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On modifications of Riemann surfaces

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I. Introduction

1. The first to introduce modifications into the theory of several complex variables were H. Behnke and K. Stein [4] and H. Hopf [7]. Meanwhile modifications have been generalized, and they developed into an important tool in the study of complex spaces ([3], 23–27).

2. It is the aim of this paper to define and study modifications of Riemann surfaces. They will be explained in an intuitive way first. A formal definition (Definition 1) will be given later.

Let R be a Riemann surface, and let N be a closed subset, such that $S=R-N$ is non-empty and connected, hence a Riemann surface. The idea of modification of R in N consists of removing N from R, thus getting S, and then inserting a set N^* instead of N, such that $S \cup N^*$ will be a Riemann surface R^*. It is understood that N^* should actually replace N; more precisely each component of N is replaced by a non-empty set, and furthermore the union of all such replacements is exactly N^*.

Hence it is possible to invert the process, i.e. one can start with R^*, then remove N^* and insert N instead, in order to get R.

3. In this study of modifications of Riemann surfaces we want to find relations between N and R^* assuming R given. We are able to characterize a class of sets N, such that for every modification of R in N, R^* is topologically equivalent to R (Definition 2 and Theorem 1). Similarly another class of sets N will be discussed, such that for every modification of R in N, R^* is conformally equivalent to R (Definition 3 and Theorem 2). It will turn out that the sets in question are simple generalizations of the sets N_{SB} and N_D, which were defined and studied by L. Ahlfors and A. Beurling [2].

We shall also deal with the converse question, namely what can be said about N, if for every modification of R in N, R^* is topologically, respectively conformally, equivalent to R. But this seems to be a rather delicate problem.

4. Nevertheless we are able to show that converses (to Theorems 1 and 2) hold for some restricted classes of Riemann surfaces. First mention has to be made of the few exceptional cases where we know that the converse (to Theorem 2) is incorrect. Then the problem will be settled for the class of compact Riemann surfaces (Theorem 3). Here use will be made of a theorem by K. Oikawa [8], stated for prolongations of Riemann surfaces. It will be shown that his problem enters into the theory of modifications.

We shall also define a second class of Riemann surfaces (Definition 4), for which complete converses can be proved (Theorem 4).

II. Modifications

1. In the introduction an informal approach to modifications of Riemann surfaces was given. Before we can formalize it, we need a few notions ([6], 63).

Let N be a closed subset of a Riemann surface R, such that $S=R-N$ is non-empty and connected. Let n be a component of N, and let $\mathfrak{U}$ be the filter of the neighborhoods U of n in R. The family whose elements are the sets $V=U\cap S$ is a filter in S, but only a filter base in R, denoted by $\mathfrak{V}$. Furthermore, if $\mathfrak{B}$ is a filter base in a Riemann surface Z and $\bar{M}$ denotes the closure of an element M of $\mathfrak{B}$, then the set of adherent points of $\mathfrak{B}$ in Z is defined as $\bigcap_{M\in\mathfrak{B}}\bar{M}$ ([6], 78).

2. The following definition of modifications of Riemann surfaces is a modification of the definition found in ([3], l.c.).

Definition 1. *A modification of a Riemann surface R in N is a quintuple $\langle R,N,S,N^*,R^*\rangle$ where*

(i) *S, R and R^* are Riemann surfaces, such that $S\subset R$, $S\subset R^*$, and $N=R-S$, $N^*=R^*-S$;*

(ii) *for every component n of N, the set of adherent points in R^* of the filter base $\mathfrak{V}$ is not empty. Here $\mathfrak{V}$ consists of the elements $V=U\cap S$, U being a neighborhood of n in R;*

(iii) *for every component n^* of N^*, the set of adherent points in R of the filter base $\mathfrak{V}^*$ is not empty. Here $\mathfrak{V}^*$ consists of the elements $V^*=U^*\cap S, U^*$ being a neighborhood of n^* in R^*.*

All the conditions, especially (ii) and (iii), are completely symmetric. Since the set of adherent points in R of the filter base $\mathfrak{V}$ is obviously the boundary of n in R, condition (ii) states that every component n of N has been replaced by a non-empty subset of N^*. But (iii) adds that conversely every component n^* of N^* is the replacement of a non-empty subset of N.

It is possible that a single component of N is replaced by several components of N^*, but also conversely several components of N might be replaced by a single component of N^*.

If $\langle R,N,S,N^*,R^*\rangle$ is a modification of R in N, then $\langle R^*,N^*,S,N,R\rangle$ is evidently a modification of R^* in N^*.

3. Let $\langle R,N,S,N^*,R^*\rangle$ be a modification of R in N. Since $S\subset R$ and $S\subset R^*$, the Riemann surface S is imbedded in both R and R^*. Or R and R^* can be considered as extensions (or prolongations) of S. The study of extensions of Riemann surfaces is therefore related to the study of modifications.

Let S be a Riemann surface of finite genus, and let R and R^* be compact Riemann surfaces of the same genus, such that $S\subset R$ and $S\subset R^*$. With the notations $N=R-S$ and $N^*=R^*-S$ one finds easily that $\langle R,N,S,N^*,R^*\rangle$ is a modification of R in N. In order to study the compact extensions of S to Riemann surfaces of the same genus, it suffices to start with one such extension R and then study the modifications of R in N in which R^* is compact and has the same genus as R.

III. Main results

1. In this section the theorems are assembled. But their proofs will be given later.

In the following T will always denote a simply connected Riemann surface of hyperbolic type, and τ will always stand for a conformal mapping τ: $T\to D$ of T

onto the unit disc $D=\{z \text{ complex: } |z|<1\}$. Also N shall be a totally disconnected, closed subset of a Riemann surface R.

2. All our theorems are built on the following generalizations of the sets N_{SB} and N_D.

Definition 2–3. *N is said to be a set N_{SB} in R [a set N_D in R], if and only if for every subset T of R and every τ any compact subset of $\tau(N \cap T)$ is a set N_{SB} [a set N_D] in the sense of L. Ahlfors and A. Beurling* [2].

As it follows immediately from the definition of the sets N_{SB} [2], one could equivalently require that for every T and every conformal mapping σ of $T-N$ into D any bounded component of the complement of $\sigma(T-N)$ consists of a single point.

Another, still equivalent characterization of the sets N_{SB} in R would be the condition that for every T any conformal mapping of $T-N$ into D admits a continuous extension to T ([10], Corollary 4).

Similarly one could demand for the definition of the sets N_D in R that for every T any conformal mapping of $T-N$ into D admits a conformal extension to T ([10], 67).

It follows from [2] but also from above that every set N_D in R is a set N_{SB} in R. On the other hand, there are sets N_{SB} in R which are not sets N_D in R ([2], Theorem 16).

Theorem 1–2. *Let R be a Riemann surface.*

If N is a set N_{SB} in R [a set N_D in R], then for every modification of R in N, R^* is homeomorphic [conformally equivalent] to R.

In addition N^* is also a set N_{SB} in R^* [a set N_D in R^*], and one such homeomorphism [one such conformal mapping] is represented by the uniquely determined extension of the identity mapping of S onto itself to a continuous mapping of R onto R^*.

3. It has to be noted that for the converses to Theorem 1 and Theorem 2 we only require that R and R^* are homeomorphic respectively conformally equivalent; we do not demand that one such homeomorphism respectively conformal mapping can be obtained as the continuous extension of the identity mapping of S onto itself. Because otherwise converses would be quite trivial.

In order to list the exceptional cases where we know that complete converses fail to exist, the class $\mathfrak{R}^-$ of Riemann surfaces will be defined. It consists of the Riemann surfaces which are conformally equivalent, either to the Riemann sphere, to the complex plane, to the unit disc, to the punctured disc, to the punctured plane, or to the doubly punctured plane. In all those six cases the converse to Theorem 2 is not correct. Because any modification of R in N, where $R \in \mathfrak{R}^-$ and N is a compact set N_{SB} in R, will lead to a Riemann surface R^* which is not only homeomorphic, but also conformally equivalent to R. On the other hand, the converse to Theorem 1 will be shown to hold also for the class $\mathfrak{R}^-$.

4. Our first partial converse is

Theorem 3. *Let R be a compact Riemann surface different from the Riemann sphere, and let N be a closed subset of R, such that $S=R-N$ is non-empty and con-nected.*

If for any modification of R in N, R^* is homeomorphic [conformally equivalent] to R, then N is a set N_{SB} in R [a set N_D in R].

5. A boundary component β of a Riemann surface R is said to be an isolated planar boundary continuum, if and only if there is a Jordan curve C partitioning R

into three sets R', R'' and C, such that (i) the boundary of R' consists of β and C and (ii) R' can be mapped conformally onto a non-degenerated annulus $\{z$ complex: $0<r<|z|<1\}$. Furthermore we recall that a planar Riemann surface is by definition conformally equivalent to some domain in the complex plane.

Definition 4. *The class* $\mathfrak{R}^+$ *consists of the planar Riemann surfaces which have finitely many but at least two isolated planar boundary continua.*

Theorem 4. *Let* R *be a Riemann surface of class* $\mathfrak{R}^+$, *and let* N *be a closed subset of* R, *such that* $S=R-N$ *is non-empty and connected.*

If for any modification of R *in* N, R^* *is homeomorphic [conformally equivalent] to* R, *then* N *is a set* N_{SB} *in* R *[a set* N_D *in* R*].*

IV. Proofs of Theorems 1 and 2

1. The proof of Theorem 1 relies essentially upon [10] and [11]. Therefore we shall first give a summary of the results we shall have to refer to.

Let S be a Riemann surface, and β be a boundary component of S. We say that S is conformally imbedded in a Riemann surface Z and that β or a part of β is realized in Z, if there is a conformal mapping of S into Z and β or a part of β corresponds (in the induced boundary correspondence) to a non-empty subset of Z. The boundary component β of S will be called point-like if and only if there exists a realization of β or of a part of β in some Z, and for every such realization, β is realized as a point.

A boundary component β of S is said to be planar, if and only if there exists a Jordan curve C in S partitioning S into three sets S^*, S' and C, such that S^* is homeomorphic to a domain in the complex plane and its boundary contains β. The family of the closed curves γ lying in S^* and separating the two boundaries C and β will be denoted by Γ.

The main result in [11] states that a boundary component β of a Riemann surface S is point-like if and only if (i) β is planar and (ii) the extremal length $\lambda(\Gamma)$ of Γ is zero.

2. Furthermore we have to make appeal to ([10], 66) in order to show that the concept of point-like boundary components is very closely related to the generalization of the sets N_{SB} given in Definition 2. Let Z be a Riemann surface, and let according to Definition 2 the set N be a set N_{SB} in Z. Then obviously every point of N is a planar boundary component of S, $S=Z-N$. And by Proposition 1 and Corollary 2 of [10] the extremal length condition mentioned in IV.1 is fulfilled in S for every such boundary point. Hence every point of N considered as a boundary component of S is point-like. But also conversely let Z be a Riemann surface, and let N be a closed subset of Z. If every component of N is point-like with respect to S, $S=Z-N$, then tracing all the steps backwards one concludes easily that N is a set N_{SB} in Z.

3. Let us now assume the hypotheses of Theorem 1 and let $\langle R,N,S,N^*,R^*\rangle$ be a modification of R in N. We want to extend the identity mapping i of $S, S\subset R$, onto $S, S\subset R^*$, to a continuous mapping of R onto R^*. Since N is a set N_{SB} in R, every point p of $N, N=R-S$, considered as a boundary component of S, is point-like as emphasized in IV.2. But S is also imbedded in R^*. Hence using IV.1 to each point $p, p\in N$, corresponds either a point p^* in R^* or no subset at all. However, the latter is excluded by condition (ii) of Definition 1. Therefore the mapping i can be extended

to N. Applying condition (iii) of Definition 1 one concludes that every component of N^* consists of a point and that $i(N)=N^*$. So i has been extended to a mapping of R onto R^*.

The mapping i is trivially one-to-one and bicontinuous at every point of S. Let p be a point of N, and let $\mathfrak{V}$ be the filter base defined for p according to Definition 1. Obviously $\mathfrak{V}$ converges in R to the point p, and in R^* to the point $p^*, p^*=i(p)$. Since every $\bar{V}$ contains an element U, i is continuous at p ([6], 82).

We shall indirectly prove that i is one-to-one. Let us therefore assume that p_1 and p_2, $p_1 \neq p_2$, $p_1 \in N$, $p_2 \in N$, are mapped by i onto p^*. Let U^* be a simply connected, open neighborhood of p^*, and let $U=i^{-1}(U^*)$. U contains p_1 and p_2 and is open since i is continuous. Let C be a Jordan curve in $U \cap S$ separating p_1 from p_2 in R. Let $q \in C$. Then there are two open curves $C_1, C_1 \subset S-C$, and $C_2, C_2 \subset S-C$, connecting q with p_1 and p_2 respectively. $i(C)$ is a Jordan curve in U^*, and $i(C_1)$ and $i(C_2)$ will connect $i(q)$ with p^*, lying entirely on different sides of $i(C)$. This is impossible since U^* is simply connected. Whence the desired contradiction.

Similarly as for i one shows that its inverse i^{-1} is continuous at any $p^*, p^* \in N^*$. So i is a homeomorphism, and R and R^* are homeomorphic. Using the filter base $\mathfrak{V}$ one also concludes that the extension of i to a continuous mapping of R is unique. Furthermore, N^* is a subset of R^*, and every point p^* of N^* considered as a boundary component of S is point-like. Hence N^* is a set N_{SB} in R^* as shown in IV.2. Thus Theorem 1 is proved.

4. Our next goal is the proof of Theorem 2. Since every set N_D in R is a set N_{SB} in R, it suffices to show that the mapping i: $R \to R^*$ of IV.3 is conformal. Let $T, T \subset R$, be simply connected and let its boundary be a Jordan curve $C, C \subset R$. Let τ: $T \to D$ map T conformally onto the unit disc D and let $T^*=i(T)$. Then T^* is also simply connected and of hyperbolic type since $C \subset R$. Denote by τ^* a conformal mapping τ^*: $T^* \to D$. The restriction of i to T is thus represented as a homeomorphism of D onto itself which is conformal in $\tau(T-N)$. By Definition 3 and ([10], 67) the set $\tau(T \cap N)$ is conformally removable as we have already stated in III.2. Hence i is conformal in any such T and therefore on R. Obviously N^* is then a set N_D in R^*.

V. Two Lemmata

Lemma 1. *Let R be a Riemann surface, and let N be a closed subset of R, such that $S=R-N$ is non-empty and connected.*

If N is not a set N_{SB} in R, then there exists a modification of R in N, such that R^* has, compared with R, one additional boundary component.

Proof. Let us first assume that N is totally disconnected. Since N is not a set N_{SB} in R, it follows by negating the definition of the sets N_{SB} in R, that there exists a compact subset N_0 of N and a simply connected neighborhood $U, U \subset R$, of N_0, such that $U-N_0$ can be mapped conformally into an annulus $A=\{z$ complex: $2<|z|<r\}$, where some point $p, p \in N_0$, corresponds (in the induced boundary correspondence) to the circle of radius 2 and the boundary of U to the circle of radius r.

In the second case where N is not totally disconnected, N contains a continuum. Choose a compact subcontinuum δ and a neighborhood $U', U' \subset R$, of δ, such that $U'-\delta$ is doubly connected and admits a conformal mapping onto an annulus $A'=$

$\{z \text{ complex: } 2<|z|<r'\}$, where some subset of the boundary of δ corresponds to the circle of radius 2.

In order to get the desired modification we shall essentially modify U and U' only.

In the first case $U-N_0$ is conformally equivalent to a subset of A. Let $B=\{z \text{ complex: } 1<|z|<r\}$, and define $R^*=(R-N_0)\cup B$ where the points in $U-N_0$ and B which correspond to each other under the conformal mapping have been identified. If we put $N^*=R^*-S$, it is easy to check that $\langle R,N,S,N^*,R^*\rangle$ is a modification of R in N. We remark that the point $p, p\in N_0$, has been replaced by a set represented by $B-A$.

Quite similarly we define $B'=\{z \text{ complex: } 1<|z|<r'\}$ in the second case. Put $R^*=(R-\delta)\cup B'$ where the points in $U'-\delta$ and B' which are mapped onto each other have been identified. Let $N^*=R^*-S$. Then $\langle R,N,S,N^*,R^*\rangle$ is a modification of R in N. And the component of N which contains δ has been replaced by a set which contains in our representation the set $B'-A'$.

In both cases a modification of R in N has been constructed, where R^* has, as it follows immediately, exactly one additional boundary component. It is represented by the unit circle, boundary of B and of B' respectively.

Lemma 2. *Let R be a Riemann surface, and let N be a closed subset of R, such that $S=R-N$ is non-empty and connected.*

If N is not a set N_{SB} in R, then there is a modification of R in N, such that R^* is a Riemann surface of infinite genus.

Proof. Let $\langle R,N,S,N',R'\rangle$ be a modification satisfying Lemma 1. According to the proof of Lemma 1 one can assume that N' contains a set represented by $W=\{z \text{ complex: } 1<|z|\leqslant 2\}$. Hence W will be regarded as a subset of R'. We shall prove Lemma 2 by removing part of W from R' and inserting in its place a Riemann surface of infinite genus.

Let H be a Riemann surface of infinite genus, and let U be a simply connected subdomain of H. Suppose g maps U conformally onto $\{z \text{ complex: } |z|>1 \text{ or } z=\infty\}$. Let H' be the surface H from which one has removed all points which are mapped by g onto $\{z \text{ complex: } |z|\geqslant 2 \text{ or } z=\infty\}$. We can consider $g(U\cap H')$ as a subset of H'.

Put $R^*=R'\cup H'$, where we agree to identify equal points in W and in $g(U\cap H')$. Let $N^*=R^*-S$. Then $\langle R,N,S,N^*,R^*\rangle$ is a modification of R in N which satisfies the requirements.

VI. Proofs of Theorems 3 and 4

1. The proofs concerning the sets N_{SB} are based upon the lemmata. Let R be a Riemann surface, which is either compact or belongs to the class $\mathcal{R}^-$ defined in III.3. Let N be a closed subset of R, such that $S=R-N$ is non-empty and connected. If N is not a set N_{SB} in R, then by Lemma 1 there is a modification of R in N, such that R^* is either not compact or is planar and has finite connectivity higher than R. Hence R and R^* cannot be homeomorphic. The partial converse to Theorem 1 stated as part of Theorem 3 is thus proved by contraposition. Furthermore, the converse to Theorem 1 holds also for the class $\mathcal{R}^-$.

In the case of the class $\mathcal{R}^+$ given by Definition 4 one concludes similarly using Lemma 2, that the converse to Theorem 1 stated as part of Theorem 4 is valid.

2. We shall now prove the second part of Theorem 3. Let R be a compact Riemann surface different from the Riemann sphere, and let N be a closed subset of R, such

that $S=R-N$ is non-empty and connected. If for any modification of R in N, R^* is conformally equivalent to R, then R^* is a compact extension of $S=R-N$. Hence it suffices to assume that all compact extensions of S are conformally equivalent. As K. Oikawa has proved ([8], Theorem 1), S is then a Riemann surface of class O_{AD}. But H. L. Royden has shown ([12], Theorem 2), that such a Riemann surface of class O_{AD} is conformally equivalent to a compact Riemann surface—which we can identify with R—from which one has removed what in our terminology is a set N_D in R. Hence the second part of Theorem 3 is proved.

3. What remains is the proof of the second part of Theorem 4. It will be done by contraposition. As it follows from III.5, it is permitted to confine oneself to domains G in the complex plane which have a finite number of isolated boundary continua. Since the first part of Theorem 4 is settled, one can assume that N is a set N_{SB} in G which is not a set N_D in G. Such sets exist ([2], l.c.) as already mentioned in III.2. Put $S=G-N$. Hence our task is to get a modification $\langle G,N,S,N',G'\rangle$ of G in N, such that G' is planar, but not conformally equivalent to G.

(α) We shall make use of the following general observation. Our goal is to construct a modification of G in N such that G' is not conformally equivalent to G. Let us assume that a modification $\langle G,N_0,S_0,N_0',G'\rangle$ of G in N_0, where N_0 is some closed subset of N, will lead to such a G'. In this case it is permissible to modify G in N_0 only in order to get the desired G'. Because one arrives at a modification $\langle G,N,S,N',G'\rangle$ of G in N with the same Riemann surface G' by simply putting $S=G-N$ and $N'=G'-S$.

(β) Consider a compact subset N_0 of N which is not a set N_D. Put $S_0=G-N_0$. Map the complement of N_0—with respect to the complex plane—by a function π conformally into the complex plane, such that the induced image $\pi(N_0)$ of the boundary N_0 has positive area. Since N_0 is not a set N_D, such mappings exist by ([2], Theorem 4). Let $N_0'=\pi(N_0)$, identify S_0 and its image $\pi(S_0)$, and put $G'=\pi(S_0)\cup\pi(N_0)$. Then $\langle G,N_0,S_0,N_0',G'\rangle$ is a modification of G in N_0. If G' is not conformally equivalent to G, we are through.

(γ) Otherwise we continue to modify G' in N_0', where N_0' is compact and has positive area. By hypothesis G and hence also G' have at least two but only finitely many isolated boundary continua. We single out two, and denote by Γ the family of the open curves lying in G' and connecting the two selected boundaries. With each pair of isolated boundary continua we thus associate an admissible family Γ. We get at least one but only finitely many such Γ. The extremal length $\lambda(\Gamma)$ of each Γ is a conformal invariant.

Take a family Γ_0, such that $\lambda(\Gamma)\leqslant\lambda(\Gamma_0)$ for every admissible family Γ. Call E_0 and E_1 the two selected boundaries which Γ_0 refers to. Let Γ_0' be the subfamily of Γ_0 which consists of the curves lying in $S_0=G'-N_0'$. Since N_0' has positive area, by ([9], § 4.1. and Theorem 2) the family Γ_0' is not normal relative to Γ_0. Therefore $\lambda(\Gamma_0)<\lambda(\Gamma_0')$.

Let σ map S_0 conformally into an annulus $A=\{z \text{ complex: } r_0<|z|<r_1\}$ such that (i) $E_0^*=\sigma(E_0)$ and $E_1^*=\sigma(E_1)$ are the circles of radii r_0 and r_1 respectively and (ii) the image $\sigma(S_0)$ of S_0 is a minimal radial slit domain in A. Identify S_0 and $\sigma(S_0)$; delete from A the set which corresponds to the boundary of G' and call that deleted domain G^*. Put in addition $N_0^*=G^*-\sigma(S_0)$. Then $\langle G',N_0',S_0,N_0^*,G^*\rangle$ is a modification of G' in N_0'.

The family Γ_0^* which consists of the open curves lying in G^* and connecting E_0^* with E_1^* has by construction an extremal length $\lambda(\Gamma_0^*)$ equal to $\lambda(\Gamma_0')$ and hence distinct from every $\lambda(\Gamma)$ defined in G'. So G^* cannot be conformally equivalent to G', because the $\lambda(\Gamma)$ are conformal invariants.

Since G' was supposed to be conformally equivalent to G, G^* and G are not conformally equivalent. To finish the proof, it suffices to remark that $\langle G, N_0, S_0, N_0^*, G^* \rangle$ is a modification of G in N_0 such that G and G^* are not conformally equivalent.

VII. Remarks

1. In the Lemmata we proved more than we actually needed for getting the converses stated as parts of the Theorems 3 and 4. Therefore one concludes from Lemma 1 and Lemma 2, that the converse to Theorem 1 holds for the class of Riemann surfaces having only a finite number of planar boundary components as well as for the class of Riemann surfaces of finite genus.

2. In the proof of the second part of Theorem 4 we needed that there was a family Γ_0 such that $\lambda(\Gamma) \leqslant \lambda(\Gamma_0)$ and Γ_0' was not normal relative to Γ_0. Using ([9], § 8.1.) it is possible to generalize the converse to Theorem 2 to the class of Riemann surfaces which are planar and have at least two but only a finite number of boundary continua which possess free subarcs.

3. We have so far treated the cases where the two Riemann surfaces R and R^* were either homeomorphic or conformally equivalent. One might consider quasiconformal equivalence too. Using III.3 and Theorem 1 combined with Teichmüller's theorem (e.g. [1], 50; [5], 107) one gets the following result: If R is a compact Riemann surface and N is a set N_{SB} in R, then for every modification of R in N, R^* is quasiconformally equivalent to R.

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REFERENCES

1. AHLFORS, L. V., The complex analytic structure of the space of closed Riemann surfaces. *Analytic Functions.* Princeton University Press, 45–66, 1960.
2. AHLFORS, L., and BEURLING, A., Conformal invariants and function-theoretic null-sets. *Acta Math. 83,* 101–129 (1950).
3. BEHNKE, H. and GRAUERT, H., Analysis in non-compact complex spaces. *Analytic Functions.* Princeton University Press, 11–44, 1960.
4. BEHNKE, H. und STEIN, K., Modifikation komplexer Mannigfaltigkeiten und Riemannsche Gebiete. *Math. Annalen 124,* 1–16 (1951).
5. BERS, L., Quasiconformal mappings and Teichmüller's theorem. *Analytic Functions.* Princeton University Press, 89–119, 1960.
6. BOURBAKI, N., *Eléments de Math. Topologie Générale.* Chap. I/II. 3e édition. Hermann, Paris, 1961.
7. HOPF, H., Schlichte Abbildungen und lokale Modifikationen 4-dimensionaler komplexer Mannigfaltigkeiten. *Comment. Math. Helvet. 29,* 132–156, (1955).
8. OIKAWA, K., On the uniqueness of the prolongation of an open Riemann surface of finite genus. *Proc. Amer. Math. Soc. 11,* 785–787 (1960).
9. RENGGLI, H., Zur konformen Abbildung auf Normalgebiete. *Comment. Math. Helvet. 31,* 5–40 (1956).
10. RENGGLI, H., Quasiconformal mappings and extremal lengths. *Amer. Journ. of Math. 86,* 63–69 (1964).
11. RENGGLI, H., On point-like boundaries of Riemann surfaces. (To appear in *Arkiv för Matematik.*)
12. ROYDEN, H. L., On a class of null-bounded Riemann surfaces. *Comment. Math. Helvet. 34,* 52–66 (1960).

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