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Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$

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1. Introduction

Let $F(x,y,z)$ be a given function, defined for $x_1 \leqslant x \leqslant x_2$ and all values of y and z. Let further $\mathfrak{F}$ be a class of functions $f(x)$, all of which are defined on $x_1 \leqslant x \leqslant x_2$ and are sufficiently regular. For every $f \in \mathfrak{F}$, we define the functional

$$H(f) = \sup_{x_1 \leqslant x \leqslant x_2} F(x, f(x), f'(x)).$$

We are interested in the problem to minimize $H(f)$ over $\mathfrak{F}$. For example, we will try to answer these questions: Does there exist a minimizing function? Is it unique? Has it any special properties? What is the value of $\inf_{f \in \mathfrak{F}} H(f)$? For reasons of brevity, many of the results are *not* given in the most general form. We shall only consider real functions and real variables.

If $g(x)$ is continuous and non-negative on $x_1 \leqslant x \leqslant x_2$, then

$$\max_{x_1 \leqslant x \leqslant x_2} g(x) = \lim_{n \to \infty} \left(\int_{x_1}^{x_2} (g(x))^n dx \right)^{1/n}.$$

This suggests that we should approximate the functional $H(f)$ with the sequence of functionals

$$H_n(f) = \left(\int_{x_1}^{x_2} [F(x, f(x), f'(x))]^n dx \right)^{1/n}, \quad n = 1, 2, 3, \ldots .$$

The Euler equation corresponding to $H_n(f) = \min$ is

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}[F(x,y,y')^n]\right) - \frac{\partial}{\partial y}[F(x,y,y')^n] = 0,$$

which can be written as

$$n(n-1)F^{n-2}\left[\frac{dF}{dx}\cdot F_{y'} + \frac{1}{n-1}\cdot F \cdot \frac{dF_{y'}}{dx} - \frac{1}{n-1} F \cdot F_y\right] = 0.$$

Let us put the expression in brackets equal to zero and then let n tend to infinity.

Then we get (formally) a new equation

$$\frac{d}{dx}(F(x,y,y'))\cdot F_{y'}(x,y,y')=0. \tag{*}$$

We want to study the connection (if there is any) between this differential equation and the minimization problem.

It might be expected that functions $f(x)$, such that $F(x, f(x), f'(x)) = \text{constant}$, should be important for the problem. This is true, as we shall see in sections 2 and 3.

In section 4, we shall introduce a class of functions which minimize the functional "on every interval" and prove that such a function must satisfy the equation (*) in a certain sense.

A similar problem has been studied in [1].

2. The special case $F = F(y, y')$

We shall start with a study of the case where F is independent of x, since this case is simpler than the general and since we are interested in the interaction between y and y'.

2 A. *The minimization problem*

Lemma 1: *Suppose that:*

1) *$f(x)$ is continuous for $a \leqslant x \leqslant b$,*
   $f'(x)$ is continuous for $a < x < b$,
2) *$f'(x) > 0$ for $a < x < b$,*
3) *$g(x)$ is absolutely continuous on $a \leqslant x \leqslant b$,*
4) *$g(a) \leqslant f(a)$, $g(b) \geqslant f(b)$ and $f \not\equiv g$.*

Then there exist numbers t_1, t_2 on the open interval (a, b) such that

I) *$f(t_1) = g(t_2)$,*
II) *$g'(t_2)$ exists and $f'(t_1) < g'(t_2)$.*

Proof: Clearly, we may assume that $g(a) = f(a)$, $g(b) = f(b)$ and $f(a) < g(x) < f(b)$ for $a < x < b$. For, if this is not the case, we define

$$p = \max\{x \mid g(x) \leqslant f(a)\},$$

$$q = \min\{x \mid x \geqslant p,\ g(x) \geqslant f(b)\},$$

and, instead of $g(x)$, we consider $g_1(x) = g(p + (q-p)(x-a)[b-a]^{-1})$. (If $g_1 \equiv f$, then the result is trivial; if $g_1 \not\equiv f$, then the proof below applies to g_1, and then the result for g follows.) Now $y = f(x)$ has a continuous inverse $x = \alpha(y)$ for $f(a) \leqslant y \leqslant f(b)$. $\alpha'(y)$ is continuous and positive for $f(a) < y < f(b)$. Hence $\alpha(y)$ is absolutely continuous on $f(a) \leqslant y \leqslant f(b)$.

Form the function $\varphi(x) = \alpha(g(x))$. It is absolutely continuous on $a \leqslant x \leqslant b$ and $\varphi'(x) = \alpha'(g(x)) \cdot g'(x)$ a.e. By assumption, there exists a number x_0, $a < x_0 < b$, such that $f(x_0) \neq g(x_0)$; say $f(x_0) < g(x_0)$ (the other case is treated similarly). Then we have $\varphi(x_0) > x_0$ and

$$\int_a^{x_0} \varphi'(x)\,dx = \varphi(x_0) - \varphi(a) = \varphi(x_0) - a > x_0 - a.$$

So there must exist a number ξ such that: $a < \xi < x_0$, $g'(\xi)$, $\varphi'(\xi)$ exist and $\varphi'(\xi) > 1$.

Thus
$$\alpha'(g(\xi)) \cdot g'(\xi) > 1.$$

But
$$\alpha'(g(\xi)) = \frac{1}{f'[\alpha(g(\xi))]}, \quad \text{which gives us}$$

$$g'(\xi) > f'[\alpha(g(\xi))].$$

Obviously, the numbers $t_1 = \alpha(g(\xi))$ and $t_2 = \xi$ will have the required properties.

Remark: If we change the conditions 2 and 4 to $f'(x) < 0$ and $g(a) \geqslant f(a)$, $g(b) \leqslant f(b)$, respectively, and the assertion II to $g'(t_2) < f'(t_1)$, then we get another form of the lemma which follows from the preceding by the substitution $t = -x$.

Now let us consider a function $F = F(y,z)$ and let us impose a few conditions upon it:

1) F is defined and continuous for all y and z.

2) $\dfrac{\partial F}{\partial z}$ exists for all y and z and $\dfrac{\partial F}{\partial z}$ is $\begin{cases} >0 & \text{if} \quad z>0 \\ =0 & \text{if} \quad z=0 \\ <0 & \text{if} \quad z<0. \end{cases}$

Let $[x_1, x_2]$ be the interval mentioned in the introduction and let y_1, y_2 be any two numbers. From now on, the class $\mathfrak{J}$ of admissible functions is defined as follows: $\mathfrak{J}$ is the class of all absolutely continuous functions on $x_1 \leqslant x \leqslant x_2$, which satisfy the boundary conditions $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $f(x) \in \mathfrak{J}$ and let E be the set where $f'(x)$ exists. It should be noticed that x_1 and x_2 belong to E if the one-sided derivatives in question exist.

Now $H(f) = \sup_{x \in E} F(f(x), f'(x))$ is well-defined and obviously

$$H(f) \geqslant \max(F(y_1, 0),\ F(y_2, 0)).$$

Therefore $\inf_{f \in \mathfrak{J}} H(f)$ is finite, and the questions mentioned in the introduction are meaningful.

With the use of Lemma 1, we can easily prove the following simple theorem:

Theorem 1: *Suppose that $F(y,z)$ satisfies the conditions* 1 *and* 2 *stated above. Suppose further that $f(x)$ is an admissible function such that*

a) *$f'(x)$ is continuous for $x_1 \leqslant x \leqslant x_2$,*
b) *$f'(x) \neq 0$ for $x_1 < x < x_2$,*
c) *$F(f(x), f'(x)) = M$ for $x_1 \leqslant x \leqslant x_2$ (M is any constant).*

Then $f(x)$ is a unique minimizing function in $\mathfrak{J}$. I.e.: if $g(x)$ is a different element of $\mathfrak{J}$, then $H(g) > H(f)$.

Proof: Consider the case $f'(x)>0$. According to Lemma 1 there exist t_1, t_2 such that $f(t_1)=g(t_2)$, $0<f'(t_1)<g'(t_2)$. This gives

$$H(g)\geqslant F(g(t_2),\, g'(t_2))>F(f(t_1),\, f'(t_1))=M=H(f).$$

Hence
$$H(g)>H(f).$$

The other case is treated analogously (see the remark to Lemma 1). This theorem should be compared with the theorems of section 3.

In order to give a more systematic treatment of the case $F=F(y,z)$, we introduce another condition on F:

3) $\lim_{|z|\to\infty} F(y,z)=+\infty$ for all y.

As is easily seen (using the conditions 1 and 2 also), this implies that the limits

$$\lim_{z\to+\infty} F(y,z)=+\infty,\quad \lim_{z\to-\infty} F(y,z)=+\infty$$

are uniform for bounded y.

In the rest of section 2, we shall always assume that $F(y,z)$ satisfies the conditions 1, 2 and 3 given above.

We now introduce two auxiliary functions:

Definition: If $F(y,0)<M$, then we set $\Phi_M(y)=$the positive number z such that $F(y,z)=M$, $\Psi_M(y)=$the negative number z such that $F(y,z)=M$, and if $F(y,0)=M$ then we set $\Phi_M(y)=\Psi_M(y)=0$. (If $F(y,0)>M$, then the equation $F(y,z)=M$ has no solution z.)

Lemma 2: *$\Phi_M(y)$ and $\Psi_M(y)$ are continuous functions of y and M on the set where they are defined.*

The proof is simple and we omit it.

We will now try to give a complete solution of the minimization problem under the assumptions 1, 2 and 3 about $F(y,z)$. We shall pay most attention to the case $y_1<y_2$ and the corresponding results for the case $y_1>y_2$ will be given later without proofs, since the reasoning is very similar in both cases. Finally, we shall consider the case $y_1=y_2$.

A) Let us now suppose $y_1<y_2$.

Integrals of the type $\int_{y_1}^{y_2}\frac{dt}{\Phi_M(t)}$ turn out to be very useful. Let us agree to call the integral above well-defined if and only if $\Phi_M(t)>0$ a.e. on $y_1\leqslant t\leqslant y_2$. Then $\frac{1}{\Phi_M(t)}$ is non-negative, measurable and finite a.e. Let us use the notation

$$\int_{y_1}^{y_2}\frac{dt}{\Phi_M(t)}=\mathfrak{L}(M).$$

Thus $\mathfrak{L}(M)>0$ always and the possibility $\mathfrak{L}(M)=+\infty$ is not excluded.

Lemma 3: *Assume that $M_\nu \to M$, $M_1 > M_2 > M_3 > \ldots$ and that, for all ν, $\mathfrak{L}(M_\nu)$ is well-defined and $\mathfrak{L}(M_\nu) \leqslant C$ (C independent of ν). Then $\mathfrak{L}(M)$ is also well-defined and $\mathfrak{L}(M) = \lim_{\nu\to\infty} \mathfrak{L}(M_\nu)$.*

Proof: If $y_1 \leqslant y \leqslant y_2$ then, clearly, $F(y,0) < M_\nu$ for all ν and $F(y,0) \leqslant M$. This means that $\Phi_M(y)$ is defined for $y_1 \leqslant y \leqslant y_2$. If $\Phi_M(y) > 0$, then, according to Lemma 2,

$$\frac{1}{\Phi_{M_\nu}(y)} \to \frac{1}{\Phi_M(y)},$$

and if $\Phi_M(y) = 0$, then
$$\frac{1}{\Phi_{M_\nu}(y)} \to \infty.$$

For every y we have
$$\frac{1}{\Phi_{M_1}(y)} < \frac{1}{\Phi_{M_2}(y)} < \frac{1}{\Phi_{M_3}(y)} < \ldots .$$

But it is also true that
$$\int_{y_1}^{y_2} \frac{dy}{\Phi_{M_\nu}(y)} \leqslant C.$$

It follows from Beppo–Levis theorem that $\lim_{\nu\to\infty} \dfrac{1}{\Phi_{M_\nu}(y)} = h(y)$ exists finite a.e. on $y_1 \leqslant y \leqslant y_2$ and that

$$\int_{y_1}^{y_2} h(y)\,dy = \lim_{\nu\to\infty} \int_{y_1}^{y_2} \frac{dy}{\Phi_{M_\nu}(y)} = \lim_{\nu\to\infty} \mathfrak{L}(M_\nu).$$

From the preceding we see that $h(y) = \dfrac{1}{\Phi_M(y)}$ a.e., which completes the proof.

Theorem 2: *If there exists an admissible function $f(x)$ such that* $\sup_x F(f(x), f'(x)) \leqslant M$, *then $\mathfrak{L}(M)$ is well-defined and $\mathfrak{L}(M) \leqslant x_2 - x_1$.*

Proof: If $M_\nu = M + 1/\nu$ for $\nu = 1, 2, 3, \ldots$, then $H(f) < M_\nu$. Hence, if $y_1 \leqslant y \leqslant y_2$, $F(y,0) < M_\nu$ and $\Phi_{M_\nu}(y) > 0$.

We may assume that $y_1 \leqslant f(x) \leqslant y_2$. Take an arbitrary ν. From the inequality $F(f(x), f'(x)) < M_\nu$ it follows that $f'(x) < \Phi_{M_\nu}(f(x))$ (a.e.). Therefore

$$\frac{f'(x)}{\Phi_{M_\nu}(f(x))} < 1. \tag{1}$$

Now $\varphi(y) = \displaystyle\int_{y_1}^{y} \frac{dt}{\Phi_{M_\nu}(t)}$ is a continuously differentiable function of y for $y_1 \leqslant y \leqslant y_2$.

Thus $\Psi(x) = \varphi(f(x))$ is absolutely continuous on $x_1 \leqslant x \leqslant x_2$. It follows from (1) that $\Psi'(x) \leqslant 1$ a.e.

So
$$\Psi(x_2) - \Psi(x_1) \leqslant x_2 - x_1.$$

But $\Psi(x_1) = \varphi(y_1) = 0$ and $\Psi(x_2) = \varphi(y_2) = \mathfrak{L}(M_\nu)$. This gives

$$\mathfrak{L}(M_\nu) \leqslant x_2 - x_1.$$

Now the assertion follows from the preceding lemma.

Theorem 3: *If $\mathfrak{L}(M)$ is well-defined and finite, then there is a function $f(x)$ with these properties:*

1) *$f(x)$ is defined and strictly increasing for $0 \leqslant x \leqslant \mathfrak{L}(M)$,*
2) *$f(0) = y_1$, $f(\mathfrak{L}(M)) = y_2$,*
3) *$f'(x)$ is continuous for $0 \leqslant x \leqslant \mathfrak{L}(M)$,*
4) *$F(f(x), f'(x)) = M$ for $0 \leqslant x \leqslant \mathfrak{L}(M)$* (i.e. $f'(x) = \Phi_M[f(x)]$).

Proof: Form the function $\varphi(y) = \int_{y_1}^{y} dt/\Phi_M(t)$. Clearly, it is continuous and strictly increasing for $y_1 \leqslant y \leqslant y_2$. Further, $\varphi(y_1) = 0$ and $\varphi(y_2) = \mathfrak{L}(M)$. If we define $f(x)$ as the inverse function of $\varphi(y)$, then it follows at once that $f(x)$ satisfies 1 and 2. In order to study $f'(x)$, take an x_0 on the interval $0 \leqslant x_0 \leqslant \mathfrak{L}(M)$ and let $\{\xi_\nu\}_1^\infty$ be a sequence such that $\xi_\nu \to x_0$ and $\xi_\nu \neq x_0$ for all ν. Set $f(\xi_\nu) = \eta_\nu$ and $f(x_0) = y_0$ ($\eta_\nu \to y_0$ but $\eta_\nu \neq y_0$). Then we have

$$\frac{f(\xi_\nu) - f(x_0)}{\xi_\nu - x_0} = \frac{\eta_\nu - y_0}{\varphi(\eta_\nu) - \varphi(y_0)} = \frac{1}{\dfrac{\varphi(\eta_\nu) - \varphi(y_0)}{\eta_\nu - y_0}} = \frac{1}{\dfrac{1}{\eta_\nu - y_0}\displaystyle\int_{y_0}^{\eta_\nu} \frac{dt}{\Phi_M(t)}}.$$

It follows from the continuity of $\Phi_M(t)$ that if $\Phi_M(y_0) > 0$, then

$$\frac{\varphi(\eta_\nu) - \varphi(y_0)}{\eta_\nu - y_0} \to \frac{1}{\Phi_M(y_0)} = \frac{1}{\Phi_M(f(x_0))},$$

but if $\Phi_M(y_0) = 0$, then
$$\frac{\varphi(\eta_\nu) - \varphi(y_0)}{\eta_\nu - y_0} \to \infty.$$

Hence
$$\frac{f(\xi_\nu) - f(x_0)}{\xi_\nu - x_0} \to \Phi_M(f(x_0))$$

in both cases, and we have
$$f'(x_0) = \Phi_M(f(x_0)).$$

But then it follows that $f'(x)$ is continuous and

$$F(f(x), f'(x)) = M \quad \text{for} \quad 0 \leqslant x \leqslant \mathfrak{L}(M).$$

This completes the proof.

The solution of the minimization problem is given in the following

Theorem 4:

a) *Given M, a necessary and sufficient condition for the existence of a function $f \in \mathfrak{F}$ such that $H(f) = M$, is that $\mathfrak{L}(M) \leqslant x_2 - x_1$;*
b) *there is a number M_0 such that $\mathfrak{L}(M) \leqslant x_2 - x_1$ if and only if $M \geqslant M_0$;*
c) *$M_0 = \inf_{f \in \mathfrak{F}} H(f)$;*

d) *there exists a minimizing function* f_0 *(which can be chosen continuously differentiable);*
e) *the minimizing function is unique if and only if* $\mathfrak{L}(M_0)=x_2-x_1$*;*
f) *if the minimizing function is not unique, then* $M_0=\max_{y_1\leqslant y\leqslant y_2} F(y,0)$, *but the converse is not true.*

Proof: a) This is seen from Theorems 2 and 3. In general, the function in Theorem 3 must be translated in the x-direction and continued as a constant to give the function mentioned above.
b) It follows from the properties of $F(y,z)$ and the definition of $\Phi_M(y)$ that $\Phi_M(y)$ is an increasing function of M and hence $\mathfrak{L}(M)$ is decreasing. Let E be the set of all numbers M such that $\mathfrak{L}(M)\leqslant x_2-x_1$. It is clear that E is bounded from below. Let $M_0=\inf E$. Lemma 3 implies that $M_0\in E$ and, since $\mathfrak{L}(M)$ is decreasing, the assertion follows.
c) and d) Consequences of a), b) and Theorem 3.
e) Suppose first that $\mathfrak{L}(M_0)<x_2-x_1$. It is clear that the function in Theorem 3 can be translated in the x-direction and continued as a constant in different ways so as to give us different minimizing functions.

Suppose then that $\mathfrak{L}(M_0)=x_2-x_1$. If $f(x)$ is the function from Theorem 3 with $M=M_0$, then we assert that $h(x)=f(x-x_1)$ is the only minimizing function. Let $g(x)\in\mathfrak{F}$ and assume for example $g(\xi)<h(\xi)$ for some ξ between x_1 and x_2. According to our choice of $h(x)$ we have

$$\int_{h(\xi)}^{y_2}\frac{dt}{\Phi_{M_0}(t)}=x_2-\xi.$$

If $H(g)=M$, then, according to Theorem 2,

$$\int_{g(\xi)}^{y_2}\frac{dt}{\Phi_M(t)}\leqslant x_2-\xi.$$

It follows that
$$\int_{h(\xi)}^{y_2}\frac{dt}{\Phi_M(t)}<\int_{h(\xi)}^{y_2}\frac{dt}{\Phi_{M_0}(t)}.$$

This implies that $M>M_0$, which completes the proof of assertion e).
f) First we shall prove that if the minimizing function is not unique, i.e. $\mathfrak{L}(M_0)<x_2-x_1$, then $M_0=\max_{y_1\leqslant y\leqslant y_2} F(y,0)$. Assume then that $M_0>\max_{y_1\leqslant y\leqslant y_2} F(y,0)$. This means that $\Phi_{M_0}(y)>0$ for $y_1\leqslant y\leqslant y_2$. Since $F(y,z)$ and $\Phi_M(y)$ are continuous functions, there must be a number $\delta>0$ such that $1/\Phi_M(y)$ is uniformly continuous for $y_1\leqslant y\leqslant y_2$ and $|M-M_0|\leqslant\delta$. But then there also must exist a number $M_1<M_0$ such that $\mathfrak{L}(M_1)<x_2-x_1$. But this contradicts b).

Hence
$$M_0=\max_{y_1\leqslant y\leqslant y_2} F(y,0). \tag{1}$$

In order to show that the converse assertion is not true, we give an example where (1) holds and the minimizing function is unique.

Choose
$$F(y,z)=y^2+z^2, x_1=y_1=0, x_2=\frac{\pi}{2}\quad\text{and}\quad y_2=1.$$

Perhaps the easiest way to see that $y = \sin x$ is the unique minimizing function is to apply Theorem 1. (But, of course, the same conclusion can be drawn with the aid of e) above.)

Clearly

$$M_0 = 1 = \max_{0 \leqslant y \leqslant 1} F(y, 0).$$

This completes the proof of Theorem 4.

B) $y_1 > y_2$.

The integral $\mathfrak{L}(M)$ is now replaced by

$$\mathfrak{L}_1(M) = \int_{y_2}^{y_1} \frac{dt}{-\Psi_M(t)} \left(= \int_{y_1}^{y_2} \frac{dt}{\Psi_M(t)} \right)$$

and our conventions are corresponding to those of the former case. Thus, for example, $0 < \mathfrak{L}_1(M) \leqslant +\infty$.

The exact analogues of Lemma 3 and Theorems 2, 3 and 4 are now obtained in a very natural manner by substituting $\mathfrak{L}_1(M)$ for $\mathfrak{L}(M)$ and (in Theorem 3) $\Psi_M(y)$ for $\Phi_M(y)$. In Theorem 3, we must also change the word increasing into decreasing. The proofs are practically the same in both cases.

C) $y_1 = y_2$.

Clearly, the function $f(x) \equiv y_1$ is a minimizing function. Let us write

$$M_0 = F(y_1, 0).$$

As regards the question of uniqueness, it follows from Theorem 4 and its analogue in case B that the minimizing function is unique if and only if *none* of the conditions α and β below is satisfied:

α) both integrals $\displaystyle\int_{y_1}^{y_1+\delta} \frac{dy}{\Phi_{M_0}(y)}$ and $\displaystyle\int_{y_1}^{y_1+\delta} \frac{dy}{-\Psi_{M_0}(y)}$ exist finite for some $\delta > 0$;

β) both integrals $\displaystyle\int_{y_1-\delta}^{y_1} \frac{dy}{\Phi_{M_0}(y)}$ and $\displaystyle\int_{y_1-\delta}^{y_1} \frac{dy}{-\Psi_{M_0}(y)}$ exist finite for some $\delta > 0$.

2 B. *Determination of the attainable cone*

Let us now leave the minimization problem and consider another question.

Suppose there are given a point (x_0, y_0) and a number $M \geqslant F(y_0, 0)$. Denote by E the set of all points (x, y) such that

a) $x \geqslant x_0$,

b) there is an absolutely continuous function $f(t)$, joining the points (x_0, y_0) and (x, y), such that $\sup_t F(f(t), f'(t)) \leqslant M$.

Our task is to determine the set E. As a convenient name for the set E we use "the attainable cone". If $x \geqslant x_0$ then clearly $(x, y_0) \in E$. If $x > x_0$ and $y > y_0$ then we know already that $(x, y) \in E$ if and only if $\int_{y_0}^{y} dt / \Phi_M(t)$ is well-defined and $\leqslant x - x_0$. Of course, there is a similar condition for the case $y < y_0$. If $(x, y_1) \in E$ and $(x, y_2) \in E$, then $(x, y) \in E$ for every y between y_1 and y_2.

We define $g(x)=\inf\{y:(x,y)\in E\}$ and $h(x)=\sup\{y:(x,y)\in E\}$. The functions $g(x)$ and $h(x)$ may take on infinite values. If $g(x)$ is finite, then $(x,g(x))\in E$ and the same is true for $h(x)$. This follows from the criterion above.

For the determination of E, it is thus sufficient to determine $g(x)$ and $h(x)$, and we shall confine our discussion to $h(x)$. Let us use the notation $\varphi(y)=x_0+\int_{y_0}^{y} dt/\Phi_M(t)$. The division into various cases below and the facts about $h(x)$ follow easily from the condition $\varphi(y)\leqslant x$ and the properties of $\varphi(y)$.

A) For every $y>y_0$, $\varphi(y)$ is infinite or not defined. Then $h(x)=y_0$ for all $x\geqslant x_0$.

B) $\varphi(y)$ is defined and finite for $y_0\leqslant y<Y<\infty$ (clearly $\varphi(Y)=\infty$).

$$h(x)=\text{inverse of } \varphi(y) \text{ for all } x\geqslant x_0$$
$$\lim_{x\to\infty} h(x)=Y.$$

C) $\varphi(y)$ defined and finite for $y_0\leqslant y\leqslant Y<\infty$

$$h(x)=\begin{cases}\text{inverse of } \varphi(y) \text{ for } x_0\leqslant x\leqslant\varphi(Y),\\ Y \text{ for } x\geqslant\varphi(Y).\end{cases}$$

D) $\varphi(y)$ defined and finite for all $y\geqslant y_0$.

1. $\lim_{y\to\infty}\varphi(y)=\infty$

   $h(x)=\text{inverse of } \varphi(y)$ for all $x\geqslant x_0$

   $\lim_{x\to\infty} h(x)=\infty$.

2. $\lim_{y\to\infty}\varphi(y)=X<\infty$

$$h(x)=\begin{cases}\text{inverse of } \varphi(y) \text{ for } x_0\leqslant x<X\\ +\infty \text{ for } x\geqslant X.\end{cases}$$

Clearly, $h(x)$ is strictly increasing on every interval where it is defined as the inverse of $\varphi(y)$.

Further, the relation

$$F(h(x),\, h'(x))=M$$

holds at every point where $h(x)$ is finite.

3. Some sufficient conditions for a given function to minimize the functional in the case $F=F(x,y,z)$

Let us now return to the general case where F is allowed to depend on x also. As we have seen in section 2, monotonic functions $f(x)$, such that $F(x, f(x), f'(x))=$ constant, are of great importance. Such a function minimizes $H(f)$, as is seen from Theorem 5. Roughly speaking, Theorem 6 shows that if $f(x)$ is also strictly monotonic, then $f(x)$ is the unique minimizing function. Compare also Theorem 1.

Of course, it can be deduced from the condition $F(x, f(x), f'(x))=$constant that $f(x)$ has a certain degree of regularity (at least for the functions $F(x,y,z)$ that we study). But since this is easy to do and not very important for the theorems, we shall omit a discussion of this and instead choose the conditions on f such as to make the proofs simple. Perhaps we shall return to this question in a later connection.

Now let us assume that the function $F(x,y,z)$ satisfies the conditions:

1) F is continuous for $x_1 \leqslant x \leqslant x_2$ and all y and z,
2) $\partial F/\partial z$ exists for $x_1 \leqslant x \leqslant x_2$ and all y and z.

Further,
$$\frac{\partial F}{\partial z} \text{ is } \begin{cases} >0 & \text{if } z>0, \\ =0 & \text{if } z=0, \\ <0 & \text{if } z<0. \end{cases}$$

We shall use the same notations as in section 2 and the same definitions of admissible functions and the functional. It follows that
$$H(g) \geqslant \max\,(F(x_1,y_1,0),\ F(x_2,y_2,0))$$
for any $g \in \mathfrak{J}$.

Theorem 5: *Suppose that F satisfies the conditions* 1 *and* 2 *above. Let f be an admissible function such that:*

a) *$f'(x)$ exists for $x_1 \leqslant x \leqslant x_2$,*
b) *$f(x)$ is monotonic (not necessarily strictly),*
c) *$F(x, f(x), f'(x)) = M$ for $x_1 \leqslant x \leqslant x_2$.*

Then $f(x)$ is a minimizing function. ($f(x)$ need *not* be the only one.)

Proof: Let us choose the case where $f(x)$ is non-decreasing. Let h be a different element of $\mathfrak{J}$. Then we have, for example, $h(\xi) < f(\xi)$ for some ξ, $x_1 < \xi < x_2$.

Then there must exist an $x_0 \leqslant x_2$ such that $h(x_0) = f(x_0)$ but $h(x) < f(x)$ for $\xi \leqslant x < x_0$.

Hence
$$\overline{\lim_{x\to x_0}}\, h'(x) \geqslant f'(x_0) \geqslant 0.$$

If $f'(x_0) = 0$, then it follows at once that
$$\overline{\lim_{x\to x_0}}\, F(x, h(x), h'(x)) \geqslant F(x_0, f(x_0), f'(x_0)) = M. \tag{1}$$

If $f'(x_0) > 0$, then we take a $\delta > 0$ and a sequence $\xi_\nu \to x_0$ such that $h'(\xi_\nu) > f'(x_0) - \delta > 0$. This means that
$$F(\xi_\nu, h(\xi_\nu), h'(\xi_\nu)) > F(\xi_\nu, h(\xi_\nu), f'(x_0) - \delta).$$

The right member tends to
$$F(x_0, f(x_0), f'(x_0) - \delta).$$

If we let $\delta \to 0$, then (1) follows again. Thus
$$H(h) \geqslant F(x_0, f(x_0), f'(x_0)) = M = H(f).$$

This completes the proof.

Theorem 6: *Suppose that $F(x,y,z)$ satisfies the conditions:*

A) $F, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$ *exist and are continuous for* $\begin{cases} x_1 \leqslant x \leqslant x_2, \\ y, z \text{ arbitrary.} \end{cases}$

B) $\dfrac{\partial F}{\partial z}$ *is* $\begin{cases} >0 & \text{if} \quad z>0, \\ =0 & \text{if} \quad z=0, \\ <0 & \text{if} \quad z<0. \end{cases}$

Suppose further that $f(x)$ is an admissible function such that:

a) *$f'(x)$ is continuous and $\neq 0$ for $x_1 \leqslant x \leqslant x_2$,*
b) *$F(x, f(x), f'(x)) = M$ for $x_1 \leqslant x \leqslant x_2$.*

Then $f(x)$ is a unique minimizing function in $\mathfrak{J}$. (Compare Theorem 1.)

Proof: Following our habit, we will carry out the proof only for the case $f'>0$. Assume now that $g \in \mathfrak{J}$, $g \not\equiv f$ and $H(g) \leqslant M$. We want to derive a contradiction from this.

Let $g(\xi) > f(\xi)$ for some ξ, $x_1 < \xi < x_2$. Then there must be a number x_0 such that $x_1 \leqslant x_0 < \xi$ and $g(x) > f(x)$ for $x_0 < x \leqslant \xi$ but $g(x_0) = f(x_0)$.

Hence
$$\overline{\lim_{x \to x_0+0}}\, g'(x) \geqslant f'(x_0).$$

But since $F(x, g, g') \leqslant M$ and $f'(x_0) > 0$ it follows with the use of B) that
$$\overline{\lim_{x \to x_0}}\, g'(x) \leqslant f'(x_0).$$

Consequently
$$\overline{\lim_{x \to x_0}}\, g'(x) = f'(x_0).$$

Form
$$\varphi(x) = g(x) - f(x).$$

If $g'(x)$ exists, then we apply the mean value theorem to the function
$$\Psi(t) = F(x, f(x) + t\varphi(x), f'(x) + t\varphi'(x))$$

between $t=0$ and $t=1$. This gives
$$\varphi(x) \cdot F_y(x, f(x) + \theta\varphi(x), f'(x) + \theta\varphi'(x)) + \varphi'(x) F_z(\ldots) \leqslant 0,$$

i.e.
$$\varphi'(x) F_z(\ldots) \leqslant -\varphi(x) F_y(\ldots). \tag{1}$$

Let δ be a number >0 such that $x_0 + \delta \leqslant \xi$, and write $J_\delta = [x_0, x_0 + \delta]$. Then $\varphi(x) \geqslant 0$ for $x \in J_\delta$.

Form the function
$$C(\delta) = \sup_{x \in J\delta} \varphi'(x).$$

Clearly
$$0 \leqslant \varphi(x) \leqslant \delta C(\delta) \quad \text{for} \quad x \in J_\delta. \tag{2}$$

Put $\delta_\nu = 1/\nu$ and select, for every $\nu \geqslant \nu_0$, $x_\nu \in J_{\delta_\nu}$ such that
$$\varphi'(x_\nu) > (1 - \delta_\nu) C(\delta_\nu) > 0.$$
$$g'(x_\nu) > f'(x_\nu) \quad \text{implies} \quad \underline{\lim_{\nu \to \infty}}\, g'(x_\nu) \geqslant f'(x_0).$$

But
$$\overline{\lim_{x\to x_0}}\, g'(x) = f'(x_0).$$

Hence
$$\lim_{\nu\to\infty} g'(x_\nu) = f'(x_0).$$

Put $x = x_\nu$ in (1).
$$f'(x_\nu) < f'(x_\nu) + \theta_\nu \varphi'(x_\nu) < g'(x_\nu).$$

We see that
$$f'(x_\nu) + \theta_\nu \varphi'(x_\nu) \to f'(x_0) > 0.$$

From the conditions on F it follows that
$$F_z(x_\nu, f(x_\nu) + \theta_\nu \varphi(x_\nu), f'(x_\nu) + \theta_\nu \varphi'(x_\nu)) \geq \alpha > 0 \quad \text{for} \quad \nu \geq \nu_1$$

and
$$|F_y(\ldots)| \leq K \quad \text{for} \quad \nu \geq \nu_2.$$

Here α and K are constants independent of ν.
(1) can now be written (for $\nu \geq \nu_3$)
$$0 < \varphi'(x_\nu) F_z(\ldots) \leq -\varphi(x_\nu) F_y(\ldots).$$

This implies that
$$|\varphi'(x_\nu)|\ |F_z(\ldots)| \leq |\varphi(x_\nu)|\ |F_y(\ldots)|.$$

Left member
$$> (1-\delta_\nu) C(\delta_\nu) \alpha.$$

Right member
$$\leq \delta_\nu C(\delta_\nu) K \ \text{(see (2))}.$$

Hence
$$(1-\delta_\nu) C(\delta_\nu) \alpha \leq \delta_\nu C(\delta_\nu) K$$

and we have
$$(1-\delta_\nu)\alpha \leq K\delta_\nu.$$

But this gives a contradiction if $\nu \to \infty$.
The case $g(\xi) < f(\xi)$ can be treated analogously, and so the proof is complete.

4. Examination of functions which minimize the functional on every interval

In this section we shall examine more closely the connection between the minimization problem and the differential equation
$$\frac{dF(x, f(x), f'(x))}{dx} \cdot F_z(x, f(x), f'(x)) = 0$$

derived in the introduction. The main results on this subject are the Theorems 8 and 9.(1)

Let us first state the conditions on $F(x, y, z)$. We shall assume that the conditions 1 and 2, given in section 3 (before Theorem 5) hold throughout section 4, but we must replace $[x_1, x_2]$ by the interval considered in each case. Later on, we shall impose further conditions on F.

Suppose that the function $f(x)$ is defined on the interval I and let $\alpha \leq x \leq \beta$ be a compact subinterval of I.

(1) See also the Theorems 5 and 6 of section 3.

Definition: If it is true, for every such interval $[\alpha,\beta]$ (inclusive of I if I is compact) that $f(x)$ is a minimizing function for $\alpha \leqslant x \leqslant \beta$ and assigned boundary values $f(\alpha)$ and $f(\beta)$, then $f(x)$ is said to minimize the functional *in the absolute sense* on the interval I. The function $f(x)$ is said to be a *minimal in the absolute sense* on I. This will be abbreviated *a.s. minimal* in the sequel.

Lemma 4: *Let I be a compact interval and suppose that $f(x)$ is absolutely continuous on I. Denote by E the subset of I where $f'(x)$ exists, including endpoints of I if the appropriate one-sided derivatives exist. Then*

$$\sup_{x\in E} F(x, f(x), f'(x)) = \operatorname{ess\,sup}_{x\in E} F(x, f(x), f'(x))$$

in the sense that if one member is finite, then so is the other and they are equal.

Proof: Let x_0 be a point of I such that $f'(x_0)$ exists and let I_1 be a subinterval of I containing x_0. Then it is true, for every $\delta > 0$, that $f'(x) > f'(x_0) - \delta$ on a subset of I_1 of positive measure. Similarly, $f'(x) < f'(x_0) + \delta$ holds on a subset of I_1 of positive measure.

From this, and from the conditions on $F(x,y,z)$, it follows easily that

$$\operatorname{ess\,sup}_{x\in I_1\cap E} F(x, f(x), f'(x)) \geqslant F(x_0, f(x_0) f'(x_0)),$$

and the rest of the proof is obvious.

Remark: It follows from this lemma that our previous definition of $H(f)$ is equivalent to the definition

$$H(f) = \operatorname{ess\,sup}_{x\in E} F(x, f(x), f'(x)).$$

It also follows that, in the definition of $H(f)$, we can exclude the endpoints of I from E without changing the functional in any way. Therefore, the conditions a) and c) in Theorem 1 may be weakened into

a′) $f'(x)$ is continuous for $x_1 < x < x_2$

and

c′) $F(f(x), f'(x)) = M$ for $x_1 < x < x_2$.

Let us now consider the minimization problem on the interval $x_1 \leqslant x \leqslant x_2$ with the boundary values y_1 and y_2, respectively. Let us use the notation

$$M(x_1, x_2; y_1, y_2) = \inf_{f\in\mathfrak{J}} H(f).$$

We shall need the following estimates:

Lemma 5: *Let L be the straight line between (x_1, y_1) and (x_2, y_2).*

Put
$$t = \frac{y_2 - y_1}{x_2 - x_1}.$$

Then
$$\min_{(x,y)\in L} F(x,y,t) \leqslant M(x_1, x_2; y_1, y_2) \leqslant \max_{(x,y)\in L} F(x,y,t).$$

Proof: The right inequality is obvious. Let $l(x)$ be the admissible linear function and let $m(x)$ be a different admissible function. Now the left inequality is proved in the same way as Theorem 5. We only have to substitute $l(x)$ for $f(x)$ and $m(x)$ for $h(x)$. This completes the proof.

Lemma 6: *Suppose that* $x_n \to x_0$, $\xi_n \to x_0$, $x_n < \xi_n$ *for all* n, $y_n \to y_0$, $\eta_n \to y_0$ *and* $(\eta_n - y_n)/(\xi_n - x_n) \to z_0$. *(Of course,* $F(x,y,z)$ *must be defined and satisfy its conditions on an interval containing all the points* x_0, $\{x_n\}_1^\infty$ *and* $\{\xi_n\}_1^\infty$.*) Then*

$$\lim_{n\to\infty} M(x_n, \xi_n; y_n, \eta_n) = F(x_0, y_0, z_0).$$

Proof: This is an immediate consequence of the preceding lemma and the continuity of $F(x,y,z)$.

Lemma 7: *Suppose that* $f(x)$ *is a minimizing function on* $x_1 \leqslant x \leqslant x_2$. *(Boundary values* y_1 *and* y_2.*) Suppose further that* $x_1 \leqslant \alpha < \beta \leqslant x_2$. *Then*

$$M(\alpha,\beta;\, f(\alpha),\, f(\beta)) \leqslant M(x_1,x_2;\, y_1,y_2).$$

Proof: $M(\alpha,\beta;\, f(\alpha), f(\beta)) \leqslant H(f;\, \alpha,\beta) \leqslant H(f;\, x_1,x_2) = M(x_1,x_2;\, y_1,y_2)$, where the meaning of the notations is obvious.

Now we must introduce a new condition on $F(x,y,z)$, and this condition is assumed to hold in the rest of section 4:

3) $\lim_{|z|\to\infty} F(x,y,z) = +\infty$ for every fixed x and y.

As is easily seen, using the conditions 1 and 2 also, this means that

$$\lim_{|z|\to\infty} \Big(\inf_{\substack{\alpha\leqslant x\leqslant\beta \\ |y|\leqslant K}} F(x,y,z) \Big) = +\infty$$

for every compact interval $[\alpha,\beta]$ (where F is defined) and for every $K > 0$.

Theorem 7: *If* $f(x)$ *is an a.s. minimal on an interval containing* x_0 *in its interior, then* $f'(x_0)$ *exists.*

Proof: Let I be a compact interval with x_0 in its interior such that $f(x)$ is an a.s. minimal on I. Then $f(x)$ and $F(x, f(x), f'(x))$ are bounded on I. But then it follows that $f'(x)$ is also bounded on I. Let α be the greatest and β the smallest of the four derivates of $f(x)$ at $x = x_0$. Then α and β are finite. Clearly, there exist sequences $\{p_n\}$ and $\{q_n\}$ such that

$$p_n < x_0 < q_n,\quad q_n - p_n \to 0$$

and

$$\frac{f(q_n) - f(p_n)}{q_n - p_n} \to \alpha.$$

Of course, corresponding sequences $\{r_n\}$, $\{s_n\}$ exist for β.

Assume now that $\alpha > \beta$ and let γ be any number such that $\alpha > \gamma > \beta$. As is easily seen, there must exist sequences $\{t_n\}$, $\{u_n\}$ such that $t_n < x_0 < u_n$, $u_n - t_n \to 0$ and $[f(u_n) - f(t_n)]/(u_n - t_n) = \gamma$. According to Lemma 6 we have

$$\lim_{n\to\infty} M(p_n, q_n; f(p_n), f(q_n)) = F(x_0, f(x_0), \alpha),$$

$$\lim_{n\to\infty} M(r_n, s_n; f(r_n), f(s_n)) = F(x_0, f(x_0), \beta),$$

and

$$\lim_{n\to\infty} M(t_n, u_n; f(t_n), f(u_n)) = F(x_0, f(x_0), \gamma).$$

Now there exist arbitrarily great numbers n_1, n_2, n_3 such that $(t_{n_1}, u_{n_1}) \subset (r_{n_2}, s_{n_2}) \subset (p_{n_3}, q_{n_3})$.

Application of Lemma 7 now gives $F(x_0, f(x_0), \gamma) \leqslant F(x_0, f(x_0), \beta) \leqslant F(x_0, f(x_0), \alpha)$. But the inclusion relations can also be chosen in the opposite way, which gives

$$F(x_0, f(x_0), \gamma) \geqslant F(x_0, f(x_0), \beta) \geqslant F(x_0, f(x_0), \alpha).$$

Hence these three numbers are equal. But this contradicts our assumption that $\alpha > \gamma > \beta$.

Consequently $\alpha = \beta$ which means that $f'(x_0)$ exists.

Remark: If $f(x)$ is an a.s. minimal on the interval $x_1 \leqslant x \leqslant x_2$, then it follows (with a few modifications in the proof) that the one-sided derivatives in question exist at x_1 and x_2.

Lemma 8: *If $f(x)$ is an a.s. minimal on an open interval containing x_0 and $f'(x_0) = 0$, then $f'(x)$ is continuous at x_0.*

Proof: Lemma 6 gives

$$\lim_{n\to\infty} M\left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}; f\left(x_0 - \frac{1}{n}\right), f\left(x_0 + \frac{1}{n}\right)\right) = F(x_0, f(x_0), 0)$$

and, since f is an a.s. minimal, we get

$$\lim_{n\to\infty} H\left(f; x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) = F(x_0, f(x_0), 0)$$

from which the assertion follows.

Remark: This result is obviously true also for an end-point of an interval.

Theorem 8: *To our previous conditions on $F(x,y,z)$ we add the following: F_x, F_y and F_z exist and are continuous for all x under consideration and all y, z.*

Suppose now that $f(x)$ is an a.s. minimal on an interval which contains x_0 in its interior, and suppose $f'(x_0) \neq 0$.

Then

1) *$f(x) \in C^2$ on an open interval I containing x_0*
2) *$F(x, f(x), f'(x)) =$ constant on I.*

$$\left(\text{Hence } \frac{dF(x, f(x), f'(x))}{dx} = 0 \text{ on I.}\right)$$

Proof: Our method of proof will be the following: We construct two solutions of the differential equation $F(x,y,y') =$ constant, the first of which is equal to $f(x)$ at x_0 and at some $x' > x_0$ and the second is equal to $f(x)$ at x_0 and at some $x'' < x_0$. Then, using Theorem 6, we prove that $f(x)$ and these two solutions are identical. We shall confine the discussion to the case $f'(x_0)>0$, since the other case is analogous.

Let us introduce the notations $y_0=f(x_0)$, $z_0=f'(x_0)$, $M_0=F(x_0,y_0,z_0)$ and $\Psi(x,y,z,M)\equiv F(x,y,z)-M$. Then $\Psi(x_0,y_0,z_0,M_0)=0$ and

$$\Psi_z(x_0,y_0,z_0,M_0)=F_z(x_0,y_0,z_0)>0.$$

Hence the equation $\Psi(x,y,z,M)=0$ can be used to define z as a function $z=\Phi(x,y,M)$ on the set R in $x\,y\,M$-space, defined by the inequalities $|x-x_0|\leqslant\delta$, $|y-y_0|\leqslant\delta$ and $|M-M_0|\leqslant\delta$, for some $\delta>0$.

We may assume that we have, for $(x,y,M)\in R$ and for some $\delta_1>0$,

$$0<z_0-\delta_1\leqslant\Phi(x,y,M)\leqslant z_0+\delta_1.$$

We shall also assume that $f(x)$ is an a.s. minimal on $|x-x_0|\leqslant\delta$. The function $\Phi(x,y,M)$ is continuously differentiable on R, and we have

$$\frac{\partial\Phi}{\partial x}=-\frac{F_x}{F_z},\quad \frac{\partial\Phi}{\partial y}=-\frac{F_y}{F_z}\quad\text{and}\quad\frac{\partial\Phi}{\partial M}=\frac{1}{F_z}.$$

Hence
$$\left|\frac{\partial\Phi}{\partial x}\right|\leqslant C_1\quad\text{and}\quad\left|\frac{\partial\Phi}{\partial y}\right|\leqslant C_2\ \text{on } R.$$

Let us consider the differential equation $y'=\Phi(x,y,M)$ together with the initial value $y(x_0)=y_0$. Here the parameter M is assumed to satisfy $|M-M_0|\leqslant\delta$.

It follows from Picard's theorem that there exists a unique solution for $|x-x_0|\leqslant\delta_2$, and δ_2 is independent of M. Let us denote the solution by $y(x,M)$. It is also true that for every x_1, such that $|x_1-x_0|\leqslant\delta_2$, the solution $y(x_1,M)$ depends continuously on M. This is proved by a standard argument. (Cf. [2], pp. 46, 65, 70.) If $|x-x_0|\leqslant\tau\leqslant\delta_2$, then, clearly, $|y(x,M)-y_0|\leqslant K\tau$, where $K=z_0+\delta_1$. If $\beta>0$ is small enough, then there is a $\tau>0$ (but $\tau\leqslant\delta_2$) such that the inequalities $|x-x_0|\leqslant\tau$, $|y-y_0|\leqslant K\tau$, imply that

$$\begin{cases}\Phi(x,y,M_0+\delta)\geqslant z_0+\beta\\ \Phi(x,y,M_0-\delta)\leqslant z_0-\beta.\end{cases}$$

and

(For Φ is continuous and $\partial\Phi/\partial M>0$ on R.) This gives the inequalities

$$y(x,\,M_0+\delta)\geqslant y_0+(z_0+\beta)(x-x_0)$$

and
$$y(x,\,M_0-\delta)\leqslant y_0+(z_0-\beta)(x-x_0),$$

valid for $x_0\leqslant x\leqslant x_0+\tau$.

Fix numbers β and τ having the above properties and, in addition, satisfying the condition

$$y_0+(z_0-\beta)\tau<f(x_0+\tau)<y_0+(z_0+\beta)\tau.$$

Then we have

$$y(x_0+\tau,M_0-\delta)<f(x_0+\tau)<y(x_0+\tau,\,M_0+\delta).$$

Since $y(x_0+\tau,M)$ depends continuously on M, there is a value M^* such that

$$y(x_0+\tau,M^*)=f(x_0+\tau).$$

But $f(x)$ is an a.s. minimal and now it follows from Theorem 6 that $f(x) = y(x, M^*)$ for $x_0 \leqslant x \leqslant x_0 + \tau$.

In a similar way one can find numbers τ' and M^{**} and prove that $f(x) = y(x, M^{**})$ for $x_0 - \tau' \leqslant x \leqslant x_0$.

Hence
$$f'(x_0) = \Phi(x_0, y_0, M_0) = \Phi(x_0, y_0, M^*) = \Phi(x_0, y_0, M^{**}),$$

which gives us $M_0 = M^* = M^{**}$. Consequently $f'(x) = \Phi(x, f(x), M_0)$ for $x_0 - \tau' \leqslant x \leqslant x_0 + \tau$.

This means that $F(x, f(x), f'(x)) = M_0$ for the same values of x, and the rest of the proof is obvious.

Remark: As is easily seen, the theorem continues to hold (but with obvious modifications) in the case when x_0 is an end-point of an interval, where $f(x)$ is an a.s. minimal.

Theorem 9: *Let $F(x,y,z)$ satisfy the same conditions as in the previous theorem. If $f(x)$ is an a.s. minimal on the interval I, then:*

1) *$f(x) \in C^1$ on I*
2) *the differential equation*

$$\frac{dF(x, f(x), f'(x))}{dx} \cdot F_z(x, f(x), f'(x)) = 0$$

is satisfied on I in the following sense: The second factor is well-defined on I and if it is different from zero at x_0, then the first factor exists and is zero in a neighbourhood of x_0.

Proof: The theorem is a consequence of Theorem 7, Lemma 8 and Theorem 8.

Remark: As is shown by Example 6 in section 5, the derivatives $f''(x)$ and $dF(x, f(x), f'(x))/dx$ need not exist at points where $f'(x) = 0$.

Corollary: *Under the present conditions on $F(x,y,z)$, suppose that $f(x)$ is a unique minimizing function on $x_1 \leqslant x \leqslant x_2$. Then*

$$f(x) \in C^1 \quad \text{on} \quad [x_1, x_2] \quad \text{and} \quad F(x, f(x), f'(x)) = \text{constant on } [x_1, x_2].$$

Proof: Obviously, $f(x)$ is an a.s. minimal on $[x_1, x_2]$. Hence, by the theorem, $f(x) \in C^1$. If it were true that $F(x_0, f(x_0), f'(x_0)) < H(f)$ for some x_0, then we could alter $f(x)$ slightly in a neighbourhood of x_0 without increasing the value of $H(f)$. But this contradicts the uniqueness, and hence we have $F(x, f(x), f'(x)) =$ constant. (Compare Theorem 1 and Theorem 6.)

In the theorems 5 and 6, the condition that $f(x)$ is monotonic plays an important role. If we impose a suitable extra condition on $F(x,y,z)$, then every a.s. minimal must be monotonic:

Theorem 10: *Suppose that $F(x,y,z)$ satisfies all the conditions in Theorem 8 together with the extra condition that $\partial F/\partial x$ does not change sign for $x \in I$ and any y,z. If now $f(x)$ is an a.s. minimal on the interval I, then $f(x)$ is monotonic on I. (But $f(x)$ need not be strictly monotonic; compare Example 3 in section 5.)*

Proof: According to Theorem 9 we have $f(x) \in C^1(I)$. Assume, for example, that there exist x_1 and x_2 on I such that $f'(x_1)>0$ and $f'(x_2)<0$. Assume also $x_1<x_2$. Obviously, there exist x_3 and x_4 such that $x_1\leqslant x_3<x_4\leqslant x_2$, $f(x_3)=f(x_4)$, $f'(x_3)>0$ and $f'(x_4)<0$. Consider the minimization problem on $x_3\leqslant x\leqslant x_4$. If $g(x)\equiv f(x_3)$, then $g(x)$ is admissible and, since F_x does not change sign, we have $H(g)=\max\,(F(x_3, f(x_3), 0), F(x_4, f(x_4), 0))$. Assume $H(g)=F(x_3, f(x_3), 0)$. But since $f'(x_3)>0$ we have $H(f)>H(g)$ which is impossible, since $f(x)$ is an a.s. minimal. Hence $f'(x)$ does not change sign, which completes the proof.

5. Examples

In this section, we want to show a few applications of some of the theorems already given. We also motivate by means of examples the introduction of condition 3 and the formulation of Theorem 9.

Example 1A: Let us use Theorem 4 to solve the minimization problem if

$$F(y,y')=y^2+y'^2, x_1=y_1=0, x_2=\frac{\pi}{4} \quad \text{and} \quad y_2=\frac{1}{\sqrt{2}}.$$

We have

$$\Phi_M(y)=\sqrt{M-y^2}, \quad \text{and}$$

$$\mathfrak{L}(M)=\int_0^{1/\sqrt{2}}\frac{dt}{\sqrt{M-t^2}}$$

is well-defined for $M\geqslant\frac{1}{2}$. For such values of M, we have $\mathfrak{L}(M)=\arcsin\,(1/\sqrt{2M})$. To find M_0, we must determine the smallest $M\geqslant\frac{1}{2}$ such that

$$\arcsin\frac{1}{\sqrt{2M}}\leqslant\frac{\pi}{4}.$$

This gives us $M_0=1$. Since $\mathfrak{L}(1)=\pi/4$, there is a unique minimizing function. In order to find it, we use Theorem 3 and form

$$\varphi(y)=\int_0^y\frac{dt}{\sqrt{1-t^2}}=\arcsin\ y.$$

Since $x_1=0$, the minimizing function is the inverse of $\varphi(y)$, namely $f_0(x)=\sin x$.

Example 1B: This will illustrate the case in Theorem 4 where there is no *unique* minimizing function.

Choose

$$F(y,y')=y'^4-16y^2,$$

$$x_1=-2, y_1=-1, x_2=2 \quad \text{and} \quad y_2=1.$$

Write $y'^4-16y^2=M$ which gives

$$\Phi_M(y)=\sqrt[4]{16y^2+M}.$$

For the existence of

$$\mathfrak{L}(M)=\int_{-1}^{1}\frac{dt}{\sqrt[4]{16t^2+M}}$$

it is clearly necessary and sufficient that $M\geqslant 0$.

We need not evaluate the integral $\mathcal{L}(M)$, since

$$\mathcal{L}(0)=\int_{-1}^{1}\frac{dt}{2\sqrt{|t|}}=<x_2-x_1.$$

We see that $M_0=0$ and that there is no unique minimizing function.

Let us determine a minimizing function! Form the function

$$g(y)=\int_0^y\frac{dt}{2\sqrt{|t|}}\quad\text{for}\quad -1\leqslant y\leqslant 1.$$

(Compare Theorem 3.) We get

$$g(y)=\begin{cases}\sqrt{y} & \text{for}\quad y\geqslant 0,\\ -\sqrt{|y|} & \text{for}\quad y\leqslant 0.\end{cases}$$

The inverse function is $f(x)=x|x|$. If we define $f(x)$ as $+1$ for $x>1$ and -1 for $x<-1$, then we get a minimizing function. The function $h(x)=\frac{1}{4}x|x|$ (for $-2\leqslant x\leqslant 2$) is also a minimizing function, but in contrast to the former one, it is continuously differentiable.

We have $M_0=\max_{y_1\leqslant y\leqslant y_2}F(y,0)$ in accordance with Theorem 4.

Example 2: Let us use the rules given in section 2B to determine the attainable cone if $F(y,y')=y'^2-y^4$, $x_0=0$, $y_0=1$ and $M=0$.

We have $\Phi_0(y)=y^2$ and hence

$$\varphi(y)=x_0+\int_{y_0}^{y}\frac{dt}{\Phi_M(t)}=\int_1^y\frac{dt}{t^2}=1-\frac{1}{y}.$$

Since $\lim_{y\to\infty}\varphi(y)=1$ we see that the present case is $D2$ and that $X=1$. The inverse of $x=\varphi(y)=1-1/y$ is $y=1/(1-x)$. Therefore,

$$h(x)=\begin{cases}\dfrac{1}{1-x} & \text{for}\quad 0\leqslant x<1,\\ +\infty & \text{for}\quad x\geqslant 1.\end{cases}$$

In order to determine $g(x)$, we form the function

$$\Psi(y)=x_0+\int_y^{y_0}\frac{dt}{-\Psi_M(t)}=\int_y^1\frac{dt}{t^2}=\frac{1}{y}-1.$$

Clearly, this corresponds to the case B. The inverse of $x=\Psi(y)=1/y-1$ is $y=1/(1+x)$. Hence $g(x)=1/(1+x)$ for all $x\geqslant 0$.

Example 3: Put $F(y,y')=y^2+y'^2$ and define the function $f_0(x)$ as

$$f_0(x)=\begin{cases}1 & \text{for} \quad x>\dfrac{\pi}{2},\\ \sin x & \text{for} \quad -\dfrac{\pi}{2}\leqslant x\leqslant\dfrac{\pi}{2},\\ -1 & \text{for} \quad x<-\dfrac{\pi}{2}.\end{cases}$$

Clearly, $f_0(x)\in C^1$ and $f_0(x)$ is monotonic on $-\infty<x<\infty$. Further, $F(f_0(x), f_0'(x))=1$ for all x. Now it follows from Theorem 5 that $f_0(x)$ is an a.s. minimal on $-\infty<x<\infty$. Observe that $f_0''(x)$ does not exist at $x=\pm\pi/2$. Compare Example 5.

Example 4: This example shows that the condition 3 cannot be omitted in Theorem 7. It will also give an idea of the case where F is independent of y. Let

$$F(x,y,y')=x^2+\frac{y'^4}{y'^4+1}.$$

The conditions 1 and 2 are satisfied, but not condition 3.

Put
$$F(x,y,y')=1.$$

This gives
$$y'=\pm\sqrt[4]{\frac{1-x^2}{x^2}}.$$

Form the primitive function

$$f(x)=\int_0^x\frac{\sqrt[4]{1-t^2}}{\sqrt{|t|}}\,dt \quad \text{for} \quad -1\leqslant x\leqslant 1.$$

It is easy to see that $f(x)$ is a unique minimizing function between $(-1, f(-1))$ and $(1, f(1))$. Hence, $f(x)$ is an a.s. minimal. But $f'(0)$ is not finite.

Example 5: This is a continuation of Example 3. We now want to determine *all* a.s. minimals for $F(y,y')=y^2+y'^2$.

Assume that $f(x)$ is an a.s. minimal on an open interval I. It follows from the theorems 9 and 10 that $f(x)\in C^1(I)$ and that $f(x)$ is monotonic.

Clearly $f(x)\equiv\text{constant}$ is possible. But assume now that we have $f'(x_1)\neq 0$ for some $x_1\in I$.

Let I_1 be the largest open interval containing x_1 where $f'(x)\neq 0$. Then $f(x)\in C^2(I_1)$ and $f(x)^2+f'(x)^2=\text{constant}$ on I_1. Differentiation gives $f''(x)+f(x)=0$. Hence there exist A and B such that $f(x)=A\sin(x+B)$ on I_1. Since $f(x)$ is monotonic, it follows that I_1 is bounded, say $I_1=(\alpha,\beta)$. Now we assert that $f'(x)=0$ on $I-I_1$ (if it is not empty). If this were not so, then we would have a different open interval $I_2=(\gamma,\delta)$ with the same properties as I_1. Let us assume that $\beta\leqslant\gamma$ and that $f'(x)>0$ on $I_1\cup I_2$.

We have $f'(\beta)=f'(\gamma)=0$. Since $f(x)$ is a sine-function on I_1, it follows that $f(\beta)>0$. From the monotonicity we conclude that $f(\gamma)\geqslant f(\beta)$. Hence $f(x)=C\sin(x+D)$ has the properties $f(\gamma)>0$, $f'(\gamma)=0$ and $f'(x)>0$ for $\gamma<x<\delta$. But this is clearly impossible. Consequently $f'(x)=0$ on $I-I_1$. This leads to the following result: There exist numbers p and q such that $f(x)=pf_0(x+q)$ for all $x\in I$, where $f_0(x)$ is the function introduced in Example 3.

Hence we have found that *the class of a.s. minimals for* $F(y,y')=y^2+y'^2$ *is the class of functions of the form* $pf_0(x+q)$ *where* p *and* q *are arbitrary real numbers, and constant functions* (which cannot be written as $pf_0(x+q)$ if I is the entire real axis).

Example 6: This example shows that the derivatives $dF(x,f(x),f'(x))/dx$ and $f''(x)$ in Theorem 9 need not exist for all x.

Choose $F(x,y,y')=y'^2-x$ and consider the function

$$f(x)=\begin{cases}0 & \text{for} \quad x\leqslant 0,\\ \frac{2}{3}x^{3/2} & \text{for} \quad x\geqslant 0.\end{cases}$$

Then

$$F(x,f(x),f'(x))=\begin{cases}|x| & \text{for} \quad x\leqslant 0,\\ 0 & \text{for} \quad x\geqslant 0.\end{cases}$$

We assert that $f(x)$ is an a.s. minimal on $-\infty<x<\infty$. To prove that, consider the minimization problem on $x_1\leqslant x\leqslant x_2$. If $x_1\geqslant 0$ then it follows from Theorem 5 that $f(x)$ is a minimizing function.

If $x_1<0$, then we have $H(f)=|x_1|$ and since every admissible function must pass through the point $(x_1,0)$, it follows that $\inf_g H(g)\geqslant F(x_1,0,0)=|x_1|$. Hence $f(x)$ is an a.s. minimal. But the derivatives $dF(\dots)/dx$ and $f''(x)$ do not exist at $x=0$.

Therefore, in Theorem 9, we can *not* assert that the differential equation $dF(\dots)/dx\cdot F_z(\dots)=0$ is satisfied in the classical sense.

REFERENCES

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In a coming paper we shall discuss some questions that have been left open here.

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