ARKIV FÖR MATEMATIK Band 6 nr 8

1.64 12 1 Communicated 14 October 1964 by O. FROSTMAN and L. CARLESON

Baire sets and Baire measures

By KENNETH A. ROSS and KARL STROMBERG

Introduction

The purpose of this paper is to ascertain to what extent certain known results about Baire sets and Baire measures on σ-compact, locally compact spaces[1] are valid in more general spaces. We frequently find that the known results carry over to paracompact, locally compact spaces, hence, in particular, to all locally compact topological groups. We also list some additional properties about locally compact groups that are not valid in all paracompact, locally compact spaces.

Let X be a topological space. A set $Z \subset X$ is a *zero-set* if $Z = f^{-1}(0)$ for some continuous real-valued function f on X; we often write $Z(f)$ for $f^{-1}(0)$. The *Baire sets* are those subsets of X belonging to the smallest σ-algebra containing all zero-sets in X. This definition of Baire sets is apparently due to Hewitt [6]. The *Borel sets* are those sets belonging to the smallest σ-algebra that contains all closed subsets of X. Clearly a Baire set is always a Borel set. In many familiar spaces, including all metric spaces, the classes of Baire sets and Borel sets coincide. Our definition of Baire sets is consistent with that of Halmos [5] whenever X is σ-compact and locally compact; see 1.1. All of our results are well known for σ-compact, locally compact spaces. We have observed a tendency among writers to assume that the results are true for general locally compact spaces; we hope that the present paper will show the limitations of these assumptions.

For a topological space X, $\mathfrak{C}(X)$ will denote the family of continuous real-valued functions on X. We also define $\mathfrak{C}_{00}(X) = \{f \in \mathfrak{C}(X) : f \text{ has compact support}\}$, $\mathfrak{C}^+(X) = \{f \in \mathfrak{C}(X) : f \geqslant 0\}$, and $\mathfrak{C}_{00}^+(X) = \{f \in \mathfrak{C}_{00}(X) : f \geqslant 0\}$.

For standard terminology and results in measure theory, topology, and topological groups, we refer the reader to Halmos [5], Kelley [9], and Hewitt and Ross [7], where he will also find references to the original sources.

1. Baire sets

We begin with an elementary observation.

Proposition 1.1. *If* X *is a σ-compact, locally compact space, the class of Baire sets is equal to the smallest σ-ring* S_0 *containing all compact G_δ's.* In other words, a set is a Baire set as defined in this paper if and only if it is a Baire set as defined in Halmos [5].

[1] All topological spaces in this paper are assumed to be Hausdorff spaces.

Proof. Since X is regular and Lindelöf, it is normal. Therefore every closed G_δ is a zero-set, and hence every member of S_0 is a Baire set.

Consider a zero-set Z; clearly Z is a G_δ. We have $X = \bigcup_{n=1}^{\infty} F_n$ where each F_n is compact. For each n, there is an f_n in $\mathfrak{C}_{00}^{+}(X)$ such that $f_n(F_n)=1$. Defining $B_n = f_n^{-1}(1)$ for each n, we obtain compact G_δ's such that $X = \bigcup_{n=1}^{\infty} B_n$. Thus $Z = \bigcup_{n=1}^{\infty} (Z \cap B_n)$ is a countable union of compact G_δ's and $Z \in \mathsf{S}_0$. Hence every zero-set belongs to S_0, and so every Baire set belongs to S_0.

The next three theorems generalize Theorem 51.D, Halmos [5].

Theorem 1.2. *A compact Baire set F in any topological space X is a zero-set.*

Proof. Exactly as in Halmos' proof of 51.D, there is a metric space M and a continuous mapping T of X onto M such that $F = T^{-1}(M_0)$ for some subset M_0 of M. Since $T(F) = M_0$, M_0 is compact and hence closed in M. Therefore M_0 is the zero-set of some g in $\mathfrak{C}(M)$. If we define $f = g \circ T$, then $f \in \mathfrak{C}(X)$ and $Z(f) = F$.

Theorem 1.3. *A closed Baire set E in a paracompact, locally compact space X is a zero-set.*

Proof. Let $\mathcal{U}$ consist of all open subsets of X having compact closure, and let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. Since $\mathcal{V}$ is also a point finite cover, there is an indexed family $\mathcal{W} = \{W_V : V \in \mathcal{V}\}$ of open sets W_V such that $W_V^- \subset V$ for all $V \in \mathcal{V}$ and $\bigcup_{V \in \mathcal{V}} W_V = X$ (see Ex. V, page 171, Kelley [9]). For $V \in \mathcal{V}$, we choose f_V in $\mathfrak{C}_{00}^{+}(X)$ such that $f_V(W_V^-) = 1$ and $f_V(X - V) = 0$. Finally, we define $B_V = f_V^{-1}(1)$ for $V \in \mathcal{V}$. Then $\bigcup_{V \in \mathcal{V}} B_V = X$ and each B_V is a compact Baire set. Hence each $B_V \cap E$ is a compact Baire set; using 1.2, it is easy to see that there exist functions $g_V \in \mathfrak{C}^{+}(X)$ such that $Z(g_V) = B_V \cap E$ and $g_V(X - V) = 1$. Define $g = \inf_{V \in \mathcal{V}} g_V$; since $\mathcal{V}$ is locally finite, g is continuous and it is obvious that $Z(g) = E$.

Examples 3.1 and 3.2 show that the conclusion of Theorem 1.3 might fail if either the hypothesis that X be paracompact or the hypothesis that X be locally compact is dropped. We have been unable to ascertain whether or not the conclusion holds when the hypothesis on X is weakened to "normal, locally compact"; however, see Theorem 1.5 below. Example 3.3 shows that the conclusion of Theorem 1.3 does hold in all well-ordered spaces.

The following routine lemma will be used in proving Theorem 1.5 and also in Section 2.

Lemma 1.4. *Let X be a normal space, and let Y be a closed Baire set in X. Then a set $E \subset Y$ is a Baire set in X if and only if it is a Baire set in Y.*

Proof. ($\Rightarrow$). Let $\mathfrak{F} = \{F \subset X : F \cap Y$ is a Baire set in $Y\}$. It is easy to verify that every zero-set in X belongs to $\mathfrak{F}$, that $F \in \mathfrak{F}$ implies $X - F \in \mathfrak{F}$, and that $\{F_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ implies $\bigcup_{n=1}^{\infty} F_n \in \mathfrak{F}$. Hence every Baire set in X belongs to $\mathfrak{F}$. Thus if $E \subset Y$ is a Baire set in X, then $E \cap Y = E$ is a Baire set in Y.

($\Leftarrow$). Let $\mathcal{E} = \{E \subset Y : E$ is a Baire set in $X\}$. Clearly $\mathcal{E}$ is a σ-algebra of subsets of Y. To show that $\mathcal{E}$ contains all Baire sets in Y, we need to show that every zero-set *in* Y belongs to $\mathcal{E}$. Suppose then that E is a zero-set *in* Y and that f in $\mathfrak{C}^{+}(Y)$ is such that $Z(f) = E$. Since X is normal, f can be extended to a function $\bar{f}$ in $\mathfrak{C}^{+}(X)$ by the Tietze-Urysohn theorem. Then $E = Y \cap \bar{f}^{-1}(0)$ is a Baire set in X and E belongs to $\mathcal{E}$.

Theorem 1.5. *A closed σ-compact Baire set E in a normal, locally compact space X is a zero-set.*

Proof. Since E is σ-compact, we can find a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets such that $E \subset \bigcup_{n=1}^{\infty} U_n$ and each U_n^- is compact. Using normality twice, we can choose open sets V and W such that $E \subset W \subset W^- \subset V \subset V^- \subset \bigcup_{n=1}^{\infty} U_n$. Choose a function k in $\mathfrak{C}^+(X)$ for which $k(V^-)=0$ and $k(X - \bigcup_{n=1}^{\infty} U_n)=1$, and define $Y=k^{-1}(0)$. Then Y is a closed Baire set and $V^- \subset Y \subset \bigcup_{n=1}^{\infty} U_n$. Also $Y = \bigcup_{n=1}^{\infty}(U_n^- \cap Y)$ is σ-compact and normal, and hence paracompact. By 1.4, E is a closed Baire set in Y. Hence by 1.3, there is an f in $\mathfrak{C}^+(Y)$ such that $Z(f)=E$. We may assume that $0 \leqslant f(x) \leqslant 1$ for $x \in Y$. Since X is normal, Urysohn's lemma shows that there is a g in $\mathfrak{C}^+(X)$ such that $g(E)=0$ and $g(X-W)=1$. Define h on X by the rule:

$$\begin{aligned} h(x) &= \sup\,(f(x), g(x)) \quad \text{for} \quad x \in Y, \\ &= 1 \quad \text{for} \quad x \notin Y. \end{aligned}$$

Clearly h is continuous on the open sets V and $X - W^-$. Hence h belongs to $\mathfrak{C}^+(X)$; it is obvious that $Z(h)=E$.

Example 3.1 shows that the hypothesis in Theorem 1.5 that X be normal is needed.

We next observe two interesting facts about Baire sets that hold for some important, but special, spaces. Example 3.4 shows that both conclusions may fail in a sufficiently bad compact space.

Theorem 1.6. *If X is a locally compact group or an arbitrary product of separable metric spaces, then the closure of a Baire set is a Baire set and the closure of any open set is a Baire set.*

We prove this theorem after proving Theorem 2.5.

2. Baire measures and functions

The next theorem generalizes Theorem 52.G, Halmos [5]. A *Baire measure* is simply a measure defined on the Baire sets which assigns finite measure to each compact Baire set.

Theorem 2.1. *If X is paracompact and locally compact and if* card (X) *is less than the first inaccessible cardinal, then every σ-finite Baire measure μ on X is regular; i.e., for each Baire set E we have*

$$\begin{aligned} \mu(E) &= \sup\,\{\mu(C) : C \subset E \text{ and } C \text{ is a compact Baire set}\} \\ &= \inf\,\{\mu(U) : U \supset E \text{ and } U \text{ is an open Baire set}\}. \end{aligned}$$

Proof. Since μ is σ-finite, there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ of finite Baire measures such that $\mu(E)=\sum_{n=1}^{\infty}\mu_n(E)$ for each Baire set E.

By Théorème 5 of chap. I, § 10, No. 12 of Bourbaki [1], we have $X=\bigcup_{\lambda \in \Lambda} X_\lambda$ where the X_λ's are pairwise disjoint open σ-compact sets.[1] For each positive integer n, we define ν_n on the set of all subsets of Λ by the rule:

$$\nu_n(\Gamma) = \mu_n\Big(\bigcup_{\lambda \in \Gamma} X_\lambda\Big) \quad \text{for} \quad \Gamma \subset \Lambda.$$

[1] The authors are indebted to Professor E. A. Michael for calling this theorem to their attention.

Note that every set $\bigcup_{\lambda\in\Gamma} X_\lambda$ is a Baire set since it is open and closed. It is evident that each ν_n is a finite measure on the set of all subsets of Λ and that card(Λ) is less than the first inaccessible cardinal. Now Theorem (A) of Ulam [16] shows that for each n there exists a countable subset Λ_n of Λ such that $\nu_n(\Lambda_n)=\nu_n(\Lambda)$. Now let $\Lambda_0=\bigcup_{n=1}^{\infty}\Lambda_n$, and let $Y=\bigcup_{\lambda\in\Lambda_0} X_\lambda$. Since Λ_0 is countable, we see that Y is σ-compact. Also

$$\mu(X-Y)=\sum_{n=1}^{\infty}\mu_n(X-Y)=\sum_{n=1}^{\infty}\nu_n(\Lambda-\Lambda_0)\leqslant\sum_{n=1}^{\infty}\nu_n(\Lambda-\Lambda_n)=0.$$

(The existence of a σ-compact set Y such that $\mu(X-Y)=0$ also follows directly from Rubin's theorem announced in [14].)

Now consider any Baire set E in X. Theorem 52.G of Halmos [5] applies to μ restricted to Y; that is, μ is regular on Y. Since Y is open and closed, Y is a Baire set in X and so $E\cap Y$ is a Baire set in X. By 1.4, $E\cap Y$ is a Baire set in Y. Now applying these facts, we have

$$\begin{aligned}\mu(E)&=\mu(E\cap Y)\\&=\inf\{\mu(U):U\supset E\cap Y,\ U \text{ is an open Baire set in } Y\}\\&=\inf\{\mu(U\cup(X-Y)):U\supset E\cap Y,\ U \text{ is an open Baire set in } Y\}\\&\geqslant\inf\{\mu(V):V\supset E,\ V \text{ is an open Baire set in } X\}\\&\geqslant\mu(E),\end{aligned}$$

and

$$\begin{aligned}\mu(E)&=\mu(E\cap Y)\\&=\sup\{\mu(C):C\subset E,\ C \text{ is a compact Baire set in } Y\}\\&\leqslant\sup\{\mu(C):C\subset E,\ C \text{ is a compact Baire set in } X\}\\&\leqslant\mu(E).\end{aligned}$$

These relations show that μ is regular on X.

Example 3.5 shows that the assumption that μ be σ-finite in Theorem 2.1 is needed. Example 3.6 shows that the hypothesis on X in Theorem 2.1 cannot be weakened to "normal and locally compact"; and Example 3.7 shows that the hypothesis that X be locally compact cannot be dropped.

A *Baire function* on a space X is a function f into a topological space Y such that $f^{-1}(U)$ is a Baire set for every open set U in Y. The next theorem shows that the definition of Baire set used in this paper is a good one. In particular, it generalizes Exercise 51.6 of Halmos [5].

Theorem 2.2. *Let X be any topological space and let $\mathcal{B}$ be the smallest class of functions on X containing all real-valued continuous functions and containing the limit of every pointwise convergent sequence of functions in it. Then $\mathcal{B}$ consists of exactly all of the real-valued Baire functions on X.*

Proof. Let $\mathcal{B}_0$ consist of all continuous functions in $\mathcal{B}$. For every ordinal $\alpha>0$ that is less than the first uncountable ordinal Ω, let $\mathcal{B}_\alpha$ consist of all functions which are pointwise limits of sequences of functions in $\bigcup_{\beta<\alpha}\mathcal{B}_\beta$. Then $\mathcal{B}=\bigcup_{\alpha<\Omega}\mathcal{B}_\alpha$. Each family $\mathcal{B}_\alpha$ is closed under finite suprema and finite linear sums, and hence the same remarks apply to $\mathcal{B}$. It follows that $\mathcal{B}$ is closed under countable suprema.

Now let Z be a zero-set in X; $Z=Z(f)$ for some $f\in C^+(X)$. Then the characteristic function χ_Z of Z is equal to $1-\sup_n n\cdot\inf(f, 1/n)$, and hence belongs to $\mathcal{B}$. If $\mathcal{E}=\{E:\chi_E\in\mathcal{B}\}$, then $\mathcal{E}$ contains all zero-sets and is closed under complements and countable unions. Thus if E is a Baire set, then $E\in\mathcal{E}$ and χ_E belongs to $\mathcal{B}$.

Finally, every Baire function belongs to $\mathcal{B}$ because it is a limit of linear combinations of characteristic functions of Baire sets. That every function in $\mathcal{B}=\bigcup_{\alpha<\Omega}\mathcal{B}_\alpha$ is a Baire function follows by transfinite induction on α, $1\leqslant\alpha<\Omega$ (clearly every function in $\mathcal{B}_0$ is a Baire function).

Theorems 2.3 and 2.5 deal with the question: Given a continuous (respectively, Baire) function g of a topological space X into a metric space Y, do there exist a continuous open mapping τ of X onto a metric space M and a continuous (respectively, Baire) function f on M such that $g=f\circ\tau$? In other words, is g essentially a function defined on a metric space?

Note that if the requirement that τ be open is dropped, then the question becomes trivial. In this case, M can be taken as $g(X)$ which is a subset of Y, τ can be taken as g, and f can be taken as the identity map on M. Example 3.8 shows that the original question does not have an affirmative answer for all compact spaces.

Theorem 2.3. *Let X be a product of separable metric spaces X_a, $a\in A$; and let g be a continuous (respectively, Baire) function of X into a separable metric space Y. Then there is a countable subset C of A such that g is essentially defined on the metric space $M=\Pi_{a\in C}X_a$. The mapping τ is the ordinary projection of X onto M.*

Proof. For g continuous, this theorem is proved by Corson and Isbell [2], Theorem 2.1. The crucial step of the proof is to show that g is determined by the indices in a countable set C:

$$x,\ y\in X \text{ and } x_a=y_a \text{ for } a\in C \text{ implies } g(x)=g(y);$$

see, for example, Theorem 4 of [13].

The proof for a Baire function g is the same once it is established that such a g is determined by a countable set of indices. This can be proved for all Baire functions by applying transfinite induction to the sequence $\mathcal{B}_\alpha$, $1\leqslant\alpha<\Omega$, defined in the proof of 2.2.

Lemma 2.4. *Let X be a σ-compact, locally compact group. If E is a Baire set in X, then there exists a compact normal subgroup N of X such that $E=EN$ and X/N is metrizable.*

Proof. Halmos [5], Theorem 64.G, states that for an appropriate compact normal subgroup N, $E=EN$ and X/N is separable. The term separable used in [5] means that there is a countable base for the topology of X/N. Thus Urysohn's metrization theorem ([9], page 125) shows that X/N is metrizable.

Theorem 2.5. *Let X be a σ-compact, locally compact group, and let g be a continuous (respectively, Baire) function on X into a separable metric space Y. Then there is a compact normal subgroup N of X such that X/N is metrizable and g is essentially defined on X/N. The mapping τ is the ordinary projection of X onto X/N.*

This theorem is a slight generalization of the theorem on page 60 of Montgomery and Zippin [12].

Proof. Let g be a Baire function on X into Y. It is easy to see that it suffices to find N and establish that

$$xy^{-1} \in N \text{ implies } g(x) = g(y).$$

Let $\mathcal{B}$ be a countable basis for Y. Each set $g^{-1}(B)$, $B \in \mathcal{B}$, is a Baire set and hence, by 2.4, there is a compact normal subgroup N_B such that $g^{-1}(B) = g^{-1}(B)N_B$ and X/N_B is metrizable. Let $N = \bigcap_{B\in\mathcal{B}} N_B$; clearly N is a compact normal subgroup of X. Since each X/N_B is metrizable, each N_B is the countable intersection of open sets. Therefore N is also the countable intersection of open sets and it follows that X/N is metrizable.

Suppose now that $xy^{-1} \in N$ and $g(x) = \alpha \in Y$. Then $\{\alpha\} = \bigcap_{j=1}^{\infty} B_j$ for appropriate $B_j \in \mathcal{B}$, and hence

$$g^{-1}(\alpha) = \bigcap_{j=1}^{\infty} g^{-1}(B_j) = \bigcap_{j=1}^{\infty} g^{-1}(B_j)N_{B_j} \supset \bigcap_{j=1}^{\infty} g^{-1}(B_j)N \supset \left(\bigcap_{j=1}^{\infty} g^{-1}(B_j)\right) N = g^{-1}(\alpha)N \supset g^{-1}(\alpha).$$

Thus $g^{-1}(\alpha) = g^{-1}(\alpha)N$, both x and y belong to $g^{-1}(\alpha)$, and $g(x) = \alpha = g(y)$. This completes the proof.

The assumption in Theorem 2.5 that X be σ-compact is necessary, at least for Abelian groups, as Theorem 2.8 below will show.

We now prove Theorem 1.6 as promised in Section 1.

Proof of Theorem 1.6. *Step* I. Suppose that E is a Baire set in a compactly generated, locally compact group X. By 2.4, there is a compact normal subgroup N of X such that $E = EN$ and X/N is metrizable. It follows that $E^- = E^-N$. If τ denotes the projection of X onto X/N, then $\tau(E^-)$ is closed in X/N and there is an f in $\mathfrak{C}(X/N)$ such that $Z(f) = \tau(E^-)$. Then $f \circ \tau \in \mathfrak{C}(X)$ and $Z(f \circ \tau) = E^-$ so that E^- is a Baire set.

Step II. Suppose that E is a Baire set in an arbitrary locally compact group X. Then X contains an open and closed, compactly generated subgroup H. Clearly every coset of H is a closed Baire set. Each $E \cap xH$ is a Baire set in xH by 1.4 and so, by Step I, each $E^- \cap xH$ is a zero-set in xH. Consequently, E^- is the zero-set of some continuous function on X.

Step III. Suppose that U is an open subset of a compactly generated, locally compact group X. Let λ denote the Haar measure on X. There exists a sequence $\{\tilde{F}_n\}_{n=1}^{\infty}$ of compact subsets of U such that $\lambda(U - \bigcup_{n=1}^{\infty} \tilde{F}_n) = 0$. Choose functions $f_n \in \mathfrak{C}^+(X)$ such that $f_n(\tilde{F}_n) = 1$ and $f_n(X - U) = 0$; define $F_n = f_n^{-1}(1)$. Then each F_n is a Baire set and $\lambda(U - \bigcup_{n=1}^{\infty} F_n) = 0$. Since $\bigcup_{n=1}^{\infty} F_n$ is a Baire set, its closure U^- is also a Baire set by Step I.

Step IV. Suppose that U is an open subset of an arbitrary locally compact group X. An argument similar to Step II and using Step III shows that U^- is a Baire set.

Step V. Suppose that E is a Baire set in a product X of separable metric spaces X_a, $a \in A$. As noted in 2.3, every Baire function is determined by countably many indices. In particular, this applies to the characteristic function of E. Hence for some countable subset C of A, $E = E_0 \times \Pi_{a\in A-C} X_a$ where E_0 is a subset of $\Pi_{a\in C} X_a$. Hence $E^- = E_0^- \times \Pi_{a\in A-C} X_a$. Since E_0^- is a zero-set in the metric space $\Pi_{a\in C} X_a$, it follows that E^- is a zero-set in X.

Step VI. Suppose that U is an open subset of $X = \Pi_{a\in A} X_a$, where each X_a is separable metric. By Theorem 3 of [13], U^- is determined by countably many indices. Arguing as in Step V, we infer that U^- is a zero-set in X.

The next two lemmas are needed to prove Theorem 2.8. Example 3.9 shows that the natural analogue of Lemma 2.7 for compact spaces is not true.

Lemma 2.6. *Let Ω_0 be the set of ordinals less than the first uncountable ordinal Ω. If A is an uncountable Abelian group, then there is a non-decreasing transfinite sequence $\{H_\alpha : \alpha \in \Omega_0\}$ of proper subgroups of A such that $\bigcup_\alpha H_\alpha = A$.*

Proof. Let J be a subgroup of A having cardinality $\aleph_1$. From Theorem 20.1 of Fuchs [3], it follows that there is an isomorphism f of J into a divisible group D such that card $(D) = \aleph_1$. By 16.1 [3], f can be extended to a homomorphism f' of A into D. Let K be the kernel of f'. Then card $(A/K) = \aleph_1$. Let $\{x_\alpha K : \alpha \in \Omega_0\}$ be a well-ordering of A/K. Define $\mathcal{H}_\alpha$ to be the countable subgroup of A/K generated by $\{x_\beta K : \beta \leqslant \alpha\}$. Then $\{\mathcal{H}_\alpha : \alpha \in \Omega_0\}$ is a non-decreasing sequence of proper subgroups of A/K and $\bigcup_\alpha \mathcal{H}_\alpha = A/K$. Finally, define $H_\alpha = \{x \in A : xK \in \mathcal{H}_\alpha\}$; the sequence $\{H_\alpha : \alpha \in \Omega_0\}$ has the desired properties.

Lemma 2.7. *Let G be a locally compact Abelian group with identity e, and assume that G is not metrizable. Then there is a set $A \subset G$ such that* card $(A) = \aleph_1$, *$e \notin A$, and every G_δ set containing e intersects A in a nonvoid set.*

Proof. There is a compactly generated open subgroup H of G. By virtue of Pontryagin's structure theorem for locally compact, compactly generated Abelian groups (see 9.8 [7]), H may be chosen to be topologically isomorphic with $R^m \times G_0$, where m is a non-negative integer, R is the real line, and G_0 is a compact group. Evidently G_0 cannot be metrizable, and it suffices to find a set A_0 in G_0 as in the lemma. (Note that G_0 is itself a G_δ set in G.) We therefore assume that G is compact.

By 23.17 and 24.48 of [7], the character group $\hat{G}$ of G is discrete and uncountable. By Lemma 2.6, there is a non-decreasing transfinite sequence $\{\mathcal{H}_\alpha : \alpha \in \Omega_0\}$ of proper subgroups of $\hat{G}$ such that $\bigcup_\alpha \mathcal{H}_\alpha = \hat{G}$. For $\alpha \in \Omega_0$, $\mathcal{H}_\alpha \neq \hat{G}$ and hence $\mathcal{H}_\alpha$ does not separate points of G by 23.20 [7]. For $\alpha \in \Omega_0$, select $x_\alpha \in G$, $x_\alpha \neq e$, such that $\psi(x_\alpha) = 1$ for all $\psi \in \mathcal{H}_\alpha$. Let A be the set $\{x_\alpha : \alpha \in \Omega_0\}$.

Sets of the form $\{x \in G : |\psi(x) - 1| < \varepsilon$ for $\psi \in \mathcal{F}\}$ constitute a base for open sets at e where $\varepsilon > 0$ and $\mathcal{F}$ is a finite subset of $\hat{G}$ (see 23.15 and 24.3 of [7]). Every G_δ set containing e contains a countable intersection of sets of this form. Hence it contains a set $\bigcap_{\psi \in C} \psi^{-1}(1)$ for some countable subset C of $\hat{G}$, and it suffices to show that A intersects $\bigcap_{\psi \in C} \psi^{-1}(1)$. To do this, choose $\alpha \in \Omega_0$ such that $C \subset \mathcal{H}_\alpha$. Then x_α belongs to A and to $\bigcap_{\psi \in C} \psi^{-1}(1)$. This property of A also shows that A must be uncountable and hence that card $(A) = \aleph_1$.

Theorem 2.8. *Let G be a locally compact Abelian group, which is not metrizable or σ-compact. There is a continuous function f of G into the circle group T such that f is not essentially defined on any metrizable G/N, where N is compact and normal.*

Proof. Let H be an open compactly generated subgroup of G. Then H is not metrizable and G/H is not countable. Let A be a subset of H as given in Lemma 2.7; card $(A) = \aleph_1$. For each $a \in A$, there is a continuous character χ_a of H for which $\chi_a(a) \neq 1$. Let $\{x_a : a \in A\}$ be a subset of G consisting of elements from distinct cosets of H. Define f on G as follows:

$$f(x) = \chi_a(xx_a^{-1}) \quad \text{if} \quad x \in x_a H,$$
$$= 1 \quad \text{if} \quad x \notin \bigcup_{a \in A} x_a H.$$

Clearly f is continuous on G.

Suppose now that N is a compact normal subgroup of G and that G/N is metrizable. Then N is a G_δ set containing e; hence there exists an element a in $A \cap N$. Plainly ax_a and x_a belong to the same coset of N. Since $f(ax_a) = \chi_a(ax_a x_a^{-1}) \neq 1$ and $f(x_a) = \chi_a(x_a x_a^{-1}) = 1$, f is not constant on cosets of N and hence is not essentially defined on G/N.

3. Examples

Example 3.1. Let N and R denote the set of integers and the real line, respectively. Let X be the space $\beta R - (\beta N - N)$ discussed in Exercise 6P of Gillman and Jerison [4]. Then X is locally compact, but not normal. Let $E = N \subset X$. Each set $\{n\}$, $n \in E$, is a compact zero-set and so E is a σ-compact Baire set. However, E is closed and E is not a zero-set by 6P.5 [4].

Example 3.2. Let X denote the real line and retopologize X so that the open sets are the sets of the form $U \cup S$, where U is open in the usual topology and S is a subset of the irrational numbers. The space X is paracompact (see Michael [10] or [11]), but not locally compact. Let E be the set of rationals in X. Each set $\{r\}$, $r \in E$, is a compact zero-set and so E is a closed σ-compact Baire set. However, E is not a zero-set in X; otherwise E would be a G_δ subset of the real line with the usual topology, which is impossible by the Baire category theorem. The space X and this property of E are discussed by Katětov [8], page 74.

Example 3.3. Let X be a well-ordered set with the order topology. Then every closed Baire set E in X is a zero-set. Note that X is always normal, and X is paracompact if and only if X has a countable cofinal set.

If X has a countable cofinal set, then E is a zero-set by 1.3. Suppose that X has no countable cofinal set. If E is bounded, then E is compact and hence a zero-set by 1.2. If E is unbounded, then $X - E$ is bounded and, for some β in X, $[\beta, \infty[\subset E$. Clearly $E \cap [1, \beta]$ is the zero-set of some function g on $[1, \beta]$. Define $\tilde{g}$ on X by the rule:

$$\tilde{g}(x) = g(x) \quad \text{for} \quad x \leqslant \beta,$$
$$= 0 \quad \text{for} \quad x \geqslant \beta + 1.$$

Since $[1, \beta]$ is open and closed, $\tilde{g}$ is continuous; and it is obvious that $Z(\tilde{g}) = E$.

Example 3.4. Let X be the compact space βR, where R is still the real line. Let $U = \bigcup_{n=1}^{\infty}]4n-1, 4n+1[\subset R$. The set U is open in R and hence is open in X. We have $U = \bigcup_{k=1}^{\infty} F_k$ where the F_k's are compact. Each F_k is a zero-set in X and therefore U is a Baire set in X.

Finally, we claim that U^- is not a Baire set. Otherwise $U^- = Z(g)$ for some $g \in \mathfrak{C}^+(X)$. Clearly $g(x) = 0$ for $x \in R$ if and only if $x \in \bigcup_{n=1}^{\infty} [4n-1, 4n+1]$. For $n = 1, 2, \ldots$, choose $x_n \in R$ such that $1 < |4n - x_n| < 2$ and $|g(x_n)| < 1/n$; let $S = \{x_1, x_2 \ldots\}$. Then S is a zero-set for some bounded continuous function on R and, by Theorem 6.4 [4], $S^- \cap U^- = \phi$. Since S is not compact, S is not closed in X and there is an element y in $S^- - S$. Then $g(y) = 0$ and $y \in Z(g)$, and yet $y \notin U^-$. This contradiction shows that U^- is not a Baire set.

Example 3.5. Let X be any space that is not σ-compact. Then the measure μ given by

$$\mu(A)=0 \text{ if } A \text{ is a subset of a } \sigma\text{-compact set,}$$
$$=\infty \text{ otherwise,}$$

defines a nontrivial infinite Baire measure on X that is not regular.

Example 3.6. Let X consist of all ordinals less than the first uncountable ordinal Ω. With the order topology, X is normal and locally compact, but not paracompact (see Ex. Y, page 172, Kelley [9]). For Baire sets $E\subset X$, define $\mu(E)=1$ if E contains an uncountable closed set, and $\mu(E)=0$ otherwise. Then μ is a finite Baire measure that is not regular; see Exercise 52.10 [5].

Example 3.7. Let X denote the Sorgenfrey line; i.e., the real line with sets $[a, b[$ as an open basis. Then X is paracompact as shown by Sorgenfrey [15]. Since the sets $[a, b[$ are Lebesgue measurable, it follows that the Baire sets of X are all Lebesgue measurable. Let μ be the Lebesgue measure restricted to $[0, 1]$ and defined for all Baire sets of X. Then μ is not regular, since every compact set, being countable, has measure zero.

Example 3.8. Let X be the compact space βN^+, where $N^+=\{1, 2, 3, \ldots\}$. Let g be the continuous function on X defined by

$$g(n)=1/n \quad \text{for} \quad n\in N^+,$$
$$g(x)=0 \quad \text{for} \quad x\in X-N^+.$$

Then there does not exist a metric space M, a continuous open mapping τ of X onto M, and a function f on M such that $f\circ\tau=g$.

Assume that such M, τ, and f exist. Then τ must be one-to-one on N^+; evidently $\tau(N^+)\neq M$. Select any x in $M-\tau(N^+)$. Then $x=\lim_k \tau(n_k)$ for some sequence $\{n_k\}_{k=1}^{\infty}$ of distinct integers. Let $U=\{n_k\in N^+ : k \text{ even}\}^-$. Since $X-U=\{n\in N^+ : n\neq n_k \text{ for any even } k\}^-$, U is open and closed in X. Plainly $\tau(U)=\{\tau(n_k) : k \text{ even}\}\cup\{x\}$. Since $x=\lim_k \tau(n_{2k+1})$, $\tau(U)$ is not open. This contradicts our assumption.

Example 3.9. Let X be the well-ordered set of ordinals less than or equal to the smallest ordinal θ whose corresponding cardinal is $\aleph_2$. Then X is compact and not first countable. If $A\subset X$ has the property that $\theta\notin\bar{A}$ and card $(A)=\aleph_1$, then A is bounded and θ has a neighborhood missing A altogether.

ACKNOWLEDGEMENT

This work was supported in part by the National Science Foundation on grants NSF-G23799 and NSF C5-1501, respectively.

REFERENCES

1. Bourbaki, N., *Topologie générale*, Actualités Sci. Ind. 1142. Paris: Hermann & Cie. 1951.
2. Corson, H. H. & Isbell, J. R., Some properties of strong uniformities, Quart. J. Math. Oxford (2) *11*, 17–33 (1960).
3. Fuchs, L., *Abelian Groups*, London: Pergamon Press 1960.
4. Gillman, L. & Jerison, M., *Rings of Continuous Functions*, Princeton, N.J.: D. Van Nostrand Co. 1960.
5. Halmos, P. R., *Measure Theory*, New York, N.Y.: D. Van Nostrand Co. 1950.

6. HEWITT, E., Linear functionals on spaces of continuous functions, Fund. Math. *37*, 161–189 (1950).
7. HEWITT, E. & ROSS, K. A., *Abstract Harmonic Analysis* I, Berlin: Springer-Verlag 1963.
8. KATĚTOV, M., Measures in fully normal spaces, Fund. Math. *38*, 73–84 (1951).
9. KELLEY, J. L., *General Topology*, New York, N.Y.: D. Van Nostrand Co. 1955.
10. MICHAEL, E. A., The product of a normal space and metric space need not be normal, Bull. Amer. Math. Soc. *69*, 375–376 (1963).
11. —— The product of a normal space and metric space need not be normal, Notices Amer. Math. Soc. *10*, 258–259 (1963).
12. MONTGOMERY, D. & ZIPPIN, L., *Topological Transformation Groups*, New York, N.Y.: Interscience Publishers, Inc. 1955.
13. ROSS, K. A. & STONE, A. H., Products of separable spaces, Amer. Math. Monthly, *71*, 398–403 (1964).
14. RUBIN, H., Almost Lindelöf measures on paracompact spaces, Notices Amer. Math. Soc. *10*, 472 (1963).
15. SORGENFREY, R. H., On the topological product of paracompact spaces, Bull. Amer. Math. Soc. *53*, 631–632 (1947).
16. ULAM, S., Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math. *16*, 140–150 (1930).

Tryckt den 21 juni 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB