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On the intersections between the trajectories of a normal stationary stochastic process and a high level

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1. Introduction

Let $\xi(t)$ be a real-valued, normal and stationary stochastic process with a continuous time parameter t, varying over $(-\infty, \infty)$. Suppose that $E\xi(t)=0$ for all t, and that the covariance function

$$r(t-u)=E\xi(t)\,\xi(u)$$

satisfies the following two conditions:

$$r(t)=1-\frac{1}{2!}\lambda_2 t^2+\frac{1}{4!}\lambda_4 t^4+o(t^4) \tag{1}$$

as $t\to 0$, and

$$r(t)=O(|t|^{-\alpha}) \tag{2}$$

for some $\alpha>0$, as $t\to\pm\infty$. We may, of course, assume $\alpha<1$.

It follows[1] from (1) that there exists an equivalent version of $\xi(t)$ having, with probability one, a continuous sample function derivative $\xi'(t)$, and it will be supposed that $\xi(t)$ has, if required, been replaced by this equivalent version.

Let $u>0$ be given, and consider the intersections between the trajectories $\eta=\xi(t)$ of the ξ process and the horizontal line, or "level", $\eta=u$ during some finite time interval, say $0<t<T$. It follows from a theorem due to Bulinskaja [2] that, with probability one, there are only a finite number of such intersections, and also that, with probability one, there is no point of tangency between the trajectory $\eta=\xi(t)$ and the level $\eta=u$ during the interval $(0,T)$. Thus, with probability one, every point with $\xi(t)=u$ can be classified as an "upcrossing" or a "downcrossing" of the level u, according as $\xi'(t)$ is positive or negative.

The present paper will be concerned with the upcrossings, and their asymptotic distribution in time, as the level u becomes large. It will be obvious that the case of a large negative u, as well as the corresponding problem for the downcrossings, can be treated in the same way.

The upcrossings may be regarded as a *stationary stream of random events* (cf.,

[1] With respect to the general theory of the normal stationary process and its sample functions we refer to the forthcoming book [4] by Cramér and Leadbetter.

e.g. Khintchine's book [5]), and it is well known that the simplest case of such a stream occurs when the successive events form a Poisson process. However, a necessary condition for this case is (Khintchine, l.c., pp. 11–12) that the numbers of events occurring in any two disjoint time intervals should be independent random variables, and it is readily seen that this condition cannot be expected to be satisfied by the stream of upcrossings. On the other hand, it seems intuitively plausible that the independence condition should be at least approximately satisfied when the level u becomes very large, provided that values of $\xi(t)$ lying far apart on the time scale can be supposed to be only weakly dependent.

Accordingly it may be supposed that, subject to appropriate conditions on $\xi(t)$, the stream of upcrossings will tend to form a Poisson process as the level u tends to infinity.

That this is actually so was first proved by Volkonskij and Rozanov in their remarkable joint paper [8]. They assumed, in addition to the condition (1) above, that $\xi(t)$ satisfies the so-called *condition of strong mixing*. This is a fairly restrictive condition, as can be seen e.g. from the analysis of the strong mixing concept given by Kolmogorov and Rozanov [6]. Moreover, in an actual case it will not always be easy to decide whether the condition is satisfied or not. On the other hand, various interesting properties of the $\xi(t)$ trajectories follow as corollaries from the asymptotic Poisson character of the stream of upcrossings (cf. Cramér [3], Cramér and Leadbetter [4]), so that it seems highly desirable to prove the latter property under simpler and less restrictive conditions.

It will be shown in the present paper that it is possible to replace the condition of strong mixing by the condition (2) above. This is considerably more general, and also simpler to deal with in most applications.

I am indebted to Dr. Yu. K. Belajev for stimulating conversations about the problem treated in this paper.

2. The main theorem

The number of upcrossings of $\xi(t)$ with the level u during the time interval (s,t) will be denoted by $N(s,t)$. This is a random variable which, by the above remarks, is finite with probability one. When $s=0$, we write simply $N(t)$ instead of $N(0,t)$. From the stationarity and the condition (1) it follows (Bulinskaja [2], Cramér and Leadbetter [4], Ch. 10) that the mean value of $N(s,t)$ is, for $s<t$,

$$EN(s,t)=EN(t-s)=\mu(t-s), \tag{3}$$

where

$$\mu=EN(1)=\frac{\sqrt{\lambda_2}}{2\pi}e^{-u^2/2}. \tag{4}$$

The quantity μ will play an important part in the sequel. We note that μ is a function of the level u, and that μ tends to zero as u tends to infinity. From (3) we obtain for any $\tau>0$

$$EN(\tau/\mu)=\tau.$$

For the study of the stream of upcrossings, it will then seem natural to choose $1/\mu$ as a scaling unit of time, thus replacing t by τ/μ. We might expect, e.g., that the probability distribution of $N(\tau/\mu)$ will tend to some limiting form when μ tends to zero while τ remains fixed. This is in fact the case, as shown by the following theorem, first proved by Volkonskij and Rozanov under more restrictive conditions, as mentioned above.

Theorem. *Suppose that the normal and stationary process $\xi(t)$ satisfies the conditions (1) and (2). Let $(a_1,b_1),\ldots,(a_j,b_j)$ be disjoint time intervals depending on u in such a way that, for $i=1,\ldots,j$,*

$$b_i - a_i = \tau_i/\mu,$$

the integer j and the positive numbers $\tau_1,\ldots\tau_j$ being independent of u. Let $k_1,\ldots,k_j$ be non--negative integers independent of u. Then

$$\lim_{u\to\infty} P\{N(a_i,b_i)=k_i \text{ for } i=1,\ldots,j\} = \prod_{i=1}^{j}\frac{\tau_i^{k_i}}{k_i!}e^{-\tau_i}.$$

Thus, when time is measured in units of $1/\mu$, the stream of upcrossings will asymptotically behave as a Poisson process as the level u tends to infinity.

We shall first give the proof for the case $j=1$, when there is one single interval (a,b) of length $b-a=\tau/\mu$. Owing to the stationarity it is sufficient to consider the interval $(0,\tau/\mu)$. Writing

$$T=\tau/\mu, \quad P_k = P\{N(T)=k\}, \tag{5}$$

we then have to prove the relation

$$\lim_{u\to\infty} P_k = \frac{\tau^k}{k!}e^{-\tau} \tag{6}$$

for any given $\tau>0$ and non-negative integer k, both independent of u. Once this has been achieved, the proof of the general case will follow in a comparatively simple way.

The proof of (6) is rather long, and will be broken up in a series of lemmas. In the following section we shall introduce some notations that will be used in the course of the proof. The lemmas will be given in sections 4 and 5, while section 6 contains the proof of the case $j=1$ of the theorem, and section 7 the proof of the general case.

3. Notations

The level u will be regarded as a variable tending to infinity, and we must now introduce various functions of u. It will be practical to define them as functions of μ, where μ is the function of u given by (4). Writing as usual $[x]$ for the greatest integer $\leqslant x$, we define

$$\left.\begin{aligned} &m_1=[\mu^{-1}], \quad m_2=[\mu^{\beta-1}],\\ &M=[T/\mu^{\beta}]. \quad n=[M/(m_1+m_2)]+1. \end{aligned}\right\} \tag{7}$$

Here β is any number satisfying the relation

$$0<(k+4)\beta<\alpha<1, \tag{8}$$

where α is the constant occurring in the condition (2), while k is the integer occurring in (6). We further write

$$q=T/M, \quad t_1=m_1 q, \quad t_2=m_2 q, \tag{9}$$

and divide the interval $(0,T)$ on the time axis into subintervals, alternatively of length t_1 and t_2, starting from the origin. We shall refer to these subintervals as t_1- and t_2-intervals respectively, the former being regarded as closed and the latter as open. Each t_i-interval $(i=1,2)$ consists of m_i subintervals of length q. The whole interval $(0,T)$, which consists of M intervals of length q, is covered by n pairs of t_1- and t_2-intervals, the nth pair being possibly incomplete. Any two distinct t_1-intervals are separated by an interval of length at least equal to t_2. An important use will be made of this remark in the proof of Lemma 5 below.

The quantities defined by (7) and (9) are all functions of u. It will be practical to express their order of magnitude for large u in terms of μ. The following relations are easily obtained from (5), (7) and (9):

$$\left.\begin{array}{ll} q\sim\mu^{\beta}, & n\sim\tau\mu^{-\beta}, \\ t_1\sim\mu^{\beta-1}, & t_2\sim\mu^{2\beta-1}. \end{array}\right\} \tag{10}$$

We now define a stochastic process $\xi_q(t)$ by taking

$$\xi_q(\nu q)=\xi(\nu q)$$

for all integers ν, and determining ξ_q by linear interpolation in the interval between two consequtive νq. To any sample function of the $\xi(t)$ process will then correspond a sample function of $\xi_q(t)$, which is graphically represented by the broken line joining the points $[\nu q, \xi(\nu q)]$. For the number of upcrossings of this broken line with the u level we use the notations $N_q(s,t)$ and $N_q(t)$, corresponding to $N(s,t)$ and $N(t)$. The probability corresponding to P_k as defined by (5) is

$$P_k^{(q)}=P\{N_q(T)=k\}. \tag{11}$$

4. Lemmas 1–3

Throughout the rest of the paper we assume that the $\xi(t)$ process satisfies the conditions of the above theorem.

Lemma 1. *If T and q are given by* (5) *and* (9), *τ and k being fixed as before, we have*

$$\lim_{u\to\infty}(P_k-P_k^{(q)})=0.$$

Evidently $N_q(T)\leqslant N(T)$. We shall prove that the non-negative and integer-valued random variable $N(T)-N_q(T)$ converges in first order mean to zero, as $u \to \infty$. It then follows that the probability that $N(T) - N_q(T)$ takes any value different from zero will tend to zero, and so the lemma will be proved.

By (3) and (5), the mean value of the number $N(T)$ of upcrossings of $\xi(t)$ in $(0, T)$ is for every u

$$EN(T) = T\mu = \tau.$$

It will now be proved that the mean value $EN_q(T)$ tends to the limit τ as $u \to \infty$, so that we have

$$\lim_{u \to \infty} E\{N(T) - N_q(T)\} = 0.$$

Since $N(T) - N_q(T)$ is non-negative, this implies convergence in first order mean to zero, so that by the above remark the lemma will be proved.

Consider first the number $N_q(q)$ of upcrossings of $\xi_q(t)$ in the interval $(0, q)$. This number is one, if $\xi(0) < u < \xi(q)$, and otherwise zero, so that

$$EN_q(q) = P\{\xi(0) < u < \xi(q)\}.$$

Now $\xi(0)$ and $\xi(q)$ have a joint normal density function, with unit variances and correlation coefficient $r = r(q)$. By (1) and (10) we have

$$r(q) = 1 - \tfrac{1}{2}\lambda_2 q^2 + O(q^4). \tag{12}$$

For the probability that $\xi(0) < u$ and $\xi(q) > u$ we obtain by a standard transformation

$$EN_q(q) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2} \Phi\left(\frac{u - rx}{\sqrt{1 - r^2}}\right) dx,$$

where as usual

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt.$$

By some straightforward evaluation of the integral from $u + (1 - r)^{\frac{1}{4}}$ to infinity we obtain, using (12), and denoting by K an unspecified positive constant,

$$EN_q(q) = \frac{1}{\sqrt{2\pi}} \int_u^{u+(1-r)^{\frac{1}{4}}} e^{-x^2/2} \Phi\left(\frac{u - rx}{\sqrt{1 - r^2}}\right) dx + O[\exp(-e^{Ku^2})].$$

For the first term in the second member we obtain, using again (12), the expression

$$\frac{1}{\sqrt{2\pi}} e^{-u^2/2} (1 + O(q^{\frac{1}{2}} u)) \int_u^{u+(1-r)^{\frac{1}{4}}} \Phi\left(\frac{u - rx}{\sqrt{1 - r^2}}\right) dx$$

$$= \frac{\sqrt{1 - r^2}}{r\sqrt{2\pi}} e^{-u^2/2} (1 + O(q^{\frac{1}{2}} u)) \left(\int_{-\infty}^0 \Phi(y)\, dy + O(qu)\right) = \frac{\sqrt{\lambda_2}}{2\pi} q e^{-u^2/2} (1 + O(q^{\frac{1}{2}} u)),$$

so that

$$EN_q(q) = q\mu + o(q\mu).$$

If ν denotes an integer, which may tend to infinity with u, it then follows from the stationarity that we have

$$EN_q(\nu q) = \nu q\mu + o(\nu q\mu). \tag{13}$$

In particular, taking $\nu = M$ we obtain from (9) and (5)

$$EN_q(T) = T\mu + o(T\mu) \to \tau.$$

According to the above remarks, this proves the lemma.

We now consider the number $N_q(t_1)$ of ξ_q upcrossings in an interval of length $t_1 = m_1 q$, observing that by (10) we have $t_1 \sim \mu^{\beta-1}$.

Lemma 2. *We have*

$$\lim_{u\to\infty} \frac{E\{N_q(t_1)[N_q(t_1)-1]\}}{EN_q(t_1)} = 0.$$

By (3) we have $EN(t_1) = t_1\mu$, while (13) gives for $\nu = m_1$

$$EN_q(t_1) = t_1\mu + o(t_1\mu) \sim EN(t_1).$$

Further, since $N_q(t_1) \leqslant N(t_1)$,

$$E\{N_q(t)[N_q(t_1)-1]\} \leqslant E\{N(t_1)[N(t_1)-1\}.$$

The truth of the lemma will then follow from the corresponding relation with N_q replaced by N. Now this latter relation is identical with the relation proved by Volkonskij and Rozanov [8] in their Lemma 3.4. It is proved by them without any mixing hypothesis, assuming only that $\xi(t)$ is regular (or purely non-deterministic) and that $r(t)$ has a fourth order derivative at $t=0$. Their proof is valid without any modification whatever, if these conditions are replaced by our conditions (1) and (2). Thus we may refer to their paper for the proof of this lemma. We note that the proof is based on the important work of S. O. Rice [7].

Lemma 3. *As $u \to \infty$, we have*

$$P\{N_q(t_1)=0\} = 1 - q + o(q),$$
$$P\{N_q(t_1)=1\} = q + o(q),$$
$$P\{N_q(t_1)>1\} = o(q).$$

For any random variable ν taking only non-negative integral values we have, writing $\pi_i = P\{\nu = i\}$ and assuming $E\nu^2 < \infty$,

$$E\nu = \pi_1 + 2\pi_2 + 3\pi_3 + \ldots,$$
$$E\nu(\nu-1) = 2\pi_2 + 6\pi_3 + \ldots,$$

and consequently

$$E\nu - E\nu(\nu-1) \leqslant \pi_1 \leqslant 1 - \pi_0 \leqslant E\nu. \tag{14}$$

Taking $\nu = N_q(t_1)$, and observing that by (10) we have $EN_q(t_1) \sim t_1\mu \sim q$, the truth of the lemma follows directly from Lemma 2.

5. Lemmas 4–5

For each $r=1,2,\dots,n$, we now define the following events, i.e. the sets of all $\xi(t)$ sample functions satisfying the conditions written between the brackets:

$$c_r=\{\text{exactly one } \xi_q \text{ upcrossing in the } r\text{th } t_1\text{-interval}\},$$
$$d_r=\{\text{at least one } \xi_q \text{ upcrossing in the } r\text{th } t_1\text{-interval}\},$$
$$e_r=\{\xi(\nu q)>u \text{ for at least one } \nu q \text{ in the } r\text{th } t_1\text{-interval}\}.$$

Further, let C_k denote the event that c_r occurs in exactly k of the t_1-intervals in $(0,T)$, while the complementary event c_r^* occurs in the $n-k$ others. D_k and E_k are defined in the corresponding way, using respectively d_r and e_r instead of c_r.

Lemma 4. *For the probability* $P_k^{(q)}$ *defined by* (11) *we have*

$$\lim_{u\to\infty}[P_k^{(q)}-P\{E_k\}]=0.$$

We shall prove that each of the differences $P_k^{(q)}-P\{C_k\}$, $P\{C_k\}-P\{D_k\}$ and $P\{D_k\}-P\{E_k\}$ tends to zero as $u\to\infty$.

By (13) and (14) the probability of at least one ξ_q upcrossing in an interval of length t_2 is at most $EN_q(t_2)=t_2\mu+o(t_2\mu)$. Thus the probability of at least one ξ_q upcrossing in at least one of the n t_2-intervals in $(0,T)$ is by (10)

$$O(nt_2\mu)=O(\mu^\beta),$$

and thus tends to zero as $u\to\infty$. It follows that we have

$$P_k^{(q)}-P\{\text{total number of } \xi_q \text{ upcrossings in all } n\ t_1\text{-intervals}=k\}\to 0. \quad (15)$$

On the other hand, by the stationarity of $\xi(t)$, Lemma 3 remains true if $N_q(t_1)$ is replaced by the number of ξ_q upcrossings in any particular t_1-interval. Since the interval $(0,T)$ contains n of these intervals, it follows from (10) that the probability of more than one ξ_q upcrossing in at least one of the t_1-intervals is $o(nq)=o(1)$, and thus tends to zero as $u\to\infty$.

From (15) and the last remark, it now readily follows that the differences $P_k^{(q)}-P\{C_k\}$ and $P\{C_k\}-P\{D_k\}$ both tend to zero as $u\to\infty$. It thus only remains to show that this is true also for $P\{D_k\}-P\{E_k\}$.

By the definitions of the events D_k and E_k we have

$$\left.\begin{aligned}P\{D_k\}&=\sum P\{d_{r_1}\dots d_{r_k}d_{s_1}^*\dots d_{s_{n-k}}^*\},\\ P\{E_k\}&=\sum P\{e_{r_1}\dots e_{r_k}e_{s_1}^*\dots e_{s_{n-k}}^*\},\end{aligned}\right\} \quad (16)$$

the summations being extended over all $\binom{n}{k}$ groups of k different subscripts $r_1,\dots,r_k$ selected among the numbers $1,\dots,n$, while in each case $s_1,\dots,s_{n-k}$ are the remaining $n-k$ subscripts.

Let $\nu_r q$ denote the left endpoint of the rth t_1-interval, and denote by g_r the event

$$g_r=\{\xi(\nu_r q)\geqslant u\}$$

of probability

$$P\{g_r\}=O\left(\frac{1}{u}e^{-u^2/2}\right).$$

Then for every $r=1,\ldots,n$

$$d_r \subset e_r \quad \text{and} \quad e_r - d_r \subset g_r,$$

so that
$$P\{e_r - d_r\}=O\left(\frac{1}{u}e^{-u^2/2}\right). \tag{17}$$

Similarly
$$e_r^* \subset d_r^*, \quad \text{and}$$
$$d_r^* - e_r^* = e_r - d_r \subset g_r,$$
$$P\{d_r^* - e_r^*\}=O\left(\frac{1}{u}e^{-u^2/2}\right).$$

By a simple recursive argument (16) then yields, using (8) and (10),

$$P\{D_k\}-P\{E_k\}=O\left(\frac{n^{k+1}}{u}e^{-u^2/2}\right)=O\left[\exp\left(-\frac{1-(k+1)\beta}{2}u^2\right)\right]\to 0,$$

which proves the lemma.

By definition, E_k is composed of certain events e_r and e_s^*. Each of these is associated with one particular t_1-interval, and it has been observed above that any two t_1-intervals are separated by an interval which is of length $\geq t_2$, and thus tends to infinity with u. By means of the condition (2) it will now be shown that the component events of E_k are asymptotically independent, as $u\to\infty$. Moreover, owing to stationarity, the probability

$$p=P\{e_r\} \tag{18}$$

is independent of r, so that by (16) the asymptotic independence will be expressed by the following lemma.

Lemma 5. *The probability p being defined by* (18) *we have, as $u\to\infty$,*

$$P\{E_k\}-\binom{n}{k}p^k(1-p)^{n-k}\to 0.$$

In order to prove this lemma, we consider the points νq on the time axis for all integers ν such that νq belongs to one of the t_1-intervals in $(0,T)$. Each t_1-interval, which we regard as closed, contains m_1+1 points νq, and there are $n-1$ complete and one possibly incomplete such interval in $(0,T)$. If L is the total number of points νq in all t_1-intervals, we thus have

$$(n-1)(m_1+1)<L\leq n(m_1+1).$$

Let $\eta_1,\ldots,\eta_L$ be the random variables $\xi(\nu q)$ corresponding to all these L points

νq, ordered according to increasing ν. The η_i sequence will consist of n groups, each corresponding to one particular t_1-interval.

Further, let $f_1(y_1, \ldots, y_L)$ be the L-dimensional normal probability density of $\eta_1, \ldots, \eta_L$, and let Λ_1 be the corresponding covariance matrix. (Our reasons for using the subscript 1 here and in the sequel will presently appear.) From (16) we obtain

$$P\{E_k\} = \int_{E_k} f_1 dy = \sum \int_{e_{r_1} \ldots e^*_{s_1} \ldots} f_1 dy, \tag{19}$$

where the abbreviated notation should be easily understood, the summation being extended as explained after (16).

Let us now consider one particular term of the sum in the last member of (19), say the term where the group of subscripts $r_1, \ldots, r_k$ coincides with the integers $1, \ldots, k$. It will be readily seen that any other term can be treated in the same way as we propose to do with this one. This term is

$$F(1) = \int_G f_1 dy,$$

where G denotes the set

$$G = e_1 \ldots e_k e^*_{k+1} \ldots e^*_n.$$

$F(1)$ may be regarded as a function of the covariances which are elements of the matrix Λ_1. Let us consider in particular the dependence of $F(1)$ on those covariances $\varrho_{ij} = E\eta_i\eta_j$ which correspond to variables η_i and η_j *belonging to different t_1-intervals*. If all covariances ϱ_{ij} having this character are replaced by $\lambda_{ij} = h\varrho_{ij}$, with $0 \leqslant h \leqslant 1$, while all other elements of Λ_1 remain unchanged, the resulting matrix will be

$$\Lambda_h = h\Lambda_1 + (1-h)\Lambda_0, \tag{20}$$

while the density function f_1 will be replaced by a certain function f_h. Evidently f_0, corresponding to the covariance matrix Λ_0, will be the normal density function that would apply if the groups of variables η_i belonging to different t_1-intervals were all mutually independent, while the joint distribution within each group were the same as before.

Thus Λ_1 and Λ_0 are both positive definite, and it then follows from (20) that the same is true for Λ_h, so that f_h is always a normal probability density. Writing

$$F(h) = \int_G f_h dy,$$

it follows from the remarks just made that we have

$$F(0) = \int_{e_1} f_0 dy \ldots \int_{e_k} f_0 dy \int_{e^*_{k+1}} f_0 dy \ldots \int_{e^*_n} f_0 dy = P\{e_1\} \ldots P\{e_k\} P\{e^*_{k+1}\} \ldots P\{e^*_n\}.$$

By stationarity this reduces to

$$F(0) = p^k (1-p)^{n-k}, \tag{21}$$

where p is given by (18).

We shall now evaluate the difference $F(1)-F(0)$ by a development of a method used by S. M. Berman [1]. We note that for any normal density function $f(x_1, \ldots, x_n)$ with zero means and covariances r_{ij} we have

$$\frac{\partial f}{\partial r_{ij}} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

In our case, $f_h(y_1, \ldots, y_L)$ is a normal density, depending on h through the covariances $\lambda_{ij} = h\varrho_{ij}$. Hence

$$F'(h) = \int_G \frac{df_h}{dh} dy = \sum \varrho_{ij} \int_G \frac{\partial f_h}{\partial \lambda_{ij}} dy = \sum \varrho_{ij} \int_G \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy, \tag{22}$$

the summation being extended over all i, j such that η_i and η_j belong to different t_1-intervals. With respect to the integral over the set $G = e_1 \ldots e_k e^*_{k+1} \ldots e^*_n$ occurring in the last sum in (22), we have to distinguish three different cases.

Case A. When η_i and η_j both belong to t_1-intervals of subscripts $> k$, say to the t_1-intervals of subscripts $k+1$ and $k+2$ respectively, integration with respect to y_i and y_j has to be performed over e^*_{k+1} and e^*_{k+2} respectively. By definition of the sets e_r, both y_i and y_j thus have to be integrated over $(-\infty, 0)$, and so we obtain by direct integration with respect to y_i and y_j

$$\int_G \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy = \int_G f_h(y_i = y_j = u)\, dy'. \tag{23}$$

The notation used in the last integral is to be understood so that we have to take $y_i = y_j = u$ in f_h, and then integrate with respect to all y's expect y_i and y_j. As $f_h > 0$ always, we have

$$0 < \int_G \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy < \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_h(y_i = y_j = u)\, dy'.$$

The last integral, where all the y's except y_i and y_j are integrated out, yields the joint density function of the random variables corresponding to η_i and η_j in the normal distribution with covariance matrix Λ_h, for the values $y_i = y_j = u$, so that

$$0 < \int_G \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy < \frac{1}{2\pi(1-h^2 \varrho_{ij}^2)^{\frac{1}{2}}} \exp\left[-u^2/(1+h|\varrho_{ij}|)\right]. \tag{24}$$

Case B. Let now η_i and η_j both belong to t_1-intervals of subscripts $\leqslant k$, say to those of subscripts 1 and 2 respectively. Then integration with respect to each of the groups of variables to which y_i and y_j belong has to be performed over e_1 and e_2 respectively. By definition, e_1 is the set of all points in the y space such that at least one of the y's associated with the first t_1-interval exceeds u, and correspondingly for e_2. Some reflection will then show that the integration indicated in the first member of (23) can still be carried out directly, and yields the same result, with the only difference that in the second member of (23) the integration has to be performed over a set G', obtained from G by replacing e_1 and e_2 by e_1^* and e_2^* respectively. It follows that the inequality (24) still holds.

Case C. Finally we have the case when η_i and η_j belong to t_1-intervals of different kinds, say to the first and the $(k+1)$st respectively. As before the integration in the first member of (23) can be carried out directly. In this case, however, we obtain the relation (23) with a changed sign of the second member, and e_1 replaced by e_1^* in the expression of the domain of integration. In this case we thus obtain the inequality (24) with changed inequality signs.

Thus in all three cases we have the inequality

$$\left|\int_G \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy\right| < \frac{1}{2\pi(1-h^2\varrho_{ij}^2)^{\frac{1}{2}}} \exp\left[-u^2/(1+h|\varrho_{ij}|)\right]. \tag{25}$$

Now ϱ_{ij} is the covariance between the variables $\eta_i = \xi(\nu_i q)$ and $\eta_j = \xi(\nu_j q)$, where the points $\nu_i q$ and $\nu_j q$ belong to different t_1-intervals, and are thus separated by an interval of length at least equal to t_2. By the condition (2) we then have

$$|\varrho_{ij}| = |r(\nu_i q - \nu_j q)| < K t_2^{-\alpha},$$

where as usual K denotes an unspecified positive constant. Further, there are less than $L^2 \leqslant n^2(m_1+1)^2$ covariances ϱ_{ij}. Owing to stationarity some of the ϱ_{ij} are equal, but it is easily seen that this does not affect our argument. It then follows from (22) and (25), using (7) and (10), that we have

$$|F'(h)| < K n^2 m_1^2 t_2^{-\alpha} e^{-u^2} < K\mu^{\alpha-4\beta},$$

$$|F(1)-F(0)| = \left|\int_0^1 F'(h)\,dh\right| < K\mu^{\alpha-4\beta}.$$

This holds for any of the $\binom{n}{k}$ terms in the last member of (19), and $F(0)$ will in all cases be given by (21), so that we finally obtain

$$\left|P\{E_k\} - \binom{n}{k} p^k (1-p)^{n-k}\right| < K\binom{n}{k}\mu^{\alpha-4\beta} < K\mu^{\alpha-(k+4)\beta}.$$

By (8) we have $(k+4)\beta < \alpha$, so that the last member tends to zero as $u \to \infty$, and the lemma is proved.

6. Proof of the case j=1 of the theorem

By (18), $p = P\{e_r\}$ is defined as the probability that at least one of the random variables $\xi(0)$, $\xi(q)$, $\xi(2q), \ldots, \xi(m_1 q)$ takes a value exceeding u. According to (17), this differs from the probability $P\{d_r\}$ of at least one ξ_q upcrossing in the first t_1-interval by a quantity of the order

$$O\left(\frac{1}{u} e^{-u^2/2}\right).$$

By Lemma 3, the latter probability is

$$P\{d_r\} = q + o(q).$$

Thus we obtain from Lemma 5, observing that by (10) we have $1/u\, e^{-u^2/2} = o(q)$,

$$P\{E_k\} - \binom{n}{k} [q + o(q)]^k [1 - q + o(q)]^{n-k} \to 0.$$

By (10) we have $nq \to \tau$, and thus

$$\lim_{u \to \infty} P\{E_k\} = \frac{\tau^k}{k!} e^{-\tau}.$$

Lemmas 1 and 4 then finally give the relation (6) that was to be proved:

$$\lim_{u \to \infty} P_k = \frac{\tau^k}{k!} e^{-\tau}.$$

Thus we have proved the simplest case of the theorem, when $j = 1$, so that there is only one interval.

7. Proof of the general case

The generalization to the case of an arbitrary number $j > 1$ of intervals is now simple.

For any $\varepsilon > 0$, it follows from the result just proved that, for every $i = 1, \ldots, j$, the random variable

$$N(a_i + \varepsilon/\mu, b_i - \varepsilon/\mu),$$

where $b_i - a_i = \tau_i/\mu$, will be asymptotically Poisson distributed with parameter $\tau_i - 2\varepsilon$. In the same way as in the proof of Lemma 5 it is shown that these j variables are asymptotically independent, so that we have

$$P\{N(a_i + \varepsilon/\mu, b_i - \varepsilon/\mu) = k_i \text{ for } i = 1, \ldots, j\} \to \prod_{i=1}^{j} \frac{(\tau_i - 2\varepsilon)^{k_i}}{k_i!} e^{-(\tau_i - 2\varepsilon)}. \tag{26}$$

From the asymptotic Poisson distributions of the variables

$$N(a_i, a_i + \varepsilon/\mu) \quad \text{and} \quad N(b_i - \varepsilon/\mu, b_i)$$

it further follows that, with a probability exceeding $1 - 2j\varepsilon$, these variables will ultimately be zero for all $i = 1, \ldots, j$. Since j is fixed, and $\varepsilon > 0$ is arbitrarily small, the truth of the theorem then follows from (26).

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