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A strong form of spectral synthesis

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Introduction

The class of sets admitting spectral synthesis has the disadvantage that one does not know whether or not the union of two such sets is always a set of the same kind. By imposing more restrictive conditions in the definitions, several subclasses have been defined that have this desirable property. One way to achieve this has been pointed out by Calderon in [2], where he introduces strongly regular sets. A subclass of the strongly regular sets is formed by the Ditkin sets studied by Rudin [9, pp. 169–171]. (See also Kahane–Salem [6, p. 183].)

In this paper we shall study a proper subclass of the Ditkin sets, which we call strong Ditkin sets. We are restricting ourselves to sets on the interval $[0, 2\pi]$ of the real line. The first three theorems tell us that finite unions of closed intervals and points are strong Ditkin sets. The rest of the paper is an examination of arbitrary strong Ditkin sets. Theorems 4–7 prove that such a set has the property that every accumulation point of it is an accumulation point of intervals of the set. Still it might be that the strong Ditkin sets are exactly the sets with finite boundary, although the author has not been able to prove it.

Professor L. Carleson suggested the subject of this paper and I wish to thank him for his valuable guidance.

Notations

We shall denote by A the Banach space of all functions f with period 2π, which have a representation

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \quad \text{where} \quad \|f\| = \sum_{k=-\infty}^{\infty} |a_k| < \infty.$$

We shall frequently use a special function in A, which we call $g_h(x, a)$ and which is defined by

$$g_h(x,a) = \begin{cases} 0 & \text{for} \quad |x-a| \geqslant h, \quad h < \pi \\ 1 & \text{for} \quad x = a \\ \text{linear for } a-h < x < a & \text{and} \quad a < x < a+h. \end{cases}$$

Then $g_h(x, a) = \sum_{k=-\infty}^{\infty} b_{k,h} e^{-ika} e^{ikx}$. This function has the following properties:

$$\|g_h(x,a)\| = 1 \text{ for every } h \text{ and } a \tag{1}$$

$$\sum_{k=-\infty}^{\infty} |b_{k+p,h} - b_{k,h}| < C_p h \text{ where } C_p \text{ depends only on } p. \tag{2}$$

For the proof of (1) and (2) see e.g. Herz [5, p. 188].

Definitions

Let E be a closed set on $[0, 2\pi]$.

Definition 1

If for every $f \in A$, such that $f(x) = 0$ on E and for every $\varepsilon > 0$ there exists a $g \in A$ such that $g(x) = 0$ on a neighbourhood of E and $\|f - g\| < \varepsilon$ then we say that E *admits spectral synthesis* or that E is a *regular set.*

Definition 2

If, in Definition 1, we can further prescribe that $g(x) = 0$ whenever $f(x) = 0$, then E is called a *strongly regular set.*

Definition 3

If, in Definition 1, we can prescribe that g shall be of the form $g = u \cdot f$ where $u(x) = 0$ on a neighbourhood of E and $u \in A$, then E is called a *Ditkin set.*

Definition 4

If, in Definition 3, we can choose a sequence of functions u, $\{u_n\}_1^\infty$, where u_n only depends on E, such that $\|u_n f - f\| \to 0$ as $n \to \infty$, for all f in question, then E is called a *strong Ditkin set.*

It is easily seen that a set which satisfies the conditions of one definition also satisfies those of the previous definitions. A famous example of Malliavin [7] shows that there are sets which do not admit spectral synthesis. It does not seem to be known if the first three definitions are equivalent. On the other hand, the only Ditkin sets that are known are those which directly are obtained from the following theorem:

(a) Every point is a Ditkin set.
(b) Enumerable unions of Ditkin sets are Ditkin sets.
(c) If the boundary of E is a Ditkin set so is E.

(a) and (c) were proved by Rudin [9, pp. 170–171] while (b) was proved by Calderon [2] for strongly regular sets. The proof for Ditkin sets is similar and we shall omit it.

If we put $v_n(x) = 1 - u_n(x)$ in Definition 4 we see that E is a strong Ditkin set if and only if we can find a sequence $\{v_n(x)\}_1^\infty$ with the properties:

$$v_n(x) \in A,\ v_n(x) = 1 \text{ on a neighbourhood of } E \text{ and}$$

$$\|v_n f\| \to 0, \text{ as } n \to \infty, \text{ for every } f \in A \text{ such that } f(x) = 0 \text{ on } E. \tag{3}$$

Strong Ditkin sets

We begin by proving three theorems which yield the existence of strong Ditkin sets. The only sets of this kind, known to the author, are those whose existence follow immediately from these theorems.

Theorem 1. [9, p. 49]. *Every point is a strong Ditkin set.*

Proof. Suppose that E consists of the point a. Let $v_n(x) = 2g_{2/n}(x,a) - g_{1/n}(x,a)$. It is sufficient to show that $\|g_h f\| \to 0$ as $h \to 0$ where f is an arbitrary function in A, such that $f(a)=0$. Since a translation does not affect the norms, we may assume that $a=0$. Now put $f(x) = \sum_{-\infty}^{\infty} a_k e^{ikx}$. Choose $\varepsilon > 0$, arbitrarily, and an integer p_0 such that

$$\sum_{|p|>p_0} |a_p| < \frac{\varepsilon}{4} \tag{4}$$

and h, according to (2), so small that

$$\sum_k |b_{k-p,h} - b_{k,h}| < \frac{\varepsilon}{2\|f\|} \quad \text{for} \quad p \leqslant p_0. \tag{5}$$

Since $\sum_{-\infty}^{\infty} a_p = f(0) = 0$ we have

$$\|g_h f\| = \sum_k \Big|\sum_p a_p b_{k-p,h}\Big| = \sum_k \Big|\sum_p a_p (b_{k-p,h} - b_{k,h})\Big|$$

and by the triangle inequality

$$\|g_h f\| \leqslant \sum_{|p|\leqslant p_0} |a_p| \sum_k |b_{k-p,h} - b_{k,h}| + \sum_{p>p_0} |a_p| \sum_k |b_{k-p,h} - b_{k,h}|.$$

This is by (1), (4) and (5) less than

$$\frac{\|f\|\cdot\varepsilon}{2\|f\|} + \frac{\varepsilon}{4} 2 \|g_h\| = \varepsilon$$

and thus $\|g_n f\| \to 0$ as $h \to 0$ q. e. d.

Theorem 2. *Every closed interval is a strong Ditkin set.*

Proof. Assume that the interval is $I = [a,b]$ and let

$$v_n(x) = 2g_{2/n}(x,a) - g_{1/n}(x,a) + 2g_{2/n}(x,b) - g_{2/n}(x,b) + k_n(x),$$

where $k_n(x) = 0$ outside I.

Let f be an arbitrary function in A such that $f(x)=0$ on I. Then $f \cdot k_n(x) \equiv 0$ and since $f(a)=f(b)=0$ Theorem 1 gives $\|v_n f\| \to 0$. Since $\{v_n(x)\}_1^{\infty}$ satisfies (3) Theorem 2 is proved.

Theorem 3. *Finite unions of strong Ditkin sets are strong Ditkin sets.*

Proof. It is sufficient to prove the theorem for two sets E_1 and E_2. Assume that $\{u_{n,1}(x)\}_1^\infty$ and $\{u_{n,2}(x)\}_1^\infty$ are sequences satisfying Definition 4 for E_1 and E_2 respectively. We claim that $\{u_{n,1}(x)\cdot u_{n,2}(x)\}_1^\infty$ is a sequence satisfying Definition 4 for the set $E=E_1\cup E_2$. The fact that $\{u_{n,1}g\}_1^\infty$ converges in A for every $g\in A$ such that $g(x)=0$ on E_1 implies, by a theorem of Banach [1, p. 80], that there exists a constant M such that $\|u_{n,1}g\|\leq M\|g\|$ for all n and every $g\in A$, which vanishes on E_1. For an arbitrary $f\in A$, such that $f(x)=0$ on E we thus obtain

$$\|u_{n,1}u_{n,2}f\|\leq\|u_{n,1}u_{n,2}f-u_{n,1}f\|+\|u_{n,1}f-f\|\leq M\|u_{n,2}f-f\|+\|u_{n,1}f-f\|.$$

The last two terms tend by assumptions to zero when n tends to infinity. Since obviously $u_{n,1}(x)\cdot u_{n,2}(x)$ is in A and is zero on a neighbourhood of E, we have proved that E is a strong Ditkin set.

For a closer examination of strong Ditkin sets, the following theorem will be of great importance.

Theorem 4. *Let E be a closed set on $[0,2\pi]$ and $\{v_n(x)\}_1^\infty$ a sequence of functions in A with the following properties:*

(1) $v_n(x)=1$ *on E for every n.*
(2) $v_n(x)\to 0$ *when $n\to\infty$ for every $x\notin E$.*

Then $\|v_n\|$ tends to infinity with n if E is not a finite set or the whole of $[0,2\pi]$.

The proof is partly analogous to Cohen's [3] approach to Littlewood's Conjecture, later modified by Davenport [4]. To make the paper self-contained, we give a complete proof although Lemma 2 is identical with Davenports's Lemma 1 and Lemma 3 is a slight modification of his Lemma 2.

Lemma 2. *Let N, A, and B be real numbevs satisfying $N\geq 3$, $A\geq -\frac{1}{2}N$ and $A^2+B^2<N^4/4$. Then*

$$\left|1-\frac{2}{N^2}-\frac{A+iB}{N^3}\right|+\frac{1}{N^{\frac{3}{2}}}(N+2A)^{\frac{1}{2}}\leq 1.$$

Proof. The expression on the left is

$$\left\{\left(1-\frac{2}{N^2}-\frac{A}{N^3}\right)^2+\frac{B^2}{N^6}\right\}^{\frac{1}{2}}+\frac{1}{N^{\frac{3}{2}}}(N+2A)^{\frac{1}{2}}$$

$$<\left\{\left(1-\frac{2}{N^2}\right)^2-2\left(1-\frac{2}{N^2}\right)\frac{A}{N^3}+\frac{1}{4N^2}\right\}^{\frac{1}{2}}+\frac{1}{N^{\frac{3}{2}}}(N+2A)^{\frac{1}{2}}$$

$$<\left(1-\frac{3}{N^2}-\frac{A}{N^3}\right)^{\frac{1}{2}}+\frac{1}{N^{\frac{3}{2}}}(N+2A)^{\frac{1}{2}}<1-\frac{3}{2N^2}-\frac{A}{2N^3}+\frac{1}{N^{\frac{3}{2}}}(N+2A)^{\frac{1}{2}}$$

$$=1-\left\{\frac{(N+2A)^{\frac{1}{2}}}{2N^{\frac{3}{2}}}-\frac{1}{N}\right\}^2-\frac{1}{4N^2}<1\ \text{q.e.d.}$$

Lemma 3. *Let $N \geqslant 3$ be a positive integer and $x_1, x_2, \ldots, x_N$ be distinct real numbers, $0 < x_i < 2\pi$. Suppose $\{b_k\}_{-\infty}^{\infty}$ is a sequence of complex numbers such that $|b_k| \leqslant 1$ for every k. Let*

$$c_k = b_k \left\{ 1 - \frac{2}{N^2} - \frac{1}{N^3} \sum_{\nu > \mu} e^{ik(x_\nu - x_\mu)} \right\} + \frac{1}{N^{\frac{5}{2}}} \sum_{\nu=1}^{N} e^{ikx_\nu}.$$

Then $|c_k| \leqslant 1$ for every k.

Proof. We have
$$\left| \sum_{\nu=1}^{N} e^{ikx_\nu} \right|^2 = N + 2\,\mathrm{Re}\left\{ \sum_{\nu < \mu} e^{ik(x_\nu - x_\mu)} \right\}$$

Putting $\sum_{\nu<\mu} e^{ik(x_\nu - x_\mu)} = A + iB$, where A and B are real

we obtain
$$\left| \sum_{\nu=1}^{N} e^{ikx_\nu} \right|^2 = N + 2A \geqslant 0$$

and
$$A^2 + B^2 \leqslant \left[\frac{N(N-1)}{2} \right]^2 < \frac{N^4}{4}.$$

The conditions of Lemma 2 are satisfied, hence

$$|c_k| \leqslant \left| 1 - \frac{2}{N^2} - \frac{A+iB}{N^3} \right| + \frac{(N+2A)^{\frac{1}{2}}}{N^{\frac{5}{2}}} \leqslant 1 \quad \text{q.e.d.}$$

Lemma 4. *Let E be a closed set on $[0, 2\pi]$ with Lebesgue measure zero and with zero as a point of accumulation. Let N be an arbitrary positive integer. Then we can construct sequences $\{P_j\}_1^{\infty}$ and $\{T_j\}_1^{\infty}$ of sets, where T_j and P_j have the following properties*

(a) $P_1 = \{x_1\} \subset E$, $T_j \subset E$ *for every* j.

(b) *For every $j \geqslant 1$, T_j consists of N points from E, $T_j = \{x_{1,j} > x_{2,j} > \ldots > x_{N,j}\}$ such that $p + x_{\nu,j} - x_{\mu,j} \notin E$ if $p \in P_j$ and $\nu < \mu$.*

(c) *P_{j+1} = the union of P_j, T_j all and points $p + x_{\nu,j} - x_{\mu,j}$, where $p \in P_j$ and $\nu < \mu$.*

Proof. We choose the numbers $x_{\nu,j}$ among the left endpoints of the complementary intervals of E. Suppose that P_j has been constructed. It then consists partly of points from E that are left endpoints of the complementary intervals of E, partly of points situated in the complement of E. Thus the points of P_j have a least distance d to the nearest point to the right that belongs to E. We choose as T_j a sequence $\{x_{\nu,j}\}_{\nu=1}^{N}$ where $x_{1,j} < d$ so that (b) is fulfilled. This is possible because zero is an accumulation point of E. The construction of T_j and, with it, that of P_{j+1} is now complete, which proves Lemma 4.

Proof of Theorem 4 in case $mE=0$

By studying sequences $\{v_n(-x)\}_1^\infty$ we see that $-E=\{-x\mid x\in E\}$ is a strong Ditkin set if and only if E is a strong Ditkin set. Therefore, without loss of generality, we may assume that zero is a point of accumulation.

We give an indirect proof. Suppose we have a sequence $\{v_n(x)\}_1^\infty$ satisfying the conditions of the theorem and

$$v_n=\sum_{-\infty}^{\infty} a_{k,n}\, e^{ikx},\quad \text{where}\quad \sum_{k=-\infty}^{\infty}|a_{k,n}|<M \text{ for every } n. \tag{6}$$

Choose an integer $N>9M^2$ and construct the sets $P_1, P_2, \ldots$ and $T_1, T_2, \ldots$ according to Lemma 4. We also construct sequences $\{b_{k,j}\}_{k=-\infty}^{\infty}$ $j=1,2,\ldots$ as follows:

$$b_{k,1}=e^{ikx_1}$$

$$b_{k,j}=b_{k,j-1}\left(1-\frac{2}{N^2}-\frac{1}{N^3}\cdot\sum_{\nu<\mu}\exp\{ik(x_{\nu,j}-x_{\mu,j})\}\right)+\frac{1}{N^{\frac{5}{2}}}\sum_{\nu=1}^{N}\exp(ikx_{\nu,j})\quad \text{for}\quad j>1.$$

Each element of the sequence $\{b_{k,j}\}_{k=-\infty}^{\infty}$ will then be of the form

$$b_{k,j}=\sum_m \alpha_{m,j}\, e^{ikx_m},$$

where $x_m\in P_{j+1}$. The cofficients $\alpha_{m,j}$ are bounded by a constant $C(j,N)$, depending only on j and N. By Lemma 3 $|b_{k,j}|\leqslant 1$ for every $j\geqslant 1$ and every k. Now define

$$I_{j,n}=\sum_{k=-\infty}^{\infty} a_{k,n}\, b_{k,j}. \tag{7}$$

We have
$$I_{1,n}=\sum_{k=-\infty}^{\infty} a_{k,n}\, e^{ikx_1}=v_n(x_1)=1$$

by condition (1) in the theorem and

$$I_{j,n}=\left(1-\frac{2}{N^2}\right)I_{j-1,n}+\frac{1}{N^{\frac{5}{2}}}\sum_{\nu=1}^{N}\sum_{k=-\infty}^{\infty} a_{k,n}\exp(ikx_{\nu,j})-\sum_p \alpha_{m_p,j}\sum_k a_{k,n}\exp(ikx_{m_p}).$$

From the construction of $\{P_j\}_1^\infty$ it follows that $\{x_{m_p}\}$ is a subset of P_{j+1}, which is disjoint from E.

By condition (1) in the theorem the second term is

$$\frac{1}{N^{\frac{5}{2}}}\sum_{\nu=1}^{N} v_n(x_{\nu,j})=\frac{1}{N^{\frac{3}{2}}}$$

for every n. By condition (2) we have

$$\sum_{k=-\infty}^{\infty} a_{k,n}\, e^{ikx_{m_p}} = v_n(x_{m_p}) \to 0 \quad \text{when} \quad n \to \infty.$$

Since $|\alpha_{m_p,j}| < C(j, N)$ we obtain

$$I_{j,n} = \left(1 - \frac{2}{N^2}\right) I_{j-1,n} + N^{-\frac{3}{2}} + \varepsilon(n)\, C_1(j, N),$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ and $C_1(j, N)$ is a constant depending only on j and N. We sum this equality for $j = 2, 3, \dots N^2$ and obtain

$$I_{N^2,n} + \frac{2}{N^2} \sum_{j=1}^{N^2-1} I_{j,n} = 1 + N^{\frac{1}{2}} - N^{-\frac{3}{2}} + \varepsilon(n)\, C_2(N),$$

where $C_2(N)$ depends on N only, and $\varepsilon(n) \to 0$, $n \to \infty$. For n sufficiently large, we obtain

$$\left| I_{N^2,n} + \frac{2}{N^2} \sum_{j=1}^{N^2-1} I_{j,n} \right| > \sqrt{N}.$$

From this inequality we deduce that

$$\sup_{1 \leqslant j \leqslant N^2} |I_{j,n}| > \frac{\sqrt{N}}{3} > M \quad \text{since} \quad N > 9M^2.$$

By (7) $\qquad |\sum_k a_{k,n} b_{k,j}| > M$ for some j in $1 \leqslant j \leqslant N^2$.

Since $|b_{k,j}| \leqslant 1$ we have $\|v_n\| = \sum_{k=-\infty}^{\infty} |a_{k,n}| > M$ which is a contradiction to (6). This completes the proof in case $mE = 0$.

Proof of Theorem 4 in case $mE > 0$

From the conditions (1) and (2) of the theorem it follows that $v_n(x)$ converges, when n tends to infinity pointwise to the characteristic function of E, which we call $v(x)$. This is discontinuous if E is not $[0, 2\pi]$ and therefore it cannot belong to A. Put

$$v_n(x) = \sum_{k=-\infty}^{\infty} a_{k,n}\, e^{ikx} \quad \text{and} \quad v(x) \sim \sum_{k=-\infty}^{\infty} a_k\, e^{ikx}.$$

Then by dominated convergence

$$\lim_{n\to\infty} a_{k,n} = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} v_n(x)\, e^{-ikx}\, dx = \frac{1}{2\pi} \int_0^{2\pi} v(x)\, e^{-ikx}\, dx = a_k.$$

Since $\sum_{k=-\infty}^{\infty} |a_k| = \infty$ it follows that $\lim_{n\to\infty} \|v_n\| = \infty$ for $mE > 0$. This concludes the proof of Theorem 4.

We shall use this theorem to prove that the only strong Ditkin sets with Lebesgue measure zero are the finite sets. It then follows that the strong Ditkin sets form a proper subclass of the Ditkin sets. We need two lemmas.

Lemma 5. *Let E be a closed set with period 2π and Lebesgue measure zero and N an arbitrary positive integer. Then there exists an open set $O \subset [0, 2\pi]$ where $mO = 2\pi$, such that*

$$\left\{\frac{2\pi p}{N} \,\middle|\, p = 1, 2, \dots N\right\} \cap E_x = \phi$$

for every $x \in O$, where $E_x = \{y - x \mid y \in E\}$.

Proof. Put
$$\bigcup_{p=1}^{\infty} E_{2\pi p/N} = F.$$

Then F has period $2\pi/N$ and $mF = 0$. Further $x \in F$ if and only if E_x contains some of the points $2\pi/N$, $p = 1, 2, \dots N$. We let O be the complement of F with respect to the interval $[0, 2\pi]$. Then $mO = 2\pi$ and $x \in O$ if and only if

$$\left\{\frac{2\pi p}{N} \,\middle|\, p = 1, 2, \dots N\right\} \cap E_x = \phi \quad \text{q.e.d.}$$

Lemma 6. *Let E be a closed set on $[0, 2\pi]$ with Lebesgue measure zero. Suppose we have a sequence of functions $\{v_n(x)\}_1^\infty$ where $v_n(x) \in A$ and $\{v_n f\}_1^\infty$ converges in A for every $f \in A$ such that $f(x) = 0$* on *E. Then there exists a constant M such that $\|v_n\| < M$ for every n.*

Proof. Put $v_n(x) = \sum_{k=-\infty}^{\infty} a_{k,n} e^{ikx}$ and choose $N(n)$ so large that $\sum_{k \geq N(n)} |a_{k,n}| < \frac{1}{3}\|v_n\|$. In the sequel we write N instead of $N(n)$. By making a suitable translation of E we may, by Lemma 4, assume that E does not contain any of the points $2\pi p/2N$, $p = 1, 2, \dots, 2N$. The translation does not affect the absolute values of the Fourier-coefficients of f and v_n.

Let $d2\pi/N$ be the distance from E to the set $\{2\pi p/2N,\ p = 1, \dots 2N\}$. Let g be an arbitrary function in A, not identically zero and with support on $[-2\pi d, 2\pi d]$. Then $g(2Nx)$ vanishes on E.

Put $f(x) = g(2Nx) = \sum_\nu b_\nu e^{i2N\nu x}$. Then $\|f\| = \|g\| < \infty$, $f = 0$ on E and

$$\|v_n f\| = \sum_k \left|\sum_\nu b_\nu a_{k-2\nu N}\right| \geq \sum_\nu |b_\nu| \sum_{|k|<N} |a_k|$$
$$- \sum_\nu |b_\nu| \sum_{k \geq N} |a_k| \geq \frac{\|f\| \cdot 2\|v_n\|}{3} - \frac{\|f\| \cdot \|v_n\|}{3} = \frac{1}{3}\|f\|\,\|v_n\|.$$

The first inequality follows from the triangle inequality and the fact that every sum of the form $\sum_\nu b_\nu a_{k-2\nu N}$ contains just one term, a_p, with an index p, such that $|p| < N$.

By a general theorem of Banach [1, p. 80] on convergence of operators there exists a constant M, such that $\|v_n f\| \leq M_1 \|f\|$ for every n and every $f \in A$ such that $f(x) = 0$ on E. We thus obtain for the special f above

$$\tfrac{1}{3}\|v_n\|\,\|f\| \leq \|v_n f\| \leq M_1 \|f\|.$$

Hence $\|v_n\| \leq 3M_1 = M$ for every n q.e.d.

We now turn to the proof of

Theorem 5. *Suppose E is a strong Ditkin set with Lebesgue measure zero. Then E is finite.*

Proof. Since E is a strong Ditkin set there exists a sequence $\{v_n(x)\}_1^\infty$ with the properties (3). This sequence then satisfies the conditions of Lemma 6. Hence there is a constant, M, such that $\|v_n\| < M$ for every n. Now the conditions of Theorem 4 are fulfilled and from it we deduce that E is finite.

Next we investigate the strong Ditkin sets with positive Lebesgue measure. We begin by proving

Lemma 7. *Suppose a closed set E with period 2π has the property that every translation of it contains at least one of the points $\{2\pi k/N\}$ $k = 1, 2, \ldots N$. Then E contains an interval.*

Proof. Define $E_x = \{p - x \mid p \in E\}$. For an arbitrary $x \in [0, 2\pi]$ we have, by assumption,

$$x + \frac{k \cdot 2\pi}{N} \in E \text{ for some } k, \quad 1 \leq k \leq N \text{ i.e. } x \in \text{some } E_{k2\pi/N}.$$

Thus
$$\bigcup_{k=1}^{N} E_{k2\pi/N} = [0, 2\pi].$$

Let $k_1 \leq N$ be the least integer such that

$$\bigcup_{1}^{k_1} E_{k2\pi/N} = [0, 2\pi]$$

and put
$$\bigcup_{1}^{k_1-1} E_{k2\pi/N} = F.$$

Then F is closed and the complement of F with respect to $(0, 2\pi)$ is open and thus a sum of intervals. Since this sum is a subset of $E_{k2\pi/N}$ it follows that E contains at least one interval.

Theorem 6. *Suppose E is a strong Ditkin set with positive Lebesgue measure. Then E contains at least one interval.*

Proof. That E is a strong Ditkin set implies the existence of a sequence $\{v_n(x)\}_1^\infty$ with the properties (3). Every translation of E then contains at least one of the points $\{2\pi k/N\}$, $k = 1, 2. \ldots N$, for each $N > N_0(E)$; otherwise we would have a sequence N_ν tending to infinity such that, for every ν, some translation of E does not contain any of the points $\{2\pi k/N_\nu\}$, $k=1, 2, \ldots N_\nu$. Using the method of proof in Lemma 6 we conclude that $\|v_n\| < M$ for every n which is false by Theorem 4. The theorem now follows from Lemma 7.

Theorem 7. *Let E be a strong Ditkin set on $[0, 2\pi]$ and $E_1 = E \cap I$, where I is an interval (a, b) such that $a \notin E$ and $b \notin E$. Then E_1 is a strong Ditkin set.*

Proof. Let $2d$ be the minimum of the distances from a and b to E. Define $\alpha(x)$ as the continuous function in A, which is zero outside I, one in the interval $a+d \leqslant x \leqslant b-d$ and linear on the remaining intervals. Since E is a strong Ditkin set there exists a sequence $\{v_n(x)\}_1^\infty$ satisfying (3). Obviously $v_n(x)\cdot\alpha(x)=1$ in a neighbourhood of E_1. Now let g be an arbitrary function in A such that $g(x)=0$ on E_1. Then

$$\|v_n \alpha g\| \to 0 \quad \text{when} \quad n\to\infty,$$

since $\alpha g = 0$ on E. Hence $\{v_n \alpha\}_1^\infty$ has the properties (3) implying that E_1 is a strong Ditkin set.

Corollary. *A strong Ditkin set E consists of* (i) *a union of closed intervals,* (ii) *the accumulation points of these intervals and* (iii) *a countable set of points clustering at the endpoints of the intervals* (i) *or the points* (ii).

Proof. Suppose that x is an accumulation point of E that does not belong to any interval of E. Then we can, for every $\varepsilon > 0$, find points a and b of the complement of E such that $a < x < b$ and $b - a < \varepsilon$. Theorem 7 implies that $E_1 = (a, b) \cap E$ is a strong Ditkin set. If $mE_1 = 0$ we conclude by Theorem 5 that E_1 is finite and so x is an isolated point of E which is a contradiction. Thus $mE_1 > 0$. This implies by Theorem 6 that E_1 contains an interval and we conclude that x is an accumulation point of intervals. This proves the corollary.

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