \), but do not change our notation. Thus \( \theta \xi = \{ |x| - 1 <\xi, \, |y - Y| < R \} \). Set \( \theta \xi = \{ |x| - 1 <\xi, \, |y - Y| < R \} \) and \( \theta \xi = \{ |x| - 1 <\xi, \, |y - Y| < 2R \} \). We choose a function \( F \), twice continuously differentiable and monotonic for \( \frac{1}{2} \leq t \leq \frac{3}{2} \), with \( F(1) = 1 \) and \( F(2) = 0 \).

Now let \( f \) be harmonic in \( D \xi \) with boundary values 0 on \( \partial D \xi - \partial \xi \), \( F(R, |y|) \) on \( \partial \xi - \partial \xi \), and 1 on \( \partial \xi \). We shall need an inequality of the following type:

\[ u(z) \leq c f(z). \]  \[(6.10)\]

Applying the method of separation of variables in the cylinder \( G \xi \), we obtain that \( u(z) \) and \( f(z) \) tend to zero in the same way when \( z \in \theta \xi - \frac{1}{2} \) tends to \( \partial \theta \xi - \frac{1}{2} \). Therefore we can, with the aid of Harnack’s inequality, establish the inequality (6.10) on \( \theta \xi - \frac{1}{2} \) and hence in \( D \xi - \frac{1}{2} \).

Let \( D \xi \) denote the subdomain of \( D \) separated from \( \theta \xi \) by \( \theta \xi \). We assume that the \( \theta \xi, \, i=1, 2, \ldots, n(x) \), are taken in such order that \( D \xi \subset D \xi + i, \, i=1, 2, \ldots, n(x) - 1 \).
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We now define

\[ D_{\varepsilon, \varepsilon} = \{ z \in D_{\varepsilon} \mid f(z) > \varepsilon \}, \quad \theta_{\varepsilon, \varepsilon} = \{ z \in \theta_{\varepsilon} \mid f(z) > \varepsilon \}, \]

\[ D'_{\varepsilon, \varepsilon} = \{ z \in D'_{\varepsilon} \mid f(z) > \varepsilon \}, \quad f_{\varepsilon} = \max (f - \varepsilon, 0), \]

and

\[ q_{\varepsilon, \varepsilon}(x) = \int_{\theta_{\varepsilon, \varepsilon}} |f_{\varepsilon}|^2 \, dy, \quad d_{\varepsilon, \varepsilon}(x) = \int_{D'_{\varepsilon, \varepsilon}} |\text{grad } f_{\varepsilon}| \, dz. \]

\{ \varepsilon \} is taken to be a sequence of the type considered in C (with respect to f).

Let \( x_{1} = \sigma \) be those hyperplanes tangent to \( \partial D \) for which one of the following situations occurs for some \( i \).

I. \( D'_{\varepsilon} \) is a proper subdomain of \( D_{\varepsilon + \varepsilon} \), and \( d_{\varepsilon, \varepsilon}(\sigma) \leq d_{\varepsilon, \varepsilon}(\sigma + 0) \).

II. \( D'_{\varepsilon} \) is a proper subdomain of \( D_{\varepsilon - \varepsilon} \), and \( d_{\varepsilon, \varepsilon}(\sigma) \leq d_{\varepsilon, \varepsilon}(\sigma - 0) \).

III. The interior of \( D'_{\varepsilon+1} - D'_{\varepsilon} \) contains no \( \theta_{\varepsilon} \) for \( j = 1, 2, \ldots, n(x) \), and \( d_{\varepsilon, \varepsilon}(\sigma) \leq d_{\varepsilon+1, \varepsilon}(\sigma) \). Cf. Fig. 3.2.

On an interval \( I \) (not containing any points \( \sigma \)) where \( D'_{\varepsilon} \) increases with \( x \)

\[ 2d_{\varepsilon, \varepsilon}(x) = q_{\varepsilon}(x) \]

by Lemma 6.2. Let \( \lambda_{1} \) be defined by \( G \). By virtue of Lemma 6.4 we obtain

\[ d_{\varepsilon, \varepsilon}(x) \geq 2\lambda_{1}(x) d_{\varepsilon, \varepsilon}(x). \quad (6.11) \]

On an interval \( J \) (not containing any points \( \sigma \)) where \( D'_{\varepsilon} \) increases with \( -x \) we obtain in the same way

\[ -d_{\varepsilon, \varepsilon}(x) \geq 2\lambda_{1}(x) d_{\varepsilon, \varepsilon}(x). \quad (6.12) \]

Now we run through \( D_{\varepsilon-1} \) from \( z_{0} \) to \( \theta_{\varepsilon-1} \) so that each \( \theta_{\varepsilon} \) is passed once. Then the Dirichlet integral of \( f_{\varepsilon} \) over \( D_{\varepsilon-1}, \varepsilon \) increases with \( D'_{\varepsilon-1, \varepsilon} \) on intervals \( I \) and \( J \) according to (6.11) and (6.12). At the points \( \sigma \) the Dirichlet integral of \( f_{\varepsilon} \) has non-negative increments as described in I–III. When running through \( D_{\varepsilon-1} \) in this manner we integrate in (6.11) and (6.12). Writing \( d_{\varepsilon} \) instead of \( d_{\varepsilon, \varepsilon} \) we obtain

\[ d_{\varepsilon}(x_{0}) \leq d_{\varepsilon}(\xi - 1) \exp \left( -2 \int_{x_{0}}^{\xi-1} \left( \sum_{i=1}^{n(x)} \lambda_{1}(x) \right) dx \right) \leq d_{\varepsilon}(\xi) \exp \left( -2 \int_{x_{0}}^{\xi-1} \lambda_{1}(x) \, dx \right). \]

By Green’s formula (with inner normal derivatives) we obtain

\[ d_{\varepsilon}(\xi) = -\int_{\theta_{\varepsilon}} f_{\varepsilon} \frac{\partial f_{\varepsilon}}{\partial n} \, dy - \int_{\theta_{\varepsilon}} f_{\varepsilon} \frac{\partial \varepsilon_{\varepsilon}}{\partial n} \, dy - \int_{\theta_{\varepsilon}} \varepsilon_{\varepsilon} \frac{\partial f_{\varepsilon}}{\partial n} \, dy, \]

when \( \varepsilon \rightarrow 0 \). By the maximum principle

\[ -\int_{\theta_{\varepsilon}} f_{\varepsilon} \frac{\partial f_{\varepsilon}}{\partial n} \, dy \leq -\int_{\theta_{\varepsilon}} \varepsilon_{\varepsilon} \frac{\partial \varepsilon_{\varepsilon}}{\partial n} \, dy, \]

where \( g \) is harmonic in \( G_{\varepsilon} \) with boundary values \( f \) on \( \theta_{\varepsilon} \) and boundary values \( 0 \) on \( \partial G_{\varepsilon} - \theta_{\varepsilon} \). Hence for sufficiently small \( \varepsilon \), \( d_{\varepsilon}(\xi) \leq c_{0} \).

Finally we estimate \( f(x_{0}) \). Since \( 2d_{\varepsilon} = q_{\varepsilon} \) is increasing,

\[ q_{\varepsilon}(x_{0}) \leq x_{0} q_{\varepsilon}(x_{0}) \leq c d_{\varepsilon}(x_{0}). \]
We let $\varepsilon$ tend to zero and then estimate $f(z_0)$ by Lemma 6.6. Thus we obtain (6.9) for $f(z_0)$. By (6.10), (6.9) is also valid for $u(z_0)$ in the modified domain and hence for the original $u(z_0) = \omega(z_0; \theta_\xi; D_\xi)$.

**Remark.** The original domain $D_\xi$ was modified so as to 1° facilitate defining auxiliary functions $f$, 2° assure the boundedness of $d_\varepsilon(\xi)$, and 3° allow estimating $u$ in terms of $f$. In the two-dimensional case we can instead begin by considering domains $D$ bounded by a finite number of analytic curves. $f$ can now be defined in the original domain $D_\xi$ as above. Discussion of $f_\varepsilon$ is unnecessary. We can use the technique of Lemma 6.9 below. We define $N$ in the following way: $\xi \in N$ if and only if an isosceles triangle $\Delta_\xi$ with base along $\theta_\xi$ and a fixed opposite angle $2\alpha$ is contained in $D_\xi$. When $\xi \in N$, the Dirichlet integral of $f$ is bounded by a fixed constant and furthermore the inequality (6.10) is correct. If $\xi \notin N$ a simple expedient is to consider $D_\xi \cup \Delta_\xi$ instead of $D_\xi$.

From Lemma 6.7 and Lemma 6.8 we now obtain the following

**Theorem 6.2.** Let $D$ satisfy A p. 13 and let $\lambda$ be defined by C p. 14. $z_0 = (x_0, y_0)$ is a fixed point in $D$. Then for $\xi > x_0 + 1$

$$\omega(z_0; \theta_\xi; D_\xi) \leq c \exp \left( - \int_{x_0}^{\xi-1} \lambda^1(x) \, dx \right).$$  

(6.13)

If $D$ satisfies A p. 13 and D p. 17 and $\lambda$ is defined by E p. 17, (6.13) is also correct. Finally, (6.13) is correct if $D$ satisfies A p. 13, D p. 17, and $D_\xi$ satisfies F p. 19, and $\lambda$ is defined by G p. 19.

The constant $c$ depends only on the geometry of $D$ near $z_0$ and constants appearing in the conditions satisfied by $D$ and $D_\xi$; when $F$ is used $c$ also depends on $x_0$.

**Remark.** The conditions involving smoothness in A and D were introduced for simplicity and are not essential. If A, D, F (or some of them) — apart from smoothness conditions—are satisfied, we can exhaust $D_\xi$ with a monotone sequence of subdomains $D_\xi^n$, $n = 1, 2, ..., \infty$, in which the theorem above can be applied. Now $\omega(z; \theta_\xi; D_\xi)$ and $\lambda(x)$ in $D_\xi$ are defined from the corresponding quantities $\omega_\varepsilon(z)$ and $\lambda_\varepsilon(x)$ in $D_\xi^\varepsilon$ by a limiting process. $\{\lambda_\varepsilon(x)\}$ is a non-increasing sequence. It follows that (6.13) is valid even though the smoothness conditions are not satisfied.

If $\lambda$ is bounded we can integrate up to $\xi$ in (6.13) and let $c$ also depend on the least upper bound of $\lambda$. We shall now show that integration up to $\xi$ in (6.13) is possible in some special cases when $\lambda$ is not bounded.

**Theorem 6.3.** Let $D$ satisfy A p. 13 and $\Theta_\xi$ be connected, $x > 0$. Let $\lambda(x)$ be defined by C p. 14. Let $\Theta(x)$ and $\lambda(x)$ be continuous, $x > 0$. $z_0 = (x_0, y_0)$ is a fixed point in $D$. Then

$$\omega(z_0; \Theta_\xi; D_\xi) \leq c \exp \left( - \int_{x_0}^{\xi} \lambda^1(x) \, dx \right)$$

(6.14)

is implied by any one of the following three conditions.

(a) $n = 2$.
(b) $n > 2$, $\lambda^1(x) \Theta'(x) \leq M_\theta$, $x > 0$, for some $r < 1$.  

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(c) $n > 2$, $\lambda^1(x) \Theta(x) \leq M_0$, $\lambda(x)$ non-decreasing, $x > 0$.

The condition (d) implies (6.15).

(d) $n > 2$, $\lambda^1(x) = \lambda^2(x_0) k(x) = \lambda^1(x_0) (\Theta(x)/\Theta(x_0))^{-\beta n - 1}$, $k(x)$ non-decreasing, $x > 0$.

$$\omega(z_0; \Theta; D_2) \leq c_0 k^{\alpha - n}(\xi) \exp \left(-\int_{x_0}^{\xi} \lambda^1(x_0) k(x) dx\right). \quad (6.15)$$

The constants $c$ depend only on the geometry of $D$ near $z_0$, $x_0$, and constants occurring in $A$, $(b)$, and $(c)$.

We first prove a lemma.

Lemma 6.9. Let $f$ be continuous and bounded, $0 < f(x) \leq K$, $x > 0$, and $f(x) = K$, $0 \leq x \leq a$. Then for $p > 1$ and $\xi \geq a$

$$\max_{x \leq \xi} f^{-p}(x) \int_0^{\xi} \exp \left(-\int_t^{\xi} f^{-1}(u) du\right) dt \geq c > 0, \quad (6.16)$$

where $c$ only depends on $p$, $a$, and $K$.

Proof. Let $\Gamma$ denote the curve $y = f(x)$, $x > 0$, in the $xy$-plane. $\Delta_x$ denotes a triangle with vertices $(x, 0)$, $(x, f(x))$, $(x - f(x) \cot \alpha, 0)$, where $\cot \alpha = q$ is a large constant. Set $N = \{x |$ the interior of $\Delta_x$ lies below $\Gamma\}$. If we choose $q > K/a$, $N$ will certainly be non-empty.

Now let $\xi$ belong to $N$. Given $\xi$, we take $t_0 = \xi - (2q)^{-1}(\xi)$.

Then

$$\int_{t_0}^{\xi} f^{-1}(x) dx \leq \int_{t_0}^{\xi} (f(\xi) - q(\xi - x))^{-1} dx \leq \log 2.$$

Hence

$$f^{-1}(\xi) \int_0^{\xi} \exp \left(-\int_t^{\xi} f^{-1}(u) du\right) dt \geq (4q)^{-1}. \quad (6.17)$$

By taking $x = \xi$ in (6.16), the truth of (6.16) follows for $\xi \in N$.

Now consider a $\xi_0 \in N$, $\xi_0 > a$. By continuity there exists a largest $b < \xi_0$ such that $b \in N$. Then (6.17) is valid for $\xi = b$ and furthermore

$$\exp \left(-\int_b^{\xi_0} f^{-1}(u) du\right) \geq (1 + q(\xi_0 - b)f^{-1}(b))^{-1/q}, \quad (6.18)$$

and $\xi_0 - b \leq K \cot \alpha$. Now take $x = b$ and $\xi = \xi_0$ in (6.16) and use (6.17) with $\xi = b$. Then it remains to determine a fixed lower bound for

$$f^{-p}(b) \exp \left(-\int_b^{\xi_0} f^{-1}(u) du\right).$$

By virtue of (6.18) this is done by choosing $q \geq (p - 1)^{-1}$. Thus the lemma is proved.
Proof of Theorem 6.3. We use the technique of Lemma 6.9. By (6.3), (6.7), and Lemma 6.6 we can reduce the proof of our theorem to determining a fixed positive lower bound for

$$\Theta^{-1}(\xi) \int_0^\xi \exp \left( -2 \int_t^\xi \lambda^1(u) \, du \right) \, dt.$$  

If this cannot be done for all large $\xi$, we can under the condition (a) obtain the desired estimate for $u(z_0)$ by a simple estimate of an auxiliary harmonic measure. Under the condition (b) we instead reduce the proof to determining a fixed positive lower bound for

$$\max_{z < \xi} \Theta^{-1}(x) \int_0^x \exp \left( -2 \int_t^x \lambda^1(u) \, du \right) \, dt.$$  

(6.19)

(a) (6.14). We use the technique of Lemma 6.9 with $f = (2\pi)^{-1} \Theta = \frac{1}{2} \lambda^{-\frac{1}{2}}$. (The assumption that $f$ be constant for small $x$ is not essential. It can be satisfied by a modification of $D$.) If $\xi \in \mathbb{N}$, (6.17) is true, and (6.14) follows. Now consider a $\xi \notin \mathbb{N}$, $\xi$ sufficiently large. By continuity there exists a largest $b < \xi$, such that $b \in \mathbb{N}$. Then (6.17) is true for $b$ instead of $\xi$. By the maximum principle

$$\omega(z_0; \Theta_b; D_b) \leq \omega(z_0; \Theta_b; D_b) \max_{z \in \Theta_b} \omega(z; \Theta_b; D_b).$$

Hence it suffices to prove that

$$\exp \left( 2 \int_b^\xi \Theta^{-1}(u) \, du \right) \max_{z \in \Theta_b} \omega(z; \Theta_b; D_b) \leq c.$$  

(6.20)

By the definition of $b$

$$2\pi \int_b^\xi \Theta^{-1}(u) \, du \leq q^{-1} \log (1 + 2\pi q(\xi - b) \Theta^{-1}(b)).$$

If $(\xi - b) \Theta^{-1}(b) \leq c_1$ (a fixed number), then (6.20) is correct. If $(\xi - b) \Theta^{-1}(b) > c_1$, we use a simple estimate of $\omega(z; \Theta_b; D_b)$. Set $l = \{z \mid x_1 = b, y \geq \Theta(b)/2 \}$. By the extension principle and an explicit conformal mapping of $G$ onto a half-plane we obtain

$$\max_{z \in \Theta_b} \omega(z; \Theta_b; D_b) \leq \omega(b; l; G) \leq c(\xi - b)^{-\frac{1}{4}} \Theta^1(b).$$

Hence (6.20) is correct, and this part of the lemma is proved.

(b) (6.14). We use Lemma 6.9 with $f = \frac{1}{2} \lambda^{-\frac{1}{2}}$. Since $\Theta(x) \leq M$, $x > 0$, $\lambda^1(x)$ is bounded from below according to the Faber-Krahn inequality. The condition (b) now implies (6.16). We thus obtain a fixed lower bound for (6.19) and (6.14) follows.

(c) (6.14), (d) (6.15). We use the technique of Lemma 6.9 with $f = \frac{1}{2} \lambda^{-\frac{1}{2}}$. For a non-increasing $f$ all sufficiently large $\xi$ are in $\mathbb{N}$ and (6.17) gives the desired results. (Instead of $f$ being non-increasing we can require that $f'$ exists and is bounded by a fixed constant.)
In a special case lower bounds of harmonic measures can be established by studying the Nevanlinna mean (7.2). Heins used this name for (7.2) in the case of a rectangle or a half-plane [7.4, p. 4].

We consider domains $D$ such that $\Theta_2=\phi$, $-\infty<x<\infty$, $\Theta_0$ is assumed to be Steiner symmetric with respect to the coordinate hyperplanes $y_i=0$, $i=1, 2, \ldots, n-1$. By this we mean that the intersection of $\Theta_0$ with a straight line perpendicular to $y_i=0$ is either a single line-segment symmetrical with respect to $y_i=0$ or empty, $i=1, 2, \ldots, n-1$. When $n>2$ $\partial\Theta_0$ is assumed to possess piecewise continuous curvature. $\Theta_x$ is obtained from $\Theta_0$ by the mapping $y\rightarrow k^{-1}(x)y$, $k(0)=1$, $k(x)>0$, $k(x)$ non-decreasing and twice continuously differentiable, $-\infty<x<\infty$. Under these assumptions we shall establish the following

Theorem 7.1. Let $\lambda$ be the principal eigenvalue of $\Theta_0$. Then

$$\max_{z\in\Theta_x} \omega(z; \Theta_0; D_\xi) \geq (k(\xi)/k(x))^{1-n} \exp \left( -\lambda \int_{\xi}^{x} k(t) dt \right).$$

We start by proving some lemmata. We use the following notation. Let $\{v_1\}_{n}^{\infty}$ be the normed eigenfunctions of

$$Av_1+\lambda v_1=0$$

in $\Theta_0$, $v=0$ on $\partial\Theta_0$, and $\{\lambda_n\}_{n}^{\infty}$ the corresponding eigenvalues. Let $\{V_x, n\}_{n}^{\infty}$ and $\{A_x, n\}_{n}^{\infty}$ be defined in the same way with respect to $\Theta_x$. Then

$$A_x, n = k^2(x) \lambda_n, \quad V_x, n(y) = k(n-1/2)(x)v_n(k(x)y).$$

In the following we write $\lambda$ and $v$ instead of $\lambda_n$ and $v_n$.

Lemma 7.1. Let $m(x)$ be the Nevanlinna mean of $u(x, y) = \omega(z; \Theta_0; D_\xi)$ over $\Theta_x$, $x<\xi$,

$$m(x) = \int_{\Theta_x} u(x, y) v(k(x)y) dy.$$  \hspace{1cm} (7.2)

Let $y \cdot \text{grad} v$ denote the scalar product in $\Theta_0$. Then

$$m'(x) = \int_{\Theta_x} u_x(x, y) v(k(x)y) dy + k'(x) \int_{\Theta_x} u(x, y) (y \cdot \text{grad} v)(k(x)y) dy, \quad x<\xi.$$  \hspace{1cm} (7.3)

Proof. Under our assumptions about $k$ and $\partial\Theta_0$, $u$ possesses continuous first derivatives up to the boundary at points where $\partial\Theta_0$ is of continuous curvature (in the case $n>2$) for $x_1<\xi$ [7.3, p. 635]. The same is true for $v$, since $\exp(\lambda_1 x_1)v_1(y)$ is harmonic in a right cylinder with base $\Theta_0$. The lemma then follows.

Lemma 7.2. $\int_{\Theta_x} u_x(x, y) v(k(x)y) dy \leq \lambda k(x) m(x), \quad x<\xi.$

Proof. Let $x$ be fixed and set $G = \{z=(s, t)|s<x, \quad t\in\Theta_x\}$. We write $\Gamma$ instead of $\partial G$ and define $\gamma=\Gamma-\Theta_x$. Let $U$ be harmonic in $G$ with boundary values $u(x, y)$ on $\Theta_0$ and $0$ on $\gamma$. $G \subset D_x$ since $\Theta_0$ is Steiner symmetric with respect to the coordinate hyperplanes and $k$ is non-decreasing. By the maximum principle the inner normal derivatives of $U$ and $u$ satisfy the inequality $\partial U/\partial n \leq \partial u/\partial n$ on $\Theta_x$. We write $V(y)$ instead of $v(k(x)y)$. Then
where $d\sigma$ is the area element on $\Gamma$. We now apply Green’s formula to the last integral and note that $U = 0$ on $\gamma$ and $\partial V / \partial n = 0$ on $\Theta_x$. (We first consider finite subdomains of $G$ and then use some majorant of $\partial U / \partial n$ in the limiting process.) Taking (7.1) into account we obtain
\[
\int_{\Gamma} V \frac{\partial U}{\partial n} d\sigma = \int_{\partial} U \Delta V dz = -\lambda k^2(x) \int_{\partial} UV dz.
\]
Hence by (7.4),
\[
\int_{\Theta_x} u_x(x, y) v(k(x) y) dy = \lambda k^2(x) \int_{\mathbb{R}} \int_{\mathbb{R}} U(s, t) v(k(x) t) ds \, dt.
\]
Now $U$ can be represented in the following way in the cylinder $G$:
\[
U(s, t) = \int_{\Theta_x} u(x, y) P(s, t; x, y) dy, \quad s < x,
\]
where
\[
P(s, t; x, y) = \sum_{n=1}^{\infty} \exp \left( (s - x) A \Lambda_{x, n} \right) V_{x, n}(y) \Lambda_{x, n}(t).
\]
We shall need the relation
\[
\int_{\Theta_x} P(s, t; x, y) v(k(x) t) dt = \exp \left( \lambda^2 (s - x) k(x) \right) v(k(x) y), \quad s < x.
\]
By (7.6) and (7.7), after first considering subdomains
\[
\{ z = (s, t) \mid z \in G, \quad -\infty < a < s < b < x \}
\]
of $G$, we obtain
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} U(s, t) v(k(x) t) ds \, dt = \lambda^2 m(x) k^{-1}(x).
\]
By (7.5), this proves our lemma.

**Lemma 7.3.** Under the assumption that $\Theta_0$ is Steiner symmetric with respect to the coordinate hyperplanes, the scalar product $y \cdot \text{grad} \, v$ is non-negative.

**Proof.** We want to prove that $v$ is symmetrically decreasing with respect to $y_i = 0$, $i = 1, 2, \ldots, n - 1$. Let us assume that this is false for some $i$. We then symmetrize $v$ with respect to $y_i = 0$ (cf. § 1) and denote the symmetrized function by $v^*$. By [7.2, pp. 184–186]\
\[
\int_{\Theta_0} v^2 dy = \int_{\Theta_0} v^{*2} dy, \int_{\Theta_0} |\text{grad} \, v|^2 dy \geq \int_{\Theta_0} |\text{grad} \, v^*|^2 dy.
\]

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By $B$ in § 6 $v$ minimizes the Rayleigh quotient and thus $v = v^*$, and the lemma is proved.

**Proof of Theorem 7.1.** Since $k'$ is non-negative it follows from the lemmata 7.1, 7.2, and 7.3 that

$$m'(x) \leq \lambda^x k(x) m(x), \ x < \xi.$$  

Hence

$$m(x) \geq m(\xi) \exp \left( -\lambda^x \int_x^\xi k(t) \, dt \right).$$

However,

$$m(\xi) = \int_{\Theta^\xi} v(k(\xi) y) \, dy = k^{1-n}(\xi) \int_{\Theta^\xi} v(y) \, dy = c_0 k^{1-n}(\xi)$$

and $v$ being positive in $\Theta_0$,

$$m(x) \leq \max_y u(x, y) \int_{\Theta^y} v(k(x) y) \, dy = c_0 k^{1-n}(x) \max_y u(x, y).$$

Hence

$$\max_y u(x, y) \geq (k(\xi)/k(x))^{1-n} \exp \left( -\lambda^x \int_x^\xi k(t) \, dt \right)$$

and Theorem 7.1 is proved.

**Remark.** Theorem 7.1 is of interest in connection with the estimate (6.15) in Theorem 6.3. Also cf. Theorem 2.1.

8. Harmonic measures and probability theory

Harmonic measures have a probabilistic interpretation in the theory of Brownian motion. A standard work of reference for this theory is that of Lévy [8.3]. Later works of Doob and others are not referred to here. Let the domain $D$ in $\mathbb{R}^n$ be bounded by a finite number of closed surfaces. $p(z; \alpha; D)$ is the probability that the Brownian motion particle which starts from the point $z$ in $D$ first reaches $\partial D$ on a subdomain $\alpha$ of $\partial D$. Then [8.3, p. 62]

$$\omega(z; \alpha; D) = p(z; \alpha; D). \quad (8.1)$$

This interpretation is useful for heuristic argument. For instance, the choice of $q$ in the proof of Theorem 3.2 appears reasonable since it takes into account those segments of $\Theta_\alpha$ that the Brownian motion particle starting from $z_0$ (and the curves in $\Gamma$) must pass through to reach $\theta_\xi$. It appears difficult to obtain majorants of harmonic measures by a study of Brownian motion or of the corresponding random walk. We can, however, prove symmetrization results for harmonic measures by considering independent components of a Brownian motion. Definitions of the different kinds of symmetrization are given in § 1.

**Theorem 8.1.** (The two-dimensional case.) Let $D_\xi$ be bounded by a finite number of simple closed curves; $\theta_\xi = \bigcup_{i=1}^n \theta_\xi^i$, $m(x) \leq M$. Assume that the $\theta_\xi^i$ vary continuously;

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it is allowed that at a finite number of points two segments come together or one splits or one vanishes. * denotes symmetrization with respect to the $x_1$-axis. Then

$$\max_{z \in \theta_\xi_x} \omega(z; \theta_\xi; D_\xi) \leq \omega(x; \theta_\xi^*; D_\xi^*).$$

**Proof.** By (8.1) is is sufficient to prove that

$$\max_{z \in \theta_\xi_x} p(z; \theta_\xi; D_\xi) \leq p(x; \theta_\xi^*; D_\xi^*). \quad (8.2)$$

Consider a two-dimensional Brownian motion \{(X(\tau), Y(\tau)) = Z(\tau), \; 0 \leq \tau < \infty\}, starting from a point $z = (x, y)$, where \{X(\tau), \; 0 \leq \tau < \infty\} and \{Y(\tau), \; 0 \leq \tau < \infty\} denote one-dimensional Brownian motions and the components are independent. $X(\tau)$ and $Y(\tau)$ are continuous with probability one [8.3, p. 10], so we assume continuity of $X(\tau)$ and $Y(\tau)$ in the following.

Let $M_X(t)$ denote $\max X(\tau)$ when $0 \leq \tau \leq t$. $M_X$ has an inverse function $T_X$ indicating the first passage time [8.3, p. 31]. We now write

$$p(z; \theta_\xi; D_\xi) = \int_{t=0}^{\infty} P\{t \leq T_X(\xi) < t + dt, \; Z(\tau) \in D_\xi\} \; \text{when} \; \; 0 \leq \tau < t$$
$$= \int_{t=0}^{\infty} P\{t \leq T_X(\xi) < t + dt\} \; P\{Y(\tau) \in \theta_{X(\tau)}\} \; \text{when} \; \; 0 \leq \tau < t \; t \leq T_X(\xi) < t + dt$$

and

$$p(z; \theta_\xi^*; D_\xi^*) = \int_{t=0}^{\infty} P\{t \leq T_X(\xi) < t + dt\} \; P\{Y(\tau) \in \theta_{X(\tau)}^*\} \; \text{when} \; \; 0 \leq \tau < t \; t \leq T_X(\xi) < t + dt.$$
Thanks to our assumptions about the $x_i$ (8.3) follows from (8.4) by e.g. [8.1, Theorem 2.2, p. 54]. Now, omitting the upper index $k$

$$P \{ Y(\tau_0) \in \mathcal{X}_x, \tau = 1, 2, \ldots, k \}$$

$$= (2\pi)^{-k/2} \int_{\mathcal{X}_x} \cdots \int_{\mathcal{X}_x} \exp \left( -\frac{\| y - y_1 \|^2}{2\tau_1} - \frac{\| y - y_2 \|^2}{2(\tau_2 - \tau_1)} - \cdots - \frac{\| y - y_k \|^2}{2(\tau_k - \tau_{k-1})} \right) dy_1 \cdots dy_k.$$  

Thus (8.4) will follow from an inequality of the following type:

$$\int_{\mathcal{X}_x} \cdots \int_{\mathcal{X}_x} \exp \left( -c_1 y_1^2 - c_2 (y_2 - y_1)^2 - \cdots - c_k (y_k - y_{k-1})^2 \right) dy_1 \cdots dy_k$$

$$\leq \int_{\mathcal{X}_x} \cdots \int_{\mathcal{X}_x} \exp \left( -c_1 y_1^2 - c_2 (y_2 - y_1)^2 - \cdots - c_k (y_k - y_{k-1})^2 \right) dy_1 \cdots dy_k$$

where the $c_i$ are positive constants and the $x_i$ are finite unions of intervals and $x_i^* = \{ y_i \in \mathbb{R}^d | y_i < l(x_i)/2 \}, l(x_i)$ being the total length of $x_i = 1, 2, \ldots, k$. Such an inequality follows from Lemma 8.1 below and thereby our theorem is proved.

**Lemma 8.1.** Let $\{ a_{r}^{(i)} \}$ be symmetrically decreasing sequences of numbers with $a_{r}^{(0)} > a_{r}^{(1)} > a_{r}^{(2)} > \cdots > 0$, $i = 1, 2, \ldots, k$. We assume that

$$b_{r}^{(i)} = \begin{cases} 1 & \text{for } 2s_i + 1 \text{ values of } r \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\sum_{s_1} \cdots \sum_{s_k} a_{r_1}^{(s_1)} a_{r_2}^{(s_2)} a_{r_3}^{(s_3)} \cdots a_{r_k}^{(s_k)} \leq \sum_{s_1} \cdots \sum_{s_k} a_{r_1}^{(s_1)} a_{r_2}^{(s_2)} \cdots a_{r_k}^{(s_k)}$$

**Proof.** Let $\{ a_r \}$ be a finite set of numbers. The rearranged set $\{ a_r^+ \}$ is defined by $a_0^+ > a_1^+ > a_2^+ > \cdots > 0$. We write $A_r = a_r^+ b_r^{(1)}$. Then $A_r^+ = 0$ and $A_r^+ \leq a_r^{(1)}$ for $r = -s_1, \ldots, s_1$. Hence it is sufficient to prove that

$$\sum_{s_1} \cdots \sum_{s_k} A_{r_1} a_{r_2}^{(s_2)} a_{r_3}^{(s_3)} \cdots a_{r_k}^{(s_k)} \leq \sum_{s_1} \cdots \sum_{s_k} A_{r_1} a_{r_2}^{(s_2)} \cdots a_{r_k}^{(s_k)}$$

When $k = 2$ (8.6) follows from Theorem 373 [8.2, p. 265]. To prove (8.6) when $k > 2$, the induction method of Theorem 374 [8.2, p. 273-274] is used. We write

$$B_{r_1} = \sum_{s_1} A_{r_1} a_{r_2}^{(s_2)} b_{r_2}^{(s_2)} \quad a_{r_2}^+ = \sum_{s_2} \cdots \sum_{s_k} a_{r_2}^{(s_2)} a_{r_3}^{(s_3)} \cdots a_{r_k}^{(s_k)}.$$
Let \( \varphi \) be a permutation function for which \( B_{\varphi(\varphi)} = B \). We write \( c_{\varphi(\varphi)} = C \) and \( d_{\varphi(\varphi)} = \theta^{(2)} \). By Theorem 373 [8.2, p. 265]

\[
\sum_{|v_1| \leq s_2} B_{v_1} c_{v_2} = \sum_{|v_1| \leq s_2} B_{v_1} C_{v_2} = \sum_{|v_1| \leq s_2} A_{v_1} d_{v_2} c_{v_1-v_2} \leq \sum_{|v_1| \leq s_2} \sum_{|v_2| \leq s_2} A_{v_1} d_{v_2} c_{v_1-v_2}.
\]

However, by Theorem 375 [8.2, p. 273] \( c_0 \geq c_1 \geq c_2 \geq c_3 \geq \ldots \). Hence \( d_{v_1} \leq c_{v_2} \) when \( |v_2| \leq s_2 \) and

\[
\sum_{|v_1| \leq s_2} B_{v_1} c_{v_2} \leq \sum_{|v_1| \leq s_2} \sum_{|v_2| \leq s_2} A_{v_1} c_{v_2} = \sum_{|v_1| \leq s_2} \ldots \sum_{|v_k| \leq s_2} A_{v_1} c_{v_2} \ldots c_{v_k}.
\]

The general result (8.6) now follows by induction and thereby the lemma is proved.

For the sake of simplicity we formulate the \( n \)-dimensional result for domains \( D \) such that \( D \) is the restriction to \( \{z : x_2 < \xi \} \) of a finite union of spheres. By an exhaustion process we can extend the result to harmonic measures in more general domains. Let \( \{D_{\xi}^{(\nu)} \} \) be a monotone sequence of subdomains of \( D \) converging to \( D \), such that each \( D_{\xi}^{(\nu)} \) is the restriction to \( \{z : x_2 < \xi \} \) of a finite union of spheres. Then by Theorem 8.2 below and the maximum principle

\[
\max_{y_i} \omega(z; \theta_{\xi}^{(\nu)}; D_{\xi}^{(\nu)}) \leq \omega(z; \theta_{\xi}^{(\nu)}; D_{\xi}^{(\nu)}).
\]

and thus

\[
\max_{y_i} \omega(z; \theta_{\xi}; D_{\xi}) \leq \omega(z; \theta_{\xi}^{*}; D_{\xi}^{*}).
\]

**Theorem 8.2.** (The \( n \)-dimensional case.) Let \( D \) be the restriction to \( \{z : x_2 < \xi \} \) of a finite union of \( n \)-dimensional spheres. \( \theta \) denotes symmetrization with respect to a coordinate hyperplane \( y_i = 0 \). Given \( z, z_i \) has the same coordinates as \( z \) except that the \( i \)-th coordinate of \( z_i \) is zero. Then

\[
\max_{y_i} \omega(z; \theta_{\xi}^{*}; D_{\xi}) \leq \omega(z; \theta_{\xi}^{*}; D_{\xi}).
\]

**Proof.** Consider an \( n \)-dimensional Brownian motion \( \{X_1(\tau), Y_1(\tau), \ldots, Y_{n-1}(\tau)\} \) starting from a point \( z = (x, y) \). \( \{X_1(\tau), 0 \leq \tau < \infty \} \), \( \{Y_i(\tau), 0 \leq \tau < \infty \}, \) denote one-dimensional Brownian motions and the components are mutually independent. The proof is analogous to that of Theorem 8.1.

When considering \( P\{Y(\tau) \in \theta_{X(\tau)} \text{ when } 0 \leq \tau < t \} \) we now translate any inaccessible line-segments on straight lines perpendicular to the hyperplane \( y_i = 0 \) in the \( (\tau, y) \)-space. We then proceed as in the proof of Theorem 8.1 up to (8.5).

Instead of (8.5) we now have, with \( Y(0) = y \),

\[
P\{Y(\tau) \in \alpha_{\nu}, \nu = 1, 2, \ldots, k\} = (2\pi)^{-n/2} (\tau_1(\tau_2 - \tau_1) \ldots (\tau_k - \tau_{k-1}))^{-(n-1)/2}
\]

\[
\times \int_{\tau_1} \ldots \int_{\tau_k} \exp \left( -\frac{|y - y^{(1)}|^2}{2\tau_1} - \frac{|y^{(1)} - y^{(2)}|^2}{2(\tau_2 - \tau_1)} - \ldots - \frac{|y^{(k-1)} - y^{(k)}|^2}{2(\tau_k - \tau_{k-1})} \right) dy^{(1)} \ldots dy^{(k)}.
\]

We now consider the integrand as a product, one factor being

\[
\exp \left( -\frac{(y_i - y_i^{(1)})^2}{2\tau_1} - \frac{(y_i^{(1)} - y_i^{(2)})^2}{2(\tau_2 - \tau_1)} - \ldots - \frac{(y_i^{(k-1)} - y_i^{(k)})^2}{2(\tau_k - \tau_{k-1})} \right).
\]
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We first integrate with respect to $dy_i^{(1)} \ldots dy_i^{(k)}$ and use Lemma 8.1. Our theorem now follows in the same way as Theorem 8.1.

**Remark 1.** The method of proof used above is not suitable for discussing the case of equality in the theorems. Cf. Theorem 4.1.

**Remark 2.** It seems reasonable that Theorem 4.2 can be generalized to higher dimensions, but a probabilistic proof does not appear easy.

**Remark 3.** In connection with the results of this paragraph we note that the principal eigenvalue occurring in $B$ in § 6 and in Theorem 6.2 is decreased by symmetrization, according to $B$.

**Remark 4.** By a limiting process we can establish the result of Theorem 8.2 for more general domains. By an infinite sequence of symmetrizations with respect to hyperplanes through the $x_i$-axis, we can thus obtain the following result, * denoting symmetrization with respect to the $x_i$-axis ($n > 2$),

$$\max_{z \in \mathbb{D}} \omega(z; \theta_i; D_i^*) \leq \omega(x; \theta_i^*; D_i^*).$$

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**BIBLIOGRAPHY**


