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On unique supports of analytic functionals

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1. Introduction

The principal aim of this paper is to prove that an analytic functional in a domain of holomorphy in $\mathbf{C}^n$ having a smooth holomorph convex compact support admits no other holomorph convex compact support. This result is derived from properties of compact weak carriers of analytic functionals which we prove by means of $\bar{\partial}$ cohomology and which are analogous to properties of compact carriers proved by Martineau using Čech cohomology.

If Ω is an open set in $\mathbf{C}^n$, we write $\mathcal{A}(\Omega)$ for the set of all analytic functions in Ω. This is a Fréchet space under the topology of uniform convergence on all compact parts of Ω. An element of the dual space $\mathcal{A}'(\Omega)$ is called an analytic functional. For an arbitrary part M of $\mathbf{C}^n$, $\mathcal{A}(M)$ denotes the inductive limit of all spaces $\mathcal{A}(\omega)$ where $\omega \supset M$ is open. If $M_1 \subset M_2$, $\mathcal{A}(M_1)$ induces in a natural way a (not necessarily Hausdorff) topology in $\mathcal{A}(M_2)$.

The Definitions 1.1 and 1.4 below are due to Martineau [4].

Definition 1.1. *A compact part K of an open set $\Omega \subset \mathbf{C}^n$ is called a weak carrier of $\mu \in \mathcal{A}'(\Omega)$ if for any open set ω such that $K \subset \omega \subset \Omega$, μ is continuous for the topology induced in $\mathcal{A}(\Omega)$ by $\mathcal{A}(\omega)$. It is equivalent to say that for every neighborhood U of K which is contained in Ω there is a constant C_U such that*

$$|\mu(f)| \leqslant C_U \sup_U |f|$$

for all $f \in \mathcal{A}(\Omega)$. Further, K is said to be a carrier of μ if μ is continuous with respect to the topology induced in $\mathcal{A}(\Omega)$ by $\mathcal{A}(K)$.

Martineau also considers carriers which need not be compact, but in this paper a carrier shall always mean a compact carrier. Every carrier of a functional $\mu \in \mathcal{A}'(\Omega)$ is a weak carrier of μ. It is unknown whether the converse is true (see Martineau [4]) so that the relation between Martineau's theorems on carriers and the corresponding results on weak carriers in Section 2 is not clarified. However, it has been proved by Martineau [4, Ch. I, Théorème 1.1'] that the two notions coincide for compact sets satisfying a certain geometric condition. In particular, it is easy to see by means of the Runge approximation theorem that a compact subset K of a domain of holomorphy Ω which is $\mathcal{A}(\Omega)$-convex in the sense of the following definition carries every analytic functional in Ω which it carries weakly. Thus the proofs of Section 3 remain equally valid if we substitute properties of carriers for the properties of weak carriers actually used.

Definition 1.2. *If M is a non-empty part of $\Omega \subset \mathbf{C}^n$ we define the $\mathcal{A}(\Omega)$-hull $\hat{M_\Omega}$ of M by*

$$\hat{M_\Omega} = \{z \in \Omega;\ \text{for all } f \in \mathcal{A}(\Omega),\ |f(z)| \leqslant \sup_M |f|\}, \tag{1.1}$$

and we say that M is $\mathcal{A}(\Omega)$-convex, or holomorph convex, if $\hat{M_\Omega} = M$. An $\mathcal{A}(\mathbf{C}^n)$-convex set is also called polynomially convex. We write ch$M$ *for the closed convex hull of M.*

The proof of Theorem 3.2 depends on the following

Theorem 1.3. *Suppose that $\Omega \subset \mathbf{C}^n$ is a domain of holomorphy and let M be a relatively compact subset of Ω. Then $\hat{M_\Omega}$ is compact and*

$$\hat{M_\Omega} = \{z \in \Omega;\ \text{for all } F \in \mathcal{P}(\Omega) \cap C^0(\Omega),\ F(z) \leqslant \sup_M F\}, \tag{1.2}$$

where $\mathcal{P}(\Omega)$ and $C^0(\Omega)$ denote, respectively, the set of all plurisubharmonic functions and the set of all continuous functions in Ω.

That $\hat{M_\Omega}$ is compact follows from the "fundamental theorem" of Cartan–Thullen [1] (see also [2, Theorem 2.5.5]). The equation (1.2) is a consequence of the solution of the Levi problem given by Oka–Norguet–Bremermann; for a proof we refer to Hörmander [2, Theorem 4.3.4]. A discussion concerning the relation of (1.2) to the Levi problem is given in Lelong [3] and Hörmander [2, Chapter IV].

Definition 1.4. *A subset K of Ω is called an $\mathcal{A}(\Omega)$-convex support of $\mu \in \mathcal{A}'(\Omega)$ if K is an $\mathcal{A}(\Omega)$-convex (weak) carrier of μ and no $\mathcal{A}(\Omega)$-convex proper subset of K carries μ (carries μ weakly). The notion of convex support is defined similarly with convexity instead of $\mathcal{A}(\Omega)$-convexity.*

In a domain of holomorphy Ω every non-zero analytic functional has at least one $\mathcal{A}(\Omega)$-convex support and, if Ω is a convex open part of $\mathbf{C}^n$, at least one convex support. Conversely, every $\mathcal{A}(\Omega)$-convex compact part of Ω is the unique $\mathcal{A}(\Omega)$-convex support of some analytic functional in Ω (Martineau [4, Ch. I, Théorème 2.1]). A natural question is thus to ask for functionals having exactly one ($\mathcal{A}(\Omega)$-)convex support, and also for compact sets K such that K is the only ($\mathcal{A}(\Omega)$-)convex support of any functional having K as an ($\mathcal{A}(\Omega)$-)convex support. A result in this spirit is the theorem of Martineau [4, Ch. I, Théorème 3.3 b] stating that an analytic functional carried by some compact set contained in $\mathbf{R}^n$ has a smallest carrier $\subset \mathbf{R}^n$ (but not necessarily a smallest $\mathcal{A}(\Omega)$-convex carrier, see the example (1.3) below). We refer the reader to [4] for some other theorems of this kind.

The Pólya representation of analytic functionals in one variable shows that any non-zero functional in $\mathcal{A}'(\mathbf{C}^1)$ has a unique convex support. However, a functional can have several polynomially convex supports even in the one dimensional case. This is shown by the example

$$\mu(f) = \int_0^1 f(z)\,dz, \quad f \in \mathcal{A}(\mathbf{C}^1). \tag{1.3}$$

Here any simple arc connecting 0 and 1 is a polynomially convex support of μ. To give a more interesting example we let $\omega = \{z \in \mathbf{C}^1;\ |z| > 1\}$, $g \in \mathcal{A}(\omega)$, and define

$$\mu(f)=\int_\Gamma f(z)g(z)\,dz, \quad f\in\mathcal{A}(\mathbf{C}^1),$$

where Γ is any Jordan curve in ω with winding number one with respect to the origin. Now write $\Gamma=\Gamma_1+\Gamma_2$ where Γ_1 and Γ_2 are simple arcs with only two points in common and define

$$\nu(f)=\mu(f)-\int_{\Gamma_1} f(z)g(z)\,dz, \quad f\in\mathcal{A}(\mathbf{C}^1).$$

Suppose $K\subset\{z\in\mathbf{C}^1;\ |z|\leqslant 1\}$ is a polynomially convex support of μ. Then $K\cup\Gamma_1$ is a polynomially convex support of ν (this follows e.g. from Corollary 2.6), and one might believe that the intersection of all polynomially convex supports of ν contains K. However,

$$\nu(f)=\int_{\Gamma_2} f(z)g(z)\,dz$$

so that ν is also carried by Γ_2. Thus, in the presence of an arbitrarily small curve Γ_1, no regularity condition on a part K of an $\mathcal{A}(\Omega)$-convex support of an analytic functional is sufficient to guarantee that K is contained in every $\mathcal{A}(\Omega)$-convex support.

On the other hand, Theorem 3.3 states that an $\mathcal{A}(\Omega)$-convex support of $\mu\in\mathcal{A}'(\Omega)$, $\Omega\subset\mathbf{C}^1$, containing (in a certain sense) no "curves" is the unique $\mathcal{A}(\Omega)$-convex support of μ.

In the case of several variables functionals may even fail to have unique convex supports. In fact, the analytic functional in $\mathbf{C}^2$ defined by

$$\mu(e^{z_1\zeta_1+z_2\zeta_2})=\cos(\zeta_1\zeta_2)^{\frac{1}{2}} \tag{1.4}$$

is carried by the polydisk (see Theorem 4.4.5 in [2])

$$K_t=\{z\in\mathbf{C}^2;\ |z_1|\leqslant t,\ |z_2|\leqslant(4t)^{-1}\},$$

where t is an arbitrary positive number. For every $t>0$ there exist convex and polynomially convex supports included in K_t, but all such sets must contain the point $a_t=(t,-(4t)^{-1})$ which proves that there are several supports of each kind (since $a_t\in K_s$ if and only if $t=s$), and also that these supports do not have continuously varying tangent planes.

The lack of smoothness of the supports of (1.4) is characteristic. We prove (Theorem 3.1) that smooth convex compact sets are unique convex supports whenever they are convex supports, and (Theorem 3.2) the analogous result for smooth $\mathcal{A}(\Omega)$-convex sets. Here Ω is supposed to be a domain of holomorphy in $\mathbf{C}^n$; however, it seems probable that the proof of the last-mentioned theorem can be modified to cover the case when Ω is a Stein manifold.

2. Some results on weak carriers of analytic functionals

Let Ω be an open set in $\mathbf{C}^n$. We write as usual $C^m(\Omega)$ for the set of all m times continuously differentiable complex-valued functions in Ω $(0\leqslant m\leqslant\infty)$, and $C_0^m(\Omega)$ for the set of all functions in $C^m(\Omega)$ having compact supports. We let $C^m_{(p,q)}(\Omega)$ stand for the set of all differential forms of type (p,q) with coefficients in $C^m(\Omega)$, i.e. $f \in C^m_{(p,q)}(\Omega)$ if and only if there exist functions $f_{i_1 \dots i_p j_1 \dots j_q} \in C^m(\Omega)$ such that

$$f = \sum_{i_1, \dots, i_p} \sum_{j_1, \dots, j_q} f_{i_1 \dots i_p j_1 \dots j_q} \quad dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

where $z_k = x_k + ix_{n+k}$, x_k and x_{n+k} real, $dz_k = dx_k + idx_{n+k}$, $d\bar{z}_k = dx_k - idx_{n+k}$. The differential of a form f is written as a sum

$$df = \partial f + \bar{\partial} f,$$

where ∂ and $\bar{\partial}$ are defined by the requirement that ∂f and $\bar{\partial} f$ be of type $(p+1,q)$ and $(p,q+1)$ respectively when f is of type (p,q). Thus, e.g. $\bar{\partial} u = \Sigma \partial u / \partial \bar{z}_k d\bar{z}_k$ if $u \in C^1(\Omega)$ where $\partial u / \partial \bar{z}_k = (\partial u / \partial x_k + i \partial u / \partial x_{n+k})/2$. In the proof of Theorem 2.1 the derivatives of the coefficients of a form are to be understood in the sense of distribution theory. Elsewhere in this paper only C^∞ forms are used. The coefficients of a form f can be chosen so that $f_{i_1 \dots i_p j_1 \dots j_q}$ is non-zero only if $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$. In this case we define $|f|$ by

$$|f|^2 = \sum \sum |f_{i_1 \dots i_p j_1 \dots j_q}|^2.$$

Theorem 2.1. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$, and K a compact subset of Ω. Then one can find a compact set $L \subset \Omega$ and a constant C such that for every form $f \in C^\infty_{(0,1)}(\Omega)$ with $\bar{\partial} f = 0$ there is a solution $u \in C^\infty(\Omega)$ of the equation $\bar{\partial} u = f$ satisfying* $\sup_K |u| \leqslant C \sup_L |f|$.

Proof. That the equation $\bar{\partial} u = f$ has a solution $u \in C^0(\Omega)$ for any $\bar{\partial}$ closed form $f \in C^0_{(0,1)}(\Omega)$ follows from Cartan's Theorem B by means of the Dolbeault isomorphism; see Schwartz [5] for a proof of the Dolbeault isomorphism when the $\bar{\partial}$ cohomology is that of forms with continuous coefficients.

However, since it is desirable in this context to rely exclusively on $\bar{\partial}$ cohomology, we wish to infer this result from Theorem 4.2.5 in Hörmander [2] according to which we can find a locally square integrable solution u of the equation $\bar{\partial} u = f$ if $f \in C^0_{(0,1)}(\Omega)$ and $\bar{\partial} f = 0$ (this is only a weak special case of Theorem 4.2.5 in [2]). We claim that $u \in C^0(\Omega)$ (after correction on a set of measure zero) if $\bar{\partial} u \in C^0_{(0,1)}(\Omega)$ and u is locally integrable. To prove this we form the regularizations $u_k = u * \varphi_k$ where φ is a positive function in $C_0^\infty(\mathbf{C}^n)$ with Lebesgue integral one and $\varphi_k(z) = k^{2n} \varphi(kz)$, $k = 1, 2, \dots$. For every open set ω, relatively compact in Ω, u_k is defined and infinitely differentiable in ω when k is large enough. It is sufficient to prove that u_k converges uniformly on every compact part K of Ω. If K is given, we choose an open neighborhood ω of K which is relatively compact in Ω. We then have that $\|u_k - u\|_\omega \to 0$ where the norm denotes the norm in $L^1(\omega)$, and that $\bar{\partial} u_k \to \bar{\partial} u$ uniformly in ω since $\bar{\partial} u$ is continuous. Applying the inequality of the next lemma to $u_j - u_k$ we find that

$$\sup_K |u_j - u_k| \leqslant C(\sup_\omega |\bar{\partial} u_j - \bar{\partial} u_k| + \|u_j - u_k\|_\omega) \tag{2.1}$$

which proves that u_k is a Cauchy sequence in $C^0(K)$, hence convergent to some function $v \in C^0(K)$, and then $v = u$ a.e. in K.

We have thus a well defined closed linear mapping

$$\{f \in C^0_{(0,1)}(\Omega);\ \bar{\partial} f = 0\} \ni f \xrightarrow{T} U \in C^0(\Omega)/\mathcal{A}(\Omega),$$

where the equivalence class U is defined by $\bar\partial u = f$ when $u \in U$. Since T is defined in a Fréchet space and has its values in an other Fréchet space, T is continuous by the closed graph theorem. In view of the definition of the topology in $C^0(\Omega)/\mathcal{A}(\Omega)$ this proves the inequality in Theorem 2.1. Finally, if $\bar\partial u \in C^\infty_{(0,1)}(\Omega)$, u is infinitely differentiable, see e.g. Hörmander [2, Theorem 4.2.5 and Corollary 4.2.6].

Remark. The estimates given in Theorem 2.1 are somewhat stronger than is actually needed in this paper. At the expense of a slightly longer proof of Theorem 2.4 we could have used weaker estimates of the type

$$\sup_K |u| \leqslant C \sup_L \sum_{|\alpha| \leqslant m} |D^\alpha f|,$$

where $D^\alpha f$ are forms whose coefficients are derivatives of those of f. These estimates follow as in the proof above if we use directly the triviality of the $\bar\partial$ cohomology of C^∞ forms.

To complete the proof of Theorem 2.1 it remains to prove the inequality (2.1) used there.

Lemma 2.2. *Let ω be an open set in $\mathbf{C}^n$ and K a compact subset of ω. Then there exists a constant C such that for all $u \in C^1(\omega)$*

$$\sup_K |u| \leqslant C \left(\sup_\omega |\bar\partial u| + \int_\omega |u|\, d\lambda_n \right),$$

where $d\lambda_n$ is the Lebesgue measure in $\mathbf{C}^n$.

Proof. First assume that K is a polycylinder in ω and choose functions $\varphi_1, \ldots, \varphi_n \in C_0^1(\mathbf{C}^1)$ such that $\Phi(z) = \varphi_1(z_1) \ldots \varphi_n(z_n) = 1$ in a neighborhood of K and the support of Φ is contained in ω. By the Cauchy integral formula applied to the function $u\Phi$ we get if $a \in K$

$$\begin{aligned} u(a) &= -\frac{1}{\pi} \int \frac{\partial(u\Phi)}{\partial \bar z_1} (z_1, a_2, \ldots, a_n)(z_1 - a_1)^{-1}\, d\lambda_1(z_1) \\ &= -\frac{1}{\pi} \int \left(\frac{\partial u}{\partial \bar z_1} \Phi \right)(z_1, a_2, \ldots, a_n)(z_1 - a_1)^{-1}\, d\lambda_1(z_1) \\ &\quad -\frac{1}{\pi} \int \left(u \frac{\partial \Phi}{\partial \bar z_1} \right)(z_1, a_2, \ldots, a_n)(z_1 - a_1)^{-1}\, d\lambda_1(z_1). \end{aligned}$$

Here $d\lambda_k$ denotes the $2k$-dimensional Lebesgue measure. By iterated application of the same formula to all coordinates $z_2, \ldots, z_n$ we obtain

$$u(a) = \sum_1^n \frac{1}{(-\pi)^k} \int \frac{\left(\frac{\partial u}{\partial \bar z_k} \frac{\partial^{k-1}\Phi}{\partial \bar z_1 \ldots \partial \bar z_{k-1}} \right)(z_1, \ldots, z_k, a_{k+1}, \ldots, a_n)}{(z_1 - a_1) \ldots (z_k - a_k)}\, d\lambda_k(z_1, \ldots, z_k)$$

$$+ \frac{1}{(-\pi)^n} \int \frac{\left(u \frac{\partial^n \Phi}{\partial \bar z_1 \ldots \partial \bar z_n} \right)(z)}{(z_1 - a_1) \ldots (z_n - a_n)}\, d\lambda_n(z).$$

The first n integrals of this sum are easily estimated by a constant times $\sup_\omega |\bar\partial u|$ since the singularities are integrable. The absolute value of the last integral does not exceed

$$\pi^{-n} \sup_z |\psi(z, a)| \int_\omega |u| \, d\lambda_n,$$

where

$$\psi(z, a) = \frac{\partial^n \Phi}{\partial \bar z_1 \dots \partial \bar z_n}(z)\,(z_1 - a_1)^{-1} \dots (z_n - a_n)^{-1} = \prod_1^n \frac{\partial \varphi_k}{\partial \bar\tau}(z_k)\,(z_k - a_k)^{-1}.$$

Since $\psi(z,a)$ is bounded as a function of z by a constant independent of $a \in K$, this proves the lemma when K is a polycylinder and hence in general since any compact set in ω can be covered by finitely many compact polycylinders contained in ω.

Theorem 2.3. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$, K an $\mathcal{A}(\Omega)$-convex compact part of Ω, and ω an arbitrary neighborhood of K. Then there exists a constant C such that for every $\varepsilon > 0$ and every form $f \in C^\infty_{(0,1)}(\Omega)$ satisfying $\bar\partial f = 0$ we can find $u \in C^\infty(\Omega)$ with $\bar\partial u = f$ and $\sup_K |u| \leqslant C \sup_\omega |f| + \varepsilon$.*

Proof. Assuming as we may that ω is compact we can find $h_1, \dots, h_m \in \mathcal{A}(\Omega)$ such that $K \subset V = \{z \in \Omega;\ |h_k(z)| < 1,\ k = 1, \dots, m\} \subset \complement \partial\omega$. Since $U = V \cap \omega$ is obviously a domain of holomorphy we can according to Theorem 2.1 choose a solution $v \in C^\infty(U)$ of the equation $\bar\partial v = f$ such that $\sup_K |v| \leqslant C \sup_U |f|$ for some constant C.

Now choose $\psi \in C_0^\infty(U)$ such that $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ in a neighborhood of K and set $g = f - \bar\partial(v\psi) \in C^\infty_{(0,1)}(\Omega)$ defined as f outside the support of ψ. Take a solution $w \in C^\infty(\Omega)$ of the equation $\bar\partial w = g$ ($\bar\partial g = 0$). Since $g = 0$ in a neighborhood of K, w is analytic there and can be approximated uniformly on K by functions in $\mathcal{A}(\Omega)$ according to the Runge approximation theorem. Adjusting w by adding a function in $\mathcal{A}(\Omega)$ we can arrange that $\sup_K |w| \leqslant \varepsilon$, and then $u = v\psi + w$ satisfies $\bar\partial u = f$ in Ω and

$$\sup_K |u| \leqslant \sup_K |v| + \sup_K |w| \leqslant C \sup_\omega |f| + \varepsilon.$$

The theorem is proved.

We can now give the main result of this section.

Theorem 2.4. *Let K_0 and K_1 be compact sets in a domain of holomorphy $\Omega \subset \mathbf{C}^n$, and denote by L the $\mathcal{A}(\Omega)$-hull of $K_0 \cup K_1$. Suppose that K is a compact set separating K_0 and K_1 in the sense that $L \backslash K = L \cap \complement K$ is a disjoint union of two sets M_0 and M_1, closed in $L \backslash K$, such that $K_j \backslash K \subset M_j$, $j = 0,1$. Then every analytic functional $\mu \in \mathcal{A}'(\Omega)$ which is weakly carried by K_0 and K_1 is also weakly carried by K.*

Proof. (i) We first choose to each open neighborhood ω of K a function $\psi \in C_0^\infty(\Omega)$, $0 \leqslant \psi \leqslant 1$, such that $\psi = j$ in $\omega_j \backslash \omega$ for some open neighborhoods ω_j of $K_j (j = 0,1)$ and ψ is constant in every component of $U \backslash \bar\omega$ for some open neighborhood U of the $\mathcal{A}(\Omega)$-hull of $\omega_0 \cup \omega_1$. In fact, putting $m_j = M_j \backslash \omega$ we get $L \backslash \omega = m_0 \cup m_1$, $m_0 \cap m_1 = \emptyset$, and m_j are closed in $L \backslash \omega$, hence compact. For some positive ε the sets $m_0^{3\varepsilon}$ and $m_1^{3\varepsilon}$ are therefore disjoint and contained in Ω. (If $B \subset \mathbf{C}^n$ and $\varepsilon \geqslant 0$, we let B^ε be the set of all points whose Euclidean distance to B is $\leqslant \varepsilon$.) We now take ψ as the convolution of the characteristic function of $m_1^{2\varepsilon}$ with a positive function in $C_0^\infty(\mathbf{C}^n)$ whose support is contained in $\{z \in \mathbf{C}^n;\ \Sigma |z_j|^2 \leqslant \varepsilon^2\}$ and whose Lebesgue integral is one. It is clear that $\psi = j$ in m_j^ε. Furthermore $m_0^\varepsilon \cup m_1^\varepsilon \cup \omega$ is a neighborhood of L and we can find two open neighborhoods U, V of L such that $\hat{V}_\Omega \subset U \subset m_0^\varepsilon \cup m_1^\varepsilon \cup \omega$. In fact, let $U \subset m_0^\varepsilon \cup m_1^\varepsilon \cup \omega$ be a relatively compact open neighborhood of L and $V = B_1 \cap U$ where $B_r = \{z \in \Omega;\ |h_k(z)| < r,\ k = 1, \dots, m\}$ and $h_k \in \mathcal{A}(\Omega)$ are chosen so that $L \subset B_1 \subset B_2 \subset U \cup \complement \hat{U}_\Omega$. With $\omega_j = (m_j^\varepsilon \cup \omega) \cap V$ we have $\omega_0 \cup \omega_1 = V$, hence the $\mathcal{A}(\Omega)$-hull of $\omega_0 \cup \omega_1$ is contained in U. Finally, since $U \setminus \bar{\omega} \subset m_0^\varepsilon \cup m_1^\varepsilon$, ψ is constant and equal to 0 or 1 in each component of $U \setminus \bar{\omega}$. Thus all claimed properties of ψ are proved.

(ii) We now prove that to each open neighborhood ω of K corresponds a constant C such that $|\mu(f)| \leqslant C \sup_\omega |f|$ for all $f \in \mathcal{A}(\Omega)$. Choose ψ according to the first part of the proof. Since U is a neighborhood of $(\omega_0 \cup \omega_1)\hat{}_\Omega$ we can by Theorem 2.3 find a constant C' such that for every $f \in \mathcal{A}(\Omega)$ and every $\varepsilon > 0$ there is a function $u \in C^\infty(\Omega)$ with $\bar{\partial} u = f \bar{\partial} \psi$ and

$$\sup_{\omega_0 \cup \omega_1} |u| \leqslant C' \sup_U |f \bar{\partial} \psi| + \varepsilon = C' \sup_\omega |f \bar{\partial} \psi| + \varepsilon. \tag{2.2}$$

The equality in (2.2) follows from the fact that $\bar{\partial}\psi = 0$ in $U \setminus \bar{\omega}$.

Now $\mu(f) = \mu(\psi f - u) + \mu((1-\psi) f + u)$ and if μ is weakly carried by K_0 and K_1 we get

$$|\mu(f)| \leqslant C_0 \sup_{\omega_0} |\psi f - u| + C_1 \sup_{\omega_1} |(1-\psi) f + u| \leqslant C_0 \sup_\omega |\psi f|$$
$$+ C_0 \sup_{\omega_0} |u| + C_1 \sup_\omega |(1-\psi) f| + C_1 \sup_{\omega_1} |u|$$

for $\psi = j$ in $\omega_j \setminus \bar{\omega}$, $j = 0,1$. From (2.2) we conclude that

$$|\mu(f)| \leqslant (C_0 + C_1)(\sup_\omega |f| + C' \sup_\omega |f \bar{\partial} \psi| + \varepsilon). \tag{2.3}$$

If f happens to be zero in ω we thus have $|\mu(f)| \leqslant (C_0 + C_1)\varepsilon$, hence $\mu(f) = 0$. Otherwise we are free to choose $\varepsilon = \sup_\omega |f|$. In both cases we obtain from (2.3)

$$|\mu(f)| \leqslant C \sup_\omega |f|$$

for some constant C since $\bar{\partial}\psi$ is bounded. This completes the proof.

Corollary 2.5. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$ and K_0, K_1 weak carriers of $\mu \in \mathcal{A}'(\Omega)$. If $K_0 \cup K_1$ is $\mathcal{A}(\Omega)$-convex, μ is weakly carried by $K_0 \cap K_1$.*

The analogous result with carriers instead of weak carriers has been proved by Martineau [4, Ch. I, Théorème 2.2].

Proof. The assumptions of Theorem 2.4 are fulfilled with $K = K_0 \cap K_1$, $M_j = K_j \setminus K (j = 0,1)$.

Remark. Theorem 2.4 can easily be deduced from the corollary. Indeed, suppose that K_j, K, and M_j satisfy the hypotheses of the theorem and put $K_j' = M_j \cup (L \cap K)$. Then K_j' is compact for it is relatively compact and since $\bar{M}_j \cap (L \setminus K) \subset M_j$ we have $\bar{K}_j' \setminus K_j' = \bar{M}_j \cap \complement (L \cap K) \cap \complement M_j \subset \complement (L \setminus K) \cap \complement (L \cap K) = \complement L$ which together with $\bar{K}_j' \subset L$ gives $\bar{K}_j' \setminus K_j' = \emptyset$. Further $K_0' \cap K_1' = (L \cap K) \cup (M_0 \cap M_1) = L \cap K$ and $K_0' \cup K_1' =$

$(L\cap K)\cup M_0\cup M_1=L$, an $\mathcal{A}(\Omega)$-convex set. Finally $K_j'\supset K_j$ so that K_j' is a weak carrier of μ. An application of Corollary 2.5 to K_0' and K_1' now proves that $K_0'\cap K_1'\subset K$ carries μ weakly.

Corollary 2.6. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$, and K_0, K_1 weak carriers of $\mu\in\mathcal{A}'(\Omega)$. Then μ is weakly carried by $K=K_0\cap(\overline{L\setminus K_0}\cup K_1)$, where $L=(K_0\cup K_1)\hat{}_\Omega$.*

Proof. We put $S=\overline{L\setminus K_0}\cup K_1$, $M_0=K_0\setminus K=K_0\setminus S=L\setminus S$, $M_1=(L\setminus K)\setminus M_0$ and shall prove that the assumptions of Theorem 2.4 are fulfilled. It is clear that $M_0\cap M_1=\emptyset$, $M_0\cup M_1=L\setminus K$. Further M_0 is closed in $L\setminus K$, for $(L\setminus K)\cap\overline{M}_0=L\cap\complement(K_0\cap\complement M_0)\cap\overline{M}_0=L\cap((\complement K_0\cap\overline{M}_0)\cup M_0)=L\cap M_0=M_0$. On the other hand, $M_0=M_0\setminus K=(L\setminus S)\setminus K=(L\setminus K)\cap\complement S$, and $\complement S$ is open so that M_0 is open in $L\setminus K$. Finally $K_0\setminus K\subset M_0$ and since $K_1\setminus K\subset\complement K_0\subset\complement M_0$ we have also that $K_1\setminus K\subset(L\setminus K)\cap\complement M_0=M_1$. An application of Theorem 2.4 now completes the proof.

For any two given $\mathcal{A}(\Omega)$-convex carriers K_0 and K_1 of a functional $\mu\in\mathcal{A}'(\Omega)$, Corollary 2.6 yields a third, $K_2=K_0\cap((L\setminus K_0)\cup K_1)\hat{}_\Omega$ contained in K_0. If K_0 is an $\mathcal{A}(\Omega)$-convex support the third carrier K_2 must be equal to K_0. This is the idea underlying the uniqueness theorems 3.2 and 3.3. More generally, Corollary 2.6 shows that the intersection of all $\mathcal{A}(\Omega)$-convex carriers of μ contains the set

$$\bigcap(K_1;\ K_1 \text{ is } \mathcal{A}(\Omega)\text{-convex and } K_0\subset(((K_0\cup K_1)\hat{}_\Omega\setminus K_0)\cup K_1)\hat{}_\Omega),$$

provided K_0 is an $\mathcal{A}(\Omega)$-convex support of μ. Similar remarks hold, of course, for convexity.

3. Unique supports

Using Corollary 2.6 we shall now prove the results concerning unique supports mentioned in the introduction. The first theorem deals with convex supports. The proof is perhaps not the shortest possible but is formulated to stress the analogy with the less perspicuous situation in Theorem 3.2.

Theorem 3.1. *Suppose K_0 is a convex compact set in $\mathbf{C}^n$ whose boundary is once continuously differentiable. Then for any domain of holomorphy Ω containing K_0 and any analytic functional $\mu\in\mathcal{A}'(\Omega)$ having K_0 as a convex support, K_0 is the only convex support of μ.*

Proof. We have to prove that every convex carrier K_1 of μ contains K_0. For this it is sufficient to find, given any convex compact set K_1 such that $K_0\setminus K_1\neq\emptyset$, a pair of convex functions F, G satisfying

$$\sup_{K_1} F\leqslant 0,\quad \sup_{K_0} F>0; \tag{3.1}$$

$$\sup_{K_0\cup K_1} G\leqslant 0,\quad \text{hence}\quad \sup_L G\leqslant 0\quad \text{where } L=\mathrm{ch}(K_0\cup K_1);\ \text{and} \tag{3.2}$$

$$z\notin K_0,\ G(z)\leqslant 0\quad \text{implies}\quad F(z)\leqslant 0. \tag{3.3}$$

In fact, suppose that K_1 is a convex carrier of μ and that $K_0\setminus K_1\neq\emptyset$. Then if $z\in L\setminus K_0$ we have $F(z)\leqslant 0$ by (3.2) and (3.3) and hence by (3.1) $\sup_{(L/K_0)\cup K_1}F\leqslant 0$ which implies $\sup_K F\leqslant 0$ where $K=K_0\cap\mathrm{ch}((L\setminus K_0)\cup K_1)$. Thus K is a convex proper subset of K_0 because $F>0$ somewhere in K_0. But Corollary 2.6 shows that K carries μ so that, contrary to hypothesis, K_0 cannot be a convex support of μ. Hence there can be no convex carrier K_1 satisfying $K_0 \setminus K_1 \neq \emptyset$ which means that K_0 is the only convex support of μ.

To prove the existence of convex functions F, G satisfying (3.1)–(3.3) we note that the assumptions on K_0 implies the existence of a continuous function N, the unit outer normal, defined on the boundary ∂K_0 of K_0, with values in $\mathbf{C}^n$ and such that $|N(z)|=1$, $\operatorname{Re}\langle z, N(z)\rangle = \sup_{w\in K_0} \operatorname{Re}\langle w, N(z)\rangle$. (We write $\langle z,\zeta\rangle = \sum_1^n z_j\bar{\zeta}_j, |\zeta| = \langle \zeta,\zeta\rangle^{\frac{1}{2}}$.) If K_1 is a convex compact set such that there exists a point $b\in K_0\setminus K_1$ we choose ζ, $|\zeta|=1$, such that $\operatorname{Re}\langle b,\zeta\rangle > \sup_{w\in K_1}\operatorname{Re}\langle w,\zeta\rangle$. Let $a\in K_0$ be a point such that $\operatorname{Re}\langle a,\zeta\rangle = \sup_{w\in K_0}\operatorname{Re}\langle w,\zeta\rangle$. Obviously $\zeta = N(a)$. We claim that for all $\varepsilon>0$ there is a $\delta>0$ such that for all $z\in\partial K_0$

$$\operatorname{Re}\langle z,\zeta\rangle \geqslant \operatorname{Re}\langle a,\zeta\rangle - \delta \Rightarrow |N(z)-\zeta| \leqslant \varepsilon. \tag{3.4}$$

Indeed, if the contrary were true we could find a sequence $(z^{(j)})$ of points on ∂K_0 and a positive number ε such that $\operatorname{Re}\langle z^{(j)},\zeta\rangle \geqslant \operatorname{Re}\langle a,\zeta\rangle - 1/j$ and $|N(z^{(j)})-\zeta| > \varepsilon$. A subsequence of $(z^{(j)})$ must then converge to some point $z\in\partial K_0$ and since $\operatorname{Re}\langle z,\zeta\rangle = \operatorname{Re}\langle a,\zeta\rangle$ we must have $N(z)=N(a)=\zeta$. Thus $|N(z^{(j)})-N(z)|>\varepsilon$ which contradicts the continuity of N.

Now choose $\varepsilon>0$ so small that $\operatorname{Re}\langle a,\theta\rangle \geqslant \sup_{w\in K_1}\operatorname{Re}\langle w,\theta\rangle$ when $|\theta-\zeta|\leqslant\varepsilon$ and then take $\delta>0$ such that (3.4) is valid for all $z\in\partial K_0$ and also $\sup_{w\in K_1}\operatorname{Re}\langle w,\zeta\rangle \leqslant \operatorname{Re}\langle a,\zeta\rangle - \delta$. Define $F(z) = \operatorname{Re}\langle z-a,\zeta\rangle + \delta$ and $G(z) = \sup(\operatorname{Re}\langle z-w, N(w)\rangle;\ w\in\partial K_0$ and $F(w)\geqslant 0)$. Then (3.1) and (3.2) are obvious. To prove (3.3), suppose that $z\notin K_0$ and $F(z)>0$. Let z' be the point closest to z in the compact set $\{w\in K_0;\ F(w)\geqslant 0\}$. Then the open segment between z' and a point w satisfying $F(w)\geqslant 0$ and $|w-z|^2 < |w-z'|^2 + |z-z'|^2$ is free from points in K_0 which proves that $\operatorname{Re}\langle w, N(z')\rangle \geqslant \operatorname{Re}\langle z', N(z')\rangle$ for all such w, and hence (since w can be arbitrarily chosen in a neighborhood of z) that $\operatorname{Re}\langle z, N(z')\rangle > \operatorname{Re}\langle z', N(z')\rangle$. We obtain $G(z) \geqslant \operatorname{Re}\langle z-z', N(z')\rangle > 0$ which proves (3.3) and so completes the proof of the theorem.

Using similar geometric ideas we prove an analogue of Theorem 3.1 for $\mathcal{A}(\Omega)$-convex sets.

Theorem 3.2. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$ and $\mu\in\mathcal{A}'(\Omega)$. If K_0 is an $\mathcal{A}(\Omega)$-convex support of μ whose boundary is twice continuously differentiable, then K_0 is the unique $\mathcal{A}(\Omega)$-convex support of μ.*

Remark. The smoothness requirement on K_0 means that there is a twice continuously differentiable real-valued function f in Ω such that grad $f(z)\neq 0$ when $f(z)=0$ and $K_0 = \{z\in\Omega;\ f(z)\leqslant 0\}$. In particular, K_0 is the closure of its interior.

Proof of Theorem 3.2. In complete analogy with the proof of the preceding theorem it suffices to construct, for every given $\mathcal{A}(\Omega)$-convex compact set K_1 with $K_0\setminus K_1\neq\emptyset$, two functions F, $G\in\mathcal{P}(\Omega)\cap C^0(\Omega)$ such that

$$\sup_{K_1} F \leqslant 0, \quad \sup_{K_0} F > 0; \tag{3.5}$$

$$\sup_{K_0\cup K_1} G \leqslant 0, \quad \text{hence} \quad \sup_L G \leqslant 0 \quad \text{where} \quad L = (K_0\cup K_1)\hat{_\Omega}; \quad \text{and} \tag{3.6}$$

$$z\notin K_0, \quad G(z)\leqslant 0 \quad \text{implies} \quad F(z)\leqslant 0. \tag{3.7}$$

(Here it is essential that the $\mathcal{A}(\Omega)$-hull of the relatively compact set $(L\backslash K_0)\cup K_1$ can be defined by (1.2).)

Now if $K_0\backslash K_1\neq\emptyset$ and K_1 is $\mathcal{A}(\Omega)$-convex, there is a function $G_0\in\mathcal{P}(\Omega)\cap C^2(\Omega)$ such that $\sup_{K_1}G_0<0$ and $\sup_{K_0}G_0\geqslant 0$ (see Theorem 2.6.11 in Hörmander [2]). The function $G_3(z)=G_0(z)+3\delta|z|^2$ satisfies the same conditions when $\delta>0$ is sufficiently small. Define $H_3=G_3-\sup_{K_0}G_3$, and choose $a\in\partial K_0$ such that $H_3(a)=0$ (by the maximum principle, the supremum is attained at the boundary). We now take $b\in K_0$ on the interior normal of ∂K_0 at a so that $z\neq a$, $|z-b|\leqslant|a-b|$ implies $z\in K_0{}^\circ$, and define

$$H_j(z)=H_3(z)-(3-j)\varepsilon(|z-b|^2-|a-b|^2),\quad j=0,1,2,$$

where $0<\varepsilon<\delta$ and ε is so small that $\sup_{K_1}H_j<0$. Then clearly $H_j\in\mathcal{P}(\Omega)\cap C^2(\Omega)$ and we claim that $H_j(z)\leqslant 0$ when $z\in K_0$ and that equality holds only at a, $j=0,1,2$. Indeed, $H_j\leqslant H_3\leqslant 0$ on the boundary of K_0, hence $H_j\leqslant 0$ in all of K_0. If $H_j(z)=0$, $z\neq a$, we have $z\in K_0{}^\circ$ by assumption and so by the maximum principle that $H_j=0$ in an open set which is impossible since H_j is strictly plurisubharmonic.

Our next step is to take a function $f\in C^2(\Omega)$ such that $K_0=\{z\in\Omega;\ f(z)\leqslant 0\}$ and $f\geqslant H_2$ in Ω. (The construction of such a function is trivial locally and follows globally by means of a partition of unity.) We claim that $f-H_1$ is convex in some open neighborhood ω' of a. In fact, we have $f-H_1\geqslant H_2-H_1$ with equality at a, and since the matrix $(\partial^2(H_2-H_1)/\partial x_j\partial x_k)_{j,k=1}^{2n}$ is positive definite at a the same is true of $(\partial^2(f-H_1)/\partial x_j\partial x_k)_{j,k=1}^{2n}$, hence the latter matrix is positive definite in some convex open neighborhood ω' of a because its coefficients are continuous ($x_1,\dots,x_{2n}$ are real coordinates in $\mathbf{C}^n$). This means that in ω' we have $f-H_1=\sup(A;\ A\leqslant f-H_1$ in $\omega')$ where the supremum is taken over all real affine functions $A(z)=\operatorname{Re}\langle z,\theta\rangle+C$. We define a norm for such functions e.g. by

$$\|A\|=\sup_{|z|\leqslant 1}|A(z)|$$

and set for arbitrary $\eta>0$

$$G_\eta=H_1+A_0+\sup_A(A;\ A_0+A\leqslant f-H_1 \text{ in } \omega' \quad\text{and}\quad \|A\|<\eta),$$

where A_0 is the affine function defined by

$$H_2(z)-H_1(z)=H_1(z)-H_0(z)=A_0(z)+o(z-a),\ z\to a.$$

By well-known properties of continuous and plurisubharmonic functions it follows that $G_\eta\in\mathcal{P}(\Omega)\cap C^0(\Omega)$. Since A_0 is also the best affine approximation of $f-H_1$ at a we have $G_\eta=f$ in some open neighborhood ω_η of a. Clearly $G_\eta\leqslant f$ in ω'. We also note that

$$H_1+A_0\leqslant G_\eta \text{ in } \Omega; \tag{3.8}$$

$$H_0+A_0\leqslant H_1 \text{ in } \Omega;\quad\text{and} \tag{3.9}$$

$$H_1+A_0\leqslant H_2 \text{ in } \Omega.$$

Now, since $G_\eta\searrow H_1+A_0$ when $\eta\searrow 0$ and $H_1+A_0\leqslant H_2<0$ in $K_0\backslash\omega'$, it follows from Dini's theorem that $G_\eta<0$ in $K_0\backslash\omega'$ if η is sufficiently small; hence $G_\eta\leqslant 0$ in K_0 because $G_\eta\leqslant f\leqslant 0$ in $K_0\cap\omega'$. In the same way we infer that $G_\eta<0$ in K_1 when η is small enough. This proves (3.6) if $G=G_\eta$ for some conveniently chosen $\eta>0$. Finally, we obtain from (3.8) if ω denotes the neighborhood of a where $G=f$

$$\begin{aligned} q &= \sup(H_0(z)+A_0(z);\ z\notin K_0 \text{ and } G(z)\leqslant 0)\\ &\leqslant \sup(H_0(z)+A_0(z);\ z\notin K_0\cup\omega \text{ and } H_1(z)+A_0(z)\leqslant 0)\\ &\leqslant \sup(-\varepsilon(|z-b|^2-|a-b|^2);\ z\notin K_0\cup\omega)<0 \end{aligned}$$

(the last inequality follows from the way b was chosen). This, together with (3.9), shows that (3.5) and (3.7) are satisfied with $F(z)=H_0(z)+A_0(z)+r$ if

$$r=\min(-q, -\sup_{K_1}(H_0+A_0))\geqslant \min(-q, -\sup_{K_1}H_1)>0.$$

The proof is complete.

When $n=1$, the smoothness assumptions of Theorems 3.1 and 3.2 are, of course, very unnatural. Indeed, convex supports are then always unique as remarked in the introduction, and $\mathcal{A}(\Omega)$-convexity is a topological notion. The following theorem generalizes Theorem 3.2 when $n=1$, and replaces the smoothness condition on K_0 there by a topological one stating intuitively that K_0 contains no curves.

We recall that if K is a compact part of $\Omega\subset\mathbf{C}^1$, $\hat{K}_\Omega$, defined by (1.1) or (1.2), is the union of K and those connected components of $\Omega\setminus K$ which are relatively compact in Ω; for a proof see [2, Theorem 1.3.3]. In particular, a connected open set which is disjoint from K and contains points outside $\hat{K}_\Omega$ is also disjoint from $\hat{K}_\Omega$.

Theorem 3.3. *Let Ω be an open set in $\mathbf{C}^1$ and $K_0\subset\Omega$ an $\mathcal{A}(\Omega)$-convex support of $\mu\in\mathcal{A}'(\Omega)$. Suppose that for any connected open set ω intersecting the boundary ∂K_0 of K_0, the interior of the union of K_0 and an arbitrary connected component of $\omega\setminus K_0$ intersects ∂K_0. Then μ has a unique $\mathcal{A}(\Omega)$-convex support.*

Remark. The following property is easier to formulate than and implies the hypothesis on K_0 in the theorem: For any $z\in\partial K_0$ there exist arbitrarily small open neighborhoods $V\ni z$ such that $V\setminus K_0$ is connected. The compact set $\{z\in\mathbf{C}^1;\ |z^2-1|\leqslant 1\}$ (whose boundary is a lemniscate) shows that the latter condition is strictly stronger. Both conditions allow K_0 to contain isolated points.

Proof of Theorem 3.3. The theorem will follow if we prove that $K_0\setminus((L\setminus K_0)\cup K_1)\hat{}_\Omega\neq\emptyset$ if K_1 is an arbitrary $\mathcal{A}(\Omega)$-convex compact set such that $K_0\setminus K_1\neq\emptyset$ and $L=(K_0\cup K_1)\hat{}_\Omega$. In fact, then every $\mathcal{A}(\Omega)$-convex carrier K_1 of μ must contain K_0. For otherwise $K_0\cap((L\setminus K_0)\cup K_1)\hat{}_\Omega$ is by Corollary 2.6 an $\mathcal{A}(\Omega)$-convex carrier which is a proper subset of K_0 contrary to the assumption that K_0 is an $\mathcal{A}(\Omega)$-convex support.

Let ω_1 be a connected component of $\Omega\setminus K_1$ intersecting K_0. Since ω_1 is not relatively compact in Ω it is not contained in the compact set L. Let ω_0 be a component of $\omega_1\setminus K_0$ not contained in L. From the remark preceding the theorem it then follows that ω_0 does not intersect L. Also, since ω_1 intersects ∂K_0, we can find an open connected set ω meeting both ∂K_0 and ω_0 such that $\bar{\omega}\subset\omega_1\subset\Omega\setminus K_1$. Indeed, ω can be defined e.g. as a sufficiently small connected open neighborhood of any curve in ω_1 which joins a point in ∂K_0 to a point in ω_0. Some component ω_2 of $\omega\setminus K_0$ intersects ω_0, and therefore $\omega_2\subset\omega_0\subset\Omega\setminus L$. According to hypothesis there exists a point $z\in\partial K_0$ such that $K_0\cup\omega_2$ contains a connected open neighborhood ω_3 of z. Since $z\in\bar{\omega}_2\subset\omega_1$, we may assume that $\omega_3\subset\omega_1$. Now ω_3 is connected, contains points outside L, and does not meet the closure of $(L\setminus K_0)\cup K_1$ so it is disjoint from $((L\setminus K_0)\cup K_1)\hat{}_\Omega$, in particular $z\notin((L\setminus K_0)\cup K_1)\hat{}_\Omega$. This completes the proof.

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