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Some problems related to iterative methods in conformal mapping

By Ingemar Lind

Introduction

A. The conformal mapping problem for domains of connectivity greater than one has been attacked in several ways. The desired mapping function can sometimes be found as the solution of an extremal problem or of an integral equation or its existence may in certain cases be proved by means of the method of continuity. Another method, sometimes called *the function-theoretical iteration process*, is to express the mapping function $F(z)$ as a composition of functions $\{f_n\}_1^\infty$:

$$F_n(z) = f_n(f_{n-1}(\dots(f_1(z))\dots)); \qquad F(z) = \lim_{n\to\infty} F_n(z), \tag{A1}$$

where the f_n (determined in some way, e.g. see Hübner below) are meromorphic and univalent in certain *simply* connected domains. An advantage of this method is that it connects the theoretical and the constructive questions about the mapping.

Hübner ([10] pp. 43–55) has constructed a process—*the general iteration process*—by means of which every function $F(z)$ conformally mapping one domain onto a domain with analytic boundary can be expressed according to (A1). Thus, theoretically several of the well-known canonical mappings can be expressed in this way. However, the determination of f_n, $n=1, 2, \dots$, as a rule requires knowledge of $F(z)$ itself and thus the process from a constructive point of view has little interest. But there exist exceptions, namely the circular ring mapping and the mapping onto the lemniscate domain, studied earlier by Walsh, Grunsky and Landau.

A detailed account of the problems referred to in this section and the following is found in [3] pp. 208–240.

B. A straightforward attempt to use the function-theoretical iteration process is sketched below. For brevity we call it *the iterative process*. Here the determination of the functions f_n causes no trouble but on the other hand the convergence question is more intricate.

Let D be a domain of connectivity $k \geq 2$ on the z-sphere with the continua $C_\nu^{(0)}$, $\nu=1, 2, \dots, k$, as boundary components. Let $D^{(n)} = F_n(D)$ (F_n to be determined later) have boundary components $C_\nu^{(n)}$ corresponding to $C_\nu^{(0)}$, $\nu=1, 2, \dots, k$. It is required to find $F(z)$, eventually restricted by some normalization conditions, conformally mapping D onto a domain Ω so that $C_\nu^{(0)}$ corresponds to L_ν, $\nu=1, 2, \dots, k$. Here L_ν

is some definite continuum (e.g. the unit circle) or some type of a continuum (e.g. a circle or a slit making a prescribed angle with the positive real axis), $\nu = 1, 2, \ldots, k$. It is of course to be understood that the mapping conditions are appropriately formulated so that the mapping is not "overdetermined". Then f_{n+1}, $n = 0, 1, 2, \ldots$, is determined in the following way. According to some rule we choose a natural number ν_n, $1 \leqslant \nu_n \leqslant k$, and look at the simply connected domain $D^{(n)}_{\nu_n}$ bounded by $C^{(n)}_{\nu_n}$ and containing the other components. This domain is mapped conformally onto a domain bounded by the continuum or the type of continuum L_{ν_n} associated with $C^{(0)}_{\nu_n}$. Together with suitable normalization conditions this determines f_{n+1} and $C^{(n+1)}_{\nu}$, $\nu = 1, 2, \ldots, k$, as well.

The iterative process has proved successful mainly in two cases, namely (*a*) the circle domain mapping and (*b*) certain simple slit mappings.

Case (*a*) is due to Koebe [12]. Suppose that the point at infinity belongs to D. We require that

$$F(z) = z + \frac{a_1}{z} + \ldots + \frac{a_n}{z_n} + \ldots \tag{B2}$$

be the Laurent expansion in the neighbourhood of infinity. We normalize f_n in the same manner and for example choose ν_n so that $\nu_n - 1 \equiv n \pmod k$, $n = 0, 1, 2, \ldots$. The proof of convergence is based on the following. It can be shown that every kernel of $\{D^{(n)}\}_1^\infty$ has a specific reflection property (compare [3] pp. 212–214) and that every domain having this property is necessarily a circle domain.

Koebe ([13] especially pp. 288–296) also gave another proof for the same theorem using a somewhat different iterative technique ("das Iterationsverfahren" as distinct from the former, called "das iterierende Verfahren") in which in order to obtain quicker convergence the reflection properties are exploited more.

Komatu [14] has used the iterative process for the circular ring mapping. Certainly this is a special case of Koebe's general theorem but Komatu uses a measure of convergence which does not involve reflection.

Case (*b*) is due to Grötzsch ([6], [7], [9]) and partly to Golusin [4]. The proofs are carried out for the parallel, circular and radial slit mappings. In the case of a parallel slit domain, where the slits are parallel to the real axis, the process can be outlined as follows: Assuming as before that D contains the point at infinity, we require that $F(z)$ and f_n are normalized according to (B2). Let $a_1^{(n)}$ be the first coefficient of the Laurent expansion of f_n in the neighbourhood of infinity and $A_1^{(n)}$ the corresponding coefficient of F_n. Then $A_1^{(n)} = \sum_{\mu=1}^{n} a_1^{(\mu)}$. Also ν_n may be chosen as above (other rules of choice may however lead to quicker convergence). The proof of convergence is based on the following. If $\{D^{(n)}\}_1^\infty$ is supposed to have a kernel which is not a parallel slit domain, then necessarily $\sum_{\mu=1}^{n} \operatorname{Re} a_1^{(\mu)} \to \infty$. On the other hand, it is clear that $|A_1^{(n)}|$ is uniformly bounded and this gives a contradiction. The well-known extremal property of the mapping in question is an immediate by-product of this. The circular and radial slit mappings are treated in a similar way. Grötzsch also gives estimates concerning the rate of convergence especially in [9].

C. The vital point of the proofs of Grötzsch and Golusin above is the very simple behaviour of the functionals $a_1^{(n)}$. It seems reasonable to suppose that proofs based on less special properties may have wider scope. A similar remark can be made about Koebe's proof.

The following paper mainly deals with constructions of such alternative proofs, discusses some possible extensions and in certain cases carries them through.

In *Chapter I* is introduced a modified iterative process in which every step involves a conformal mapping of a $(k-1)$-connected domain—the original domain being k-connected. It is proved that this process can be successfully applied to a number of slit mappings. The proofs of convergence are essentially dependent on the maximum principle for harmonic functions. Besides two examples of extremal properties connected to the mappings referred to in this paragraph are proved with the aid of the iterative process. We conclude the chapter with an estimate of the rate of convergence of Grötzsch's process.

In *Chapter II* the iterative process is applied to the case where the boundary components of Ω (see **B**) are required to be simple, closed and analytic, for the rest they are of arbitrarily prescribed types. It is proved that the process converges if D is subjected to certain geometrical conditions, depending on ∂D and $\partial\Omega$, and essentially meaning that the boundary components are sufficiently separated from each other.

Chapter III deals with two eigenvalue problems, which may have a certain interest in themselves. Some partial results are deduced and the connection between these problems and the classification problem of certain Riemann surfaces is discussed. Especially this problem is studied for a surface, which can be associated with a circle domain and which also plays an important part in Koebe's proofs referred to above.

The aim of *Chapter IV* is to discuss an application of the process of Chapter I in the case when Ω is a mixed rectilinear slit domain of connectivity 2, that is its boundary consists of two rectilinear slits making a non-zero angle with one another. D is supposed to be of "nearly right" shape. We state a sufficient convergence condition which is connected with the eigenvalue problems of Chapter III.

Professor Lennart Carleson suggested the subject of this paper. I wish to express my gratitude for his generously given advice and kind interest in my work.

I. Conformal mappings onto certain domains of slit type

1. *Definitions*

By a k-connected domain D we shall in this paper understand a domain bounded by k disjoint continua $C_1, C_2, \dots, C_k$, $\partial D = \bigcup_{\nu=1}^{k} C_\nu$.

Let D be a domain on the z-sphere. Then we make the following definitions.

Definition 1.1. *$\Sigma(D)$ is the class of functions $f(z)$ meromorphic and univalent in D which have the Laurent expansion*

$$f(z) = z + a_0 + \frac{a_1}{z} + \dots + \frac{a_n}{z^n} + \dots$$

in the neighbourhood of infinity. It is to be understood that D contains the point at infinity.

Definition 1.2. *$\Sigma'(D)$ is the subclass of $\Sigma(D)$ consisting of those functions for which $a_0 = 0$.*

Definition 1.3. *$\Sigma_0(D)$ is the subclass of $\Sigma(D)$ consisting of those functions for which $f(0)=0$. It is to be understood that D in this case contains the origin.*

2. *The modified iterative process*

Our aim is to express a canonical mapping $F(z)$ in the form

$$z_n(z) = z_n(z_{n-1}(\dots(z_1(z))\dots)); \quad F(z) = \lim_{n\to\infty} z_n(z) \tag{2.1}$$

where $z_n(z_{n-1})$, $n=1, 2, \dots, (z_0 = z)$, is a conformal mapping of a certain $(k-1)$-connected domain on the z_{n-1}-sphere—the original domain being k-connected. We write $z_n(z_m) = z_n(z_{n-1}(\dots(z_m)\dots))$, $n > m$, and denote the inverse of $z_n(z_m)$ by $z_m(z_n)$.

Some formal notation will be used throughout this chapter. Let $D = D^{(0)}$ be a k-connected domain on the z-sphere and let $\partial D = \bigcup_{\mu=1}^{k} C_\mu$ where $C_\mu = C_\mu^{(0)}$, $\mu = 1, 2, \dots, k$, are the boundary components. We write $D^{(n)} = z_n(D)$ and $\partial D^{(n)} = \bigcup_{\mu=1}^{k} C_\mu^{(n)}$ where $C_\mu^{(n)}$ corresponds to $C_\mu^{(0)}$ under the mapping $z_n(z)$, $\mu = 1, 2, \dots, k$. By $D_\nu^{(n)}$ is meant the $(k-1)$-connected domain such that $\partial D_\nu^{(n)} = \bigcup_{\mu\neq\nu} C_\mu^{(n)}$ and $C_\nu^{(n)} \subset D_\nu^{(n)}$. Unless otherwise mentioned the letter k stands for connectivity and D for domain.

The precise description of the modified iterative process differs slightly from one case to another. The differences however are purely formal and therefore it is sufficient to describe a typical situation.

Our result is that for any finitely-connected domain D on the z-sphere ($\infty \in D$) there exists $F(z) \in \Sigma'(D)$ mapping D onto a canonical domain of some type A (e.g. a parallel slit domain). Our proof runs in three steps. (a) The statement is proved true for $k=1$. (b) Assuming it true for $k-1$ we determine $z_{n+1}(z_n)$ as follows: Let $z_{n+1}(z_n) \in \Sigma'(D_{\nu_n}^{(n)})$ and let it map $D_{\nu_n}^{(n)}$ onto a domain of type A, $n = 0, 1, 2, \dots$. Here $\{\nu_n\}_0^\infty$ is any sequence of the numbers $1, 2, \dots, k$ such that $\nu_{n+1} \neq \nu_n$, $n = 0, 1, 2, \dots$. Then it is proved that $F(z) = \lim_{n\to\infty} z_n(z) \in \Sigma'(D)$ exists, and hence necessarily maps D onto a domain of type A. (c) By induction the statement is true.

Fig. 1 shows the first steps in a modified iterative process in the case $k=3$. The canonical domain in question is a parallel slit domain.

3. *Slit mapping theorems*

The proof of Theorem 3.1—with some formal differences—is found in [15]. For the convenience of the reader it is repeated here. Theorems 3.1–3.10 are classical and of course there exist several proofs of them, (see e.g. [11]). Throughout this paper a slit is to be thought of as having two different edges.

Theorem 3.1. *For each θ, $0 \leq \theta < \pi$, there exists a unique function $\varphi_\theta(z) \in \Sigma'(D)$ mapping D onto a domain bounded by rectilinear slits making the angle θ with the positive real axis.*

Remark. We call such a domain a θ-angled parallel slit domain.

Proof. Suppose that Ω is a θ-angled parallel slit domain on the ω-sphere and suppose that $f(\omega) \in \Sigma'(\Omega)$ maps Ω onto another domain of the same type. Then

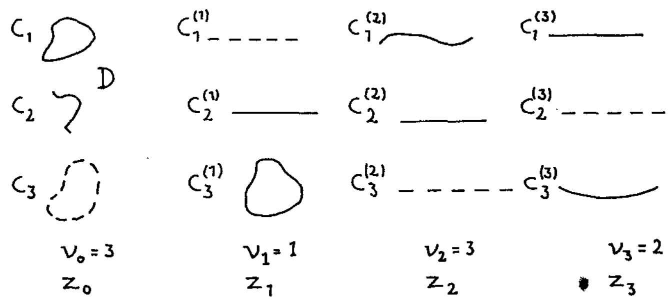

Fig. 1.

$e^{-i\theta}(f(\omega)-\omega)$ is analytic and bounded in Ω, it is zero at infinity and its imaginary part is constant on each boundary component. Hence $f(\omega)\equiv\omega$. Thus $\varphi_\theta(z)$ is unique if it exists.

The function $\varphi_\theta(z)$ exists if $k=1$; this follows from the Riemann mapping theorem plus an elementary transformation.

Applying the process described in the preceding section with $z_{n+1}(z_n)\in\Sigma'(D_{\nu_n}^{(n)})$ we observe that

$$u_n(z_n)=\operatorname{Im} e^{-i\theta}(z_{n+1}(z_n)-z_n),\ n=2,3,\ldots, \tag{3.1}$$

is harmonic and bounded in $D_{\nu_n}^{(n)}$, that $u_n(\infty)=0$ and that it is constant on each boundary component (not necessarily the same constant on each) but one, $C_{\nu_{n-1}}^{(n)}$. (It may happen that it takes a constant value on this component too but in this case the problem is solved.) Since $u_n(z_n)$ is the imaginary part of a bounded and analytic function in $D_{\nu_n}^{(n)}$ this means that $u_n(z_n)$ attains its maximum and minimum values on $C_{\nu_{n-1}}^{(n)}$. Write $\gamma_n=C_{\nu_{n-1}}^{(n)}$, $\Gamma_n=C_{\nu_n}^{(n)}$ and

$$\Delta_n=\max_{z_n',z_n''\in\gamma_n}\left|\operatorname{Im} e^{-i\theta}\{z_n'-z_n''\}\right|=\max_{z_n',z_n''\in\gamma_n}\left|u_n(z_n')-u_n(z_n'')\right|.$$

In particular it follows that

$$|u_n(z_n)|\leqslant\Delta_n,\ z_n\in\gamma_n.$$

From the maximum principle we deduce that there exists a number q_n, $0<q_n<1$, such that

$$\max_{z_n',z_n''\in\Gamma_n}\left|u_n(z_n')-u_n(z_n'')\right|\leqslant\Delta_n q_n.$$

From [1], Ch. IV, 26E, p. 263, combined with the compactness of $\Sigma'(D)$, it follows that it is possible to find a number q, $0<q<1$, such that q_n can be chosen $\leqslant q$ for all $n\geqslant 2$.

Thus
$$\max_{z_n',z_n''\in\Gamma_n}\left|u_n(z_n')-u_n(z_n'')\right|\leqslant\Delta_n q.$$

But since $\operatorname{Im} e^{-i\theta} z_n$ is constant on Γ_n this means that

$$\Delta_{n+1} \leqslant \Delta_n q.$$

Thus when $n \to \infty$ we have

$$\Delta_n = O(q^n) \to 0. \tag{3.2}$$

This implies that $\lim_{n\to\infty} z_n(z)$ exists. For suppose the contrary. Since $\Sigma'(D)$ is a compact family it is then possible to select two subsequences converging to different functions $\omega_1(z), \omega_2(z) \in \Sigma'(D)$. From (3.2) it follows that $\omega_1(D)$ and $\omega_2(D)$ are both θ-angled parallel slit domains. They are conformally equivalent under a mapping belonging to Σ' and thus according to the uniqueness $\omega_1(z) \equiv \omega_2(z)$ which gives a contradiction. Theorem 3.1 is now proved.

Remark 1. The proof indicates that one can estimate the rate of convergence of the process in terms of quantities which depend only on D. Let $\Omega = \varphi_\theta(D)$ and let $\varphi_n(\omega) = z_n(\varphi_\theta^{-1}(\omega))$ which thus maps Ω onto $D^{(n)}$. From $|\operatorname{Im} e^{-i\theta}(\varphi_n(\omega) - \omega)| \leqslant \Delta_n$, $\omega \in \partial\Omega$, it follows that in any closed subset A of Ω we have

$$|\varphi_n(\omega) - \omega| \leqslant K_A \Delta_n,\ \omega \in A,$$

where K_A depends on A and Ω only. The only non-rectilinear boundary component of $\partial D^{(n)}$ is $\gamma_n = z_n(\Gamma_{n-1})$. Let $\tilde{z}_{n-1}$ be the reflection of z_{n-1} with respect to Γ_{n-1} $(n \geqslant 3)$. Then we define $\tilde{z}_n = z_n(\tilde{z}_{n-1}(z_n))$ to be the reflection of z_n with respect to γ_n. It now follows from the reflection principle that $\varphi_n(\omega)$ can be analytically continued over each of the boundary components of $\partial\Omega$ onto a suitably chosen k-connected domain G on a many-sheeted Riemann surface branched at the endpoints of the slits of $\partial\Omega$. The continuations are given by $\tilde{\varphi}_n(\omega^*)$, where the symbols $\tilde{}$ and $*$ denote the reflection operators with respect to a slit and its image respectively. From the compactness of $\Sigma'(\Omega)$ it follows that ∂G can be chosen so that its projection onto the ω-sphere is a fixed, closed subset A of Ω independent of n and such that $|\tilde{\omega}^* - \omega| \leqslant K\Delta_{n-1}$, $\omega \in A$, (K independent of n). Observing that $\Delta_{n-1} \leqslant K' q^{n-1}$ we finally deduce from the maximum principle for analytic functions that

$$|\varphi_n(\omega) - \omega| \leqslant C q^n,\ \omega \in \partial\Omega,$$

where C and q depend only on D (or Ω). Similar remarks can be made in connection with the following theorems of this chapter.

Remark 2. The argument in the proof of Theorem 3.1 built on [1], 26E, p. 263 and the compactness of $\Sigma'(D)$ will be frequently used in various forms in this chapter. All of these forms are essentially the same as the following. Let F be a compact family of k-connected domains D on the z-sphere such that (1°) ∂D is contained in $|z| \leqslant R$, (2°) $\infty \in D$, (3°) $z_D \in D$ and (4°) z_D is distant at least $d > 0$ from ∂D. Further let $u(z)$ be any harmonic function in D such that $u(\infty) = 0$ and $\overline{\lim}_{z \in D} |u(z)| = 1$. Then $|u(z_D)| \leqslant q < 1$ where q depends on F and d only.

Theorem 3.2. *There exists a unique function $\Phi(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on concentric circles with the origin as centre.*

Proof. The proof is analogous to the preceding one. With the same notation we observe that the branch of $\log(f(\omega)/\omega)$ which is zero at infinity is analytic in Ω and that its real part is constant on each boundary component. Thus uniqueness follows. Prescribing $z_{n+1}(z_n) \in \Sigma_0(D_{\nu_n}^{(n)})$ we choose the branch of $\log (z_{n+1}(z_n)/z_n)$ which is zero at infinity. We can now apply the same argument to

$$u_n(z_n) = \operatorname{Re} \log \frac{z_{n+1}(z_n)}{z_n}, \quad n = 2, 3, \ldots, \tag{3.3}$$

as we did to the functions (3.1).

Theorem 3.3. *There exists a unique function $\Psi(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on half-rays emanating from the origin.*

Proof. The proof is analogous to the preceding one. Here the imaginary part of $\log (f(\omega)/\omega)$ is constant on each boundary component and instead of (3.3) we use

$$u_n(z_n) = \operatorname{Im} \log \frac{z_{n+1}(z_n)}{z_n}. \tag{3.4}$$

Theorem 3.4. *For each θ, $0 < \theta < \pi$, $\theta \neq \pi/2$, there exists a unique function $f_\theta(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on logarithmic spirals making the angle θ with half-rays emanating from the origin.*

Proof. The proof is analogous to the preceding ones. Instead of (3.3) we use

$$u_n(z_n) = \operatorname{Im} e^{-i\theta} \log \frac{z_{n+1}(z_n)}{z_n}. \tag{3.5}$$

Theorem 3.5. *There exists a unique function $\omega = \Phi_1(z)$ conformally mapping D onto a domain contained in $|\omega| < 1$, bounded by $|\omega| = 1$ corresponding to C_1 and slits on circles centred at the origin and such that*

$$\Phi_1(0) = 0, \quad \Phi_1'(0) > 0, \quad (0 \in D).$$

Proof. The proof (see e.g. [11] p. 74) may be based on the possibility of mapping a certain domain of connectivity $2(k-1)$ conformally onto a circular slit domain and this mapping can be represented in iterative terms according to Theorem 3.2. The proof can be carried through in a direct way too. It is then more suitable to change the conditions so that $\Phi_1'(0) = 1$ and to let the radius of the outer circle be unspecified. Then whenever $\nu_n = 1$, we define $z_{n+1}(z_n) \equiv \Phi(z_n)/\Phi'(0)$ where Φ is the function of Theorem 3.2 with respect to $D_1^{(n)}$. The proof is analogous to that of Theorem 3.2.

Theorem 3.6. *There exists a unique function $\omega = \Psi_1(z)$ conformally mapping D onto a domain contained in $|\omega| < 1$, bounded by $|\omega| = 1$ corresponding to C_1 and slits on half-rays emanating from the origin and such that*

$$\Psi_1(0) = 0, \quad \Psi_1'(0) > 0, \quad (0 \in D).$$

Proof. The proof (see e.g. [11] p. 74) may be based on the possibility of mapping a certain domain of connectivity $2(k-1)$ conformally onto a radial slit domain and this mapping can be represented in iterative terms according to Theorem 3.3. On the other hand, the proof can be carried through in a direct way. As in the preceding proof we change the conditions so that $\Psi_1'(0)=1$. For $k=2$, $\zeta=1/z$ maps D onto D'. Then it is possible to find a θ such that $\varphi_\theta(\zeta)$ (Theorem 3.1) maps D' onto D'' bounded by two slits on one and the same line. A suitable, elementary root transformation then maps D'' onto a domain of the desired type. For $k>2$ we may assume that D is contained in $|z|<r$ and that C_1 is the circle $|z|=r$. In the iterative process it is then always possible to prescribe $\nu_n \neq 1$, $n=1, 2, \ldots$, and $z_{n+1}(z_n)\equiv\Psi_1(z_n)$ $(\Psi_1'(0)=1)$ with respect to $D_{\nu_n}^{(n)}$. Observing that the function

$$f_n(z_n)=\log\frac{z_{n+1}(z_n)}{z_n};\quad f_n(0)=0,$$

is analytic and bounded in $D_{\nu_n}^{(n)}$, that its real part is constant on $C_1^{(n)}$ and that its imaginary part is constant on each other boundary component but one, γ_n, we conclude that $\operatorname{Im} f_n(z_n)$ attains its maximum and minimum on γ_n. Then the proof is analogous to that of Theorem 3.3.

Thereom 3.7. *Suppose that $1\in C_1$ (analytic) and $k\geqslant 2$. There exists a unique function $\omega=\Phi_2(z)$ mapping D conformally onto a domain bounded by $|\omega|=1$ corresponding to C_1, $|\omega|=r<1$ (r may not be prescribed) corresponding to C_2 and slits on circles centred at the origin, and such that $\Phi_2(1)=1$.*

Proof. The proof of uniqueness is analogous to that of Theorem 3.2.

In the iterative process we choose ν_n to be alternately 2 and 1. We identify $z_{n+1}(z_n)$ alternately with $\Phi_1(z_n)/\Phi_1(1)$ and $\Phi_1(1)/\Phi_1(1/z_n)=\Phi_1^*(z_n)$. Here $\Phi_1(\zeta)$ is the function of Theorem 3.5 with respect to $D_2^{(n)}$ in the former case and to $D_1^{(n)}$ inverted in $|\zeta|=1$ in the latter case. Thus $\Phi_1^*(z_n)$ maps $D_1^{(n)}$ onto a domain contained in $|z_{n+1}|>r_{n+1}(<1)$ and such that $C_2^{(n+1)}$ is the circle $|z_{n+1}|=r_{n+1}$. Further $\Phi_1^*(\infty)=\infty$ and $\Phi_1^*(1)=1$. The functions

$$u_n(z_n)=\operatorname{Re}\log\frac{z_{n+1}(z_n)}{z_n},\quad n=2,3,\ldots,\tag{3.6}$$

being zero at $z_n=1$ and being bounded and harmonic in $D_{\nu_n}^{(n)}$ behave like the functions (3.3). The $z_n(z)$ belong to a compact family of univalent functions. It follows that every kernel of $\{D^{(n)}\}_0^\infty$ must be an annulus with circular slits of the type described in the theorem. Together with the uniqueness this implies that $\Phi_2(z)=\lim_{n\to\infty} z_n(z)$. Theorem 3.7 is proved.

Remark. Unlike the preceding proofs the above is not inductive. For $k=2$ it is essentially the same as that of Komatu (see the introduction p. 102).

Theorem 3.8. *Suppose that $1\in C_1$ (analytic) and $k\geqslant 2$. There exists a unique function $\omega=\Psi_2(z)$ mapping D conformally onto a domain bounded by $|\omega|=1$ corresponding to C_1, $|\omega|=r<1$ (r may not be prescribed) corresponding to C_2 and slits on half-rays emanating from the origin and such that $\Psi_2(1)=1$.*

Proof. The proof is analogous to the preceding one. The iterative process is based on the function $\Psi_1(\zeta)$ of Theorem 3.6. As above we use the functions

$$u_n(z_n) = \operatorname{Re}\log\frac{z_{n+1}(z_n)}{z_n}, \quad n=2,3,\ldots, \tag{3.7}$$

to prove the convergence. It is to be observed that $u_n(z_n)$ attains its extremal values on $C^{(n)}_{\nu_{n-1}}$ ($k>2$; for $k=2$ Theorems 3.7 and 3.8 are identical) since $\log(z_{n+1}(z_n)/z_n)$ is analytic and bounded in $D^{(n)}_{\nu_n}$ and its imaginary part is constant on $C^{(n)}_\nu$, $\nu \neq 1, 2$.

Theorem 3.9. *For each θ, $0 \leqslant \theta < 2\pi$, there exists a unique function $P_\theta(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on confocal co-axial parabolas with the origin as focus and the axis making the angle θ with the positive real axis.*

Proof. Let Ω be a domain on the ζ-sphere, $0, \infty \in \Omega$. By $\tilde{\Omega}$ is meant the two-sheeted covering surface of Ω branched at zero and infinity.

Supposing Ω on the ω-sphere and Ω' being parabolic slit domains of the type in question conformally equivalent under a mapping $f(\omega) \in \Sigma_0(\Omega)$ we can define a bounded analytic function in $\tilde{\Omega}$ to take the values

$$\sqrt{e^{-i\theta} f(\omega)} - \sqrt{e^{-i\theta}\omega}$$

at points lying over ω. It is zero at the branch points and its imaginary part is constant on each boundary component. Hence $f(\omega) \equiv \omega$ which proves the uniqueness.

The theorem is true for $k=1$ (see e.g. [11], pp. 78–80).

The iterative process is constructed in the standard way ($z_{n+1}(z_n) \in \Sigma_0(D^{(n)}_{\nu_n})$). We observe that the functions $u_n(P_n)$ taking the values

$$\operatorname{Im}\left\{\sqrt{e^{-i\theta} z_{n+1}(z_n)} - \sqrt{e^{-i\theta} z_n}\right\}, \quad n=2,3,\ldots, \tag{3.8}$$

at points $P_n \in \tilde{D}^{(n)}_{\nu_n}$ lying over z_n, are bounded and harmonic, are zero at the branch points and are constant on each boundary component (of $\partial\tilde{D}^{(n)}_{\nu_n}$) except two which lie one over the other, with the boundary values differing only in sign. Thus the functions $u_n(P_n)$, $n=2, 3, \ldots$, behave like the functions (3.1) and it follows as in Theorem 3.1 that every kernel of $\{D^{(n)}\}_0^\infty$ must be a parabolic slit domain of the type in question. From the uniqueness it then follows that $P_\theta(z) = \lim_{n\to\infty} z_n(z)$. Theorem 3.9 is proved.

Theorem 3.10. *For each $a \in D$, $a \neq 0$, and α, $0 \leqslant \alpha < \pi$, there exists a unique function $\omega = G(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on curves belonging to the family*

$$\omega = \frac{G(a)}{4}\left(e^{(c+it)e^{-i\alpha}} + e^{-(c+it)e^{-i\alpha}}\right)^2,$$

where t is a real parameter and c a real constant.

Remark. The family of curves of Theorem 3.10 are trajectories of the family of ellipses with foci at 0 and $G(a)$. For $\alpha = 0$ we obtain these ellipses and for $\alpha = \pi/2$ hyperbolas with foci at 0 and $G(a)$.

Proof. Let Ω be a domain on the ζ-sphere, $0,\ a \in \Omega\ (a \neq 0)$. By $\tilde{\Omega}(a)$ is meant the two-sheeted covering surface of Ω branched at 0 and a. Further we write formally

$$g(\zeta, A) = \frac{2}{A}\left[\zeta - \frac{A}{2} + \sqrt{\zeta(\zeta - A)}\right], \quad A \neq 0.$$

The uniqueness follows mainly as in the preceding proof. With analogous notation it is possible to define a bounded analytic function in $\tilde{\Omega}(a)$ to take the values

$$e^{i\alpha} \log \frac{g(f(\omega), f(a))}{g(\omega, a)}$$

at points over ω, being zero at points over ∞. Since its real part is constant on each boundary component it follows that $f(\omega) \equiv \omega$.

In the iterative process we choose $z_{n+1}(z_n) \in \Sigma_0(D_{\nu_n}^{(n)})$ and the associated parameter is $a_n = z_n(a)$. The functions $u_n(P_n)$ taking the values

$$\operatorname{Re} e^{i\alpha} \log \frac{g(z_{n+1}(z_n), a_{n+1})}{g(z_n, a_n)}, \quad n = 2, 3, \ldots, \tag{3.9}$$

at points $P_n \in \tilde{D}_{\nu_n}^{(n)}(a_n)$ then have properties similar to the functions (3.8) and the argument runs as in the preceding proof.

As regards the case $k=1$, see e.g. [5], p. 128. Theorem 3.10 is proved.

Theorem 3.11. *For each $a \in D\ (a \neq 0)$ there exists a unique function $\omega = C_1(z) \in \Sigma_0(D)$ mapping D onto a domain bounded by slits on circles going through 0 and $C_1(a)$.*

Proof. Suppose that Ω on the ω-sphere is a slit domain of the type described in the theorem (associated parameter a) and suppose that there exists $f(\omega) \in \Sigma_0(\Omega)$ mapping Ω onto another domain of the same type (associated parameter $f(a)$). Then

$$g(\omega) = \log \frac{\omega(f(\omega) - f(a))}{f(\omega)\,(\omega - a)}, \quad g(\infty) = 0$$

is analytic and bounded in Ω and $\operatorname{Im} g(\omega)$ is constant on each boundary component and this implies that $f(\omega) \equiv \omega$. Thus $C_1(z)$ is unique if it exists.

In the case $k=1$ we must prove that there exists a function $\omega = f(\zeta)$, univalent and meromorphic in $E: |\zeta| < 1$, having a simple pole with residue 1 at the origin, and mapping E onto a domain with boundary of the following type: Let $\alpha_i \in E$, $\alpha_i \neq 0\ (i = 1, 2)$, $\alpha_1 \neq \alpha_2$, with α_1, α_2 otherwise arbitrary. Then the required boundary is to be a slit on a circle through $0 = f(\alpha_1)$ and $f(\alpha_2)$. Suppose first that the straight line through α_1 and α_2 contains the origin and makes the angle θ with the positive real axis $(0 \leqslant \theta < \pi)$. Then the desired mapping is

$$f(\zeta) = \zeta^{-1} - \alpha_1^{-1} + e^{-2i\theta}(\zeta - \alpha_1),$$

which maps E conformally onto a domain bounded by a slit on the straight line through $0 = f(\alpha_1)$ and $f(\alpha_2)$. In another case the system

$$\left|\frac{u-\alpha_i}{1-\bar{u}\alpha_i}\right| = |\alpha_i|, \quad i=1,2,$$

which is equivalent to

$$\left|u-\frac{\alpha_i}{1+|\alpha_i|^2}\right| = \frac{|\alpha_i|}{1+|\alpha_i|^2}, \quad i=1,2,$$

has a unique non-zero solution u, $|u|<1$ and

$$f(\zeta)=\frac{u-\zeta}{u\zeta(1-\bar{u}\zeta)}-\frac{u-\alpha_1}{u\alpha_1(1-\bar{u}\alpha_1)}$$

is the desired mapping. If we write $T(\zeta)=(u-\zeta)/(u\zeta(1-\bar{u}\zeta))$ we can readily verify that $f(|\zeta|=1)$ is a slit on the circle $|\omega+T(\alpha_1)|=|u|^{-1}$. This circle contains $0=f(\alpha_1)$ and $T(\alpha_2)-T(\alpha_1)=f(\alpha_2)$ since $|uT(\alpha_i)|=1$, $i=1, 2$, according to the choice of u above. Thus the theorem is true for $k=1$.

As regards the iterative process, we have $z_{n+1}(z_n)\in\Sigma_0(D^{(n)}_{\nu_n})$ and the associated parameter is $a_n=z_n(a)$. The functions

$$u_n(z_n)=\operatorname{Im}\log\frac{(z_{n+1}(z_n)-a_{n+1})\,z_n}{(z_n-a_n)\,z_{n+1}(z_n)}, \quad n=2,3,\ldots, \tag{3.10}$$

behave like the functions (3.3) and thus together with the uniqueness lead to $C_1(z)=\lim_{n\to\infty} z_n(z)$. Theorem 3.11 is proved.

Theorem 3.12. *For each $a\in D$ $(a\neq 0)$ there exists a unique function $\omega=C_2(z)\in\Sigma_0(D)$ mapping D onto a domain bounded by slits on the circles $|(\omega-C_2(a))/\omega|=$const.*

Proof. The proof is analogous to the preceding one. As regards the uniqueness we observe that with analogous notation $\operatorname{Re} g(\omega)$ is constant on each boundary component.

For the iterative process, instead of the functions (3.10) we use

$$u_n(z_n)=\operatorname{Re}\log\frac{(z_{n+1}(z_n)-a_{n+1})\,z_n}{(z_n-a_n)\,z_{n+1}(z_n)}, \quad n=2,3,\ldots.$$

In the case $k=1$ the theorem is proved in a way similar to that of the preceding proof. With the same notation we first suppose that $|\alpha_1|=|\alpha_2|$. Then it is possible to find θ and t, $0\leqslant\theta<\pi$, $t\neq 0$, so that $\alpha_1=\varrho e^{i(\theta+t)}$, $\alpha_2=\varrho e^{i(\theta-t)}$. Then the desired mapping is

$$f(\zeta)=\zeta^{-1}-\alpha_1^{-1}+e^{-2i\theta}(\zeta-\alpha_1),$$

which maps E conformally onto a domain bounded by a slit on the perpendicular bisector of the straight line segment joining $0=f(\alpha_1)$ and $f(\alpha_2)$. If $|\alpha_1|\neq|\alpha_2|$ then

$$f(\zeta)=\frac{u-\zeta}{u\zeta(1-\bar{u}\zeta)}-\frac{u-\alpha_1}{u\alpha_1(1-\bar{u}\alpha_1)}$$

is the mapping where $u=re^{i\varphi}$ is the (unique) solution of

$$r=\frac{e^{-i\varphi}\alpha_2+e^{i\varphi}\bar{\alpha}_1}{1+\bar{\alpha}_1\alpha_2} \tag{3.12}$$

such that $0<r<1$. To verify the existence of the solution we may suppose that $|\alpha_1|<|\alpha_2|$; the other case is symmetrical to this. Taking imaginary parts of (3.12) we obtain

$$\operatorname{Im} e^{-i\varphi}(\alpha_1(1-|\alpha_2|^2)-\alpha_2(1-|\alpha_1|^2))=0. \tag{3.13}$$

It is easily verified that there exists φ satisfying (3.13) and such that $\operatorname{Re} e^{-i\varphi}\alpha_2>0$ since otherwise we obtain a contradiction to $|\alpha_1|<|\alpha_2|$. Choosing this value we observe that

$$\omega(z)=\frac{z+\alpha_2 e^{-i\varphi}}{1+z\alpha_2 e^{-i\varphi}}$$

maps $|z|<|\alpha_2|$ onto a disc intersecting the real axis along the segment $0<\operatorname{Re}\omega<(2\operatorname{Re} e^{-i\varphi}\alpha_2)/(1+|\alpha_2|^2)<1$. Further $r=\omega(e^{i\varphi}\bar{\alpha}_1)$ is real. Hence $0<r<1$. If $T(\zeta)=(u-\zeta)/(u\zeta(1-\bar{u}\zeta))$ then (3.12) is equivalent to $|u|^2T(\alpha_1)\bar{T}(\alpha_2)=1$. It is readily verified that $f(|\zeta|=1)$ is a slit on the circle $\omega(t)=-T(\alpha_1)+|u|^{-1}e^{it}$, $0\leqslant t<2\pi$. The choice of u implies that

$$\left|\frac{\omega(t)-f(\alpha_2)}{\omega(t)-f(\alpha_1)}\right|=|u|\,|T(\alpha_2)|\left|\frac{e^{it}-|u|\,T(\alpha_2)}{1-e^{it}|u|\,\bar{T}(\alpha_2)}\right|=|u|\,|T(\alpha_2)|.$$

Theorem 3.12 is proved.

Remark. Letting $a\to\infty$ we see that every kernel of $\{C_1(D,a)\}$ is a radial slit domain and that every kernel of $\{C_2(D,a)\}$ is a circular slit domain, $(C_i(z,a)\equiv C_i(z)$, $i=1,2)$. From the uniqueness of the mappings $\Phi(z)$ and $\Psi(z)$ of Theorems 3.2 and 3.3 it follows that $\Phi(z)=\lim_{a\to\infty} C_2(z,a)$ and $\Psi(z)=\lim_{a\to\infty} C_1(z,a)$.

Letting $a\to 0$ we see that every kernel of $\{C_1(D,a)\}$ and $\{C_2(D,a)\}$ is a domain bounded by slits on circles $\operatorname{Re} e^{i\varphi}\omega^{-1}=\text{const.}$ $(0\leqslant\varphi<\pi)$. It is readily verified that there exist kernels corresponding to any φ, $0\leqslant\varphi<\pi$.

We conclude this section with a brief discussion on the application of the modified iterative process to the following problem: Does there exist $\omega=f(z)\in\Sigma'(D)$ mapping D onto a domain bounded by slits on lemniscates $|\omega-f(a_1)|\,|\omega-f(a_2)|=\text{const.}$, where a_1 and a_2 are given points in D? If the choices of a_1 and a_2 are restricted in a certain way depending only on D then such a mapping exists.

Suppose first that $k=1$. The sets $E_i=\{f(a_i)\,|\,f\in\Sigma'(D)\}$, $i=1,2$, are certain closed discs ([5] p. 129) and let $E=\{\frac{1}{2}(\omega_1+\omega_2)\,|\,\omega_i\in E_i,\ i=1,2\}$. Further there exists a closed disc K such that the boundary of $f(D)$ is contained in K for all $f\in\Sigma'(D)$ ([5] p. 178). We now make the assumption that $K\cap E=\phi$. This condition is for example satisfied if the points a_i, $i=1,2$, are at a sufficiently great distance from ∂D. Let $b_i\in E_i$, $i=1,2$. We indicate the construction of a family F of simple closed Jordan curves in the ω-plane, as follows (Fig. 2). Let L be the perpendicular bisector of the straight line segment through b_1 and b_2 and let K' be the reflection of K in the segment. Now L divides the plane in the half-planes H_1 and H_2. We demand that $L\subset H_1$ (so that H_1 is closed and H_2 open) and suppose for example that the centre of K is in H_1. We define H_1' to be $H_1\cup K\cup K'$ and let H_2' be its complement. A curve belonging to F is to consist of an arc of a lemniscate $|\omega-b_1|\,|\omega-b_2|=\text{const.}$, $\omega\in H_1'$, if this does not intersect $\partial H_1'$ and such an arc completed by drawing a circular arc $|\omega-\frac{1}{2}(b_1+b_2)|=\text{const.}$, $\omega\in H_2'$, if the lemniscate does intersect $\partial H_1'$. Through each

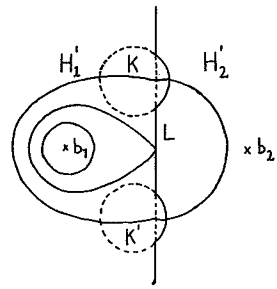

Fig. 2.

point in the plane there passes precisely one member of the family. A slit on a curve in F (which in the present case may be along the whole closed curve) is uniquely determined by three real parameters. A standard application of the method of continuity proves that there exists a unique function $f(z; b_1, b_2) \in \Sigma'(D)$ which maps D onto a domain bounded by a slit on a curve in F (see [8]).

From the choice of K it follows that this slit must be situated within K. Further it is clear that $f(z_0; b_1, b_2)$, $z_0 \in D$, $b_i \in E_i$, $i=1, 2$, is continuous with respect to b_1 and b_2. Regarding $b_i' = f(a_i; b_1, b_2)$, $i=1, 2$, as a transformation of (b_1, b_2) the conditions for an application of Brouwer's fixed point theorem are fulfilled. Hence there exist $b_i \in E_i$, such that $b_i = f(a_i; b_1, b_2)$, $i=1, 2$. Thus our statement is true for $k=1$.

If the choices of a_i, $i=1, 2$, are appropriate (depending in particular on k, compare [11], p. 96) the iterative method can be used. The proof of convergence is based on the functions

$$u_n(z_n) = \operatorname{Re} \log \prod_{i=1}^{2} \frac{z_{n+1}(z_n) - a_i^{(n+1)}}{z_n - a_i^{(n)}}, \quad n = 2, 3, \ldots,$$

where $z_{n+1}(z_n) \in \Sigma'(D_{\nu_n}^{(n)})$ and $a_i^{(n)} = z_n(a_i)$, $i=1, 2$.

As regards the uniqueness we obtain the condition

$$(f(\omega) - f(a_1))(f(\omega) - f(a_2)) = (\omega - a_1)(\omega - a_2),$$

with obvious notation. Since $f(\omega) \in \Sigma'$ it follows that $a_1 + a_2 = f(a_1) + f(a_2)$. Further

$$f'(\omega)\,[f(\omega) - \tfrac{1}{2}(f(a_1) + f(a_2))] = \omega - \tfrac{1}{2}(a_1 + a_2).$$

Since $\frac{1}{2}(a_1 + a_2) \in D$ it necessarily follows that $f(\frac{1}{2}(a_1 + a_2)) = \frac{1}{2}(f(a_1) + f(a_2))$. It now easily follows that $a_1 - a_2 = f(a_1) - f(a_2)$. Thus $f(a_i) = a_i$, $i=1, 2$, and finally $f(\omega) \equiv \omega$.

4. *Extremal properties*

Most of the mappings of the previous section have simple extremal properties, which as a matter of fact uniquely characterize them. The modified iterative process may enable the actual calculation of the extremal quantities in certain cases. We give two examples of the extremal properties.

Corollary 4.1. $\mathrm{Max}_{f\in\Sigma'(D)}\,\mathrm{Re}\,\{e^{-2i\theta}a_1\} = \mathrm{Re}\,\{e^{-2i\theta}a_\theta\} \geqslant 0$, $0 \leqslant \theta < \pi$, *where* $f(z) = z + (a_1/z) + \ldots$ *and* $\varphi_\theta(z) = z + (a_\theta/z) + \ldots$ *are the Laurent expansions near the point at infinity and* $\varphi_\theta(z)$ *is the function of Theorem* 3.1. *Equality occurs if and only if D is a θ-angled parallel slit domain.*

Proof. Since $\Sigma'(D)$ is compact the existence of a solution within the class is guaranteed. Thus it is sufficient to prove that $\mathrm{Re}\,\{e^{-2i\theta}a_\theta\} \geqslant 0$ with equality if and only if D is a θ-angled parallel slit domain. Supposing this done we let $\zeta = f(z) = z + (a_1/z) + \ldots$ map D onto Ω, which is not a θ-angled parallel slit domain, and let $\omega = \varphi_\theta(\zeta) = \zeta + (b_\theta/\zeta) + \ldots$ be the function of Theorem 3.1 with respect to Ω. Then $\varphi_\theta(f(z)) = z + (a_1 + b_\theta)/z + \ldots \in \Sigma'(D)$ and

$$\mathrm{Re}\,e^{-2i\theta}(a_1 + b_\theta) > \mathrm{Re}\,e^{-2i\theta}a_1.$$

Thus $f(z)$ cannot be extremal.

The corollary is true for $k = 1$ which for example can be proved with the aid of the area theorem. Supposing it true for $k-1$ we express $\varphi_\theta(z)$ iteratively as $\lim_{n\to\infty} z_n(z)$ according to Theorem 3.1. Writing $z_{n+1}(z_n) = z_n + a_\theta^{(n)}/z_n + \ldots$ we deduce that

$$a_\theta = \sum_{n=0}^{\infty} a_\theta^{(n)}.$$

Hence we have

$$\mathrm{Re}\,e^{-2i\theta}a_\theta = \sum_{n=0}^{\infty} \mathrm{Re}\,e^{-2i\theta}a_\theta^{(n)} \geqslant 0. \tag{4.1}$$

Clearly equality occurs if and only if all terms are zero, which means that D is a θ-angled parallel slit domain.

By induction Corollary 4.1 is true for all k.

Remark. As far as Corollary 4.1 is regarded as a purely qualitative statement the proof may be simplified. Then the only essential points are the compactness of $\Sigma'(D)$ and the truth of the corollary for $k = 1$. Considering the remark of Theorem 3.1, (4.1) however enables us to estimate the extremal value. For example given D we can find constants B and $q < 1$ such that $\mathrm{Re}\,e^{-2i\theta}a_\theta^{(n)} \leqslant Bq^n$ and hence $\mathrm{Re}\,e^{-2i\theta}a_\theta = \sum_{n=0}^{N-1}\mathrm{Re}\,e^{-2i\theta}a_\theta^{(n)} + O(q^N)$. The extremal properties of $\Phi(z)$, $\Psi(z)$, $f_\theta(z)$, $\Phi_1(z)$, $\Psi_1(z)$ (compare [11] pp. 72–77) and $G(z)$ ([5] p. 128) of Theorems 3.1–3.6 and 3.10 can be treated in a similar way.

We return to the extremal property of $\varphi_\theta(z)$ in Section 5.

As a less obvious example we now prove the extremal property of $\Phi_2(z)$ in Theorem 3.7 which in fact also gives an alternative proof of the existence of the mapping.

Corollary 4.2. *Let $F_{12} = F_{12}(D)$ be the class of functions, $\omega = f(z)$, regular and univalent in D such that C_1 corresponds to $|\omega| = 1$ and C_2 to $|\omega| = r < 1$ $(k \geqslant 3)$. Then*

$$\max_{f\in F_{12}} r = \bar{r}$$

where $|\omega| = \bar{r}$ corresponds to C_2 under $\Phi_2(z)$ and where $\Phi_2(z)$ is the function of Theorem 3.7. The maximum is attained for the functions $e^{i\gamma}\Phi_2(z)$ (γ real) only.

Proof. F_{12} is compact which guarantees the existence of a maximum. We may suppose that D is contained in the annulus $r' < |z| < 1$ $(C_1: |z| = 1, C_2: |z| = r')$ and that at least one of the remaining boundary components is not a slit on a circle of centre the origin. We now apply an iterative process, which mainly is that of Theorem 3.7 but with the difference that $z_{n+1}(z_n)$ is identified alternately with $\Phi_1(z_n)/\Phi_1'(0)$ and $\Phi_1'(0)/\Phi_1(1/z_n)$ (p. 107). Let $\underline{\varrho}_i^{(n)}$ be the inner radius (with respect to he origin) of the finite domain bounded by $C_i^{(n)}$ and let $\bar{\varrho}_i^{(n)}$ be the outer radius of the infinite domain bounded by $C_i^{(n)}$, $i = 1, 2$; $n = 0, 1, 2, \ldots$. Then $\underline{\varrho}_i^{(n)} \leqslant \bar{\varrho}_i^{(n)}$ where equality occurs if and only if $C_i^{(n)}$ is a circle of centre the origin. Further, we construct the iterative process so that $C_1^{(2n+1)}$ and $C_2^{(2n)}$ are circles. We denote their radii R_{2n+1} and r_{2n} respectively. According to the extremal property of $\Phi_1(\zeta)$ we obtain

$$R_{2n-1} = \bar{\varrho}_1^{(2n)} > \underline{\varrho}_1^{(2n)} \geqslant R_{2n+1},$$

$$r_{2n-2} = \underline{\varrho}_2^{(2n-1)} < \bar{\varrho}_2^{(2n-1)} \leqslant r_{2n}, \quad n = 1, 2, \ldots.$$

Thus $r_{2n} \to r > r'$ and $R_{2n+1} \to R < 1$ (since $R_1 < R_0 = 1$). This proves in particular that $\{z_n(z)\}_1^\infty$ belongs to a compact family of univalent functions and as in Theorem 3.7 it follows that every kernel of $\{D^{(n)}\}$ must be an annulus $r < |\omega| < R$ with circular slits. Further $r' < r/R$ which proves the extremal property. Corollary 4.2 is proved.

A remark similar to that of Corollary 4.1 concerning the possibility of estimating $\bar{r}$ can be made here.

5. *On rates of convergence*

Grötzsch indicates in [9] (for the case of circular slits) a method of obtaining estimates of the rate of convergence of the iterative method used by him. In the present section we give a similar method concerning mappings onto zero-angled parallel slit domains, a method which gives an explicit estimate of the rate of convergence. This is not exponential ($k \geqslant 3$), contrary to the rate of convergence of the modified iterative process (Theorem 3.1, see Remark 1, p. 106). The precise rate is not known, (compare also [3], pp. 236–238).

Let the width of a slit Γ be defined as $\mathrm{Max}_{a,b \in \Gamma} \operatorname{Im}(a-b)$ and let Δ_n be the maximal width of the slits $\{C_\nu^{(n)}\}_1^k$. Each step in the iterative process is determined in the following way. Consider the simply connected domain which is bounded by a slit of maximal width (we denote one such slit by γ_n). This domain is mapped onto a zero-angled parallel slit domain. The mapping functions are normalized in the usual manner:

$$z_{n+1}(z_n) = z_n + \frac{a_1^{(n)}}{z_n} + \ldots, \quad n = 0, 1, 2, \ldots,$$

and we write $\operatorname{Re} a_1^{(n)} = \mu_n$ for short. We pose the following problem ($k \geqslant 3$):

Given $\varepsilon > 0$, find a number N_ε such that $\Delta_n < \varepsilon$ whenever $n \geqslant N_\varepsilon$.

Suppose that $\gamma_n = C_\nu^{(n)}$ $(n > n_0)$. Let n^* be the greatest index $< n$ such that $\gamma_{n^*-1} = C_\nu^{(n^*-1)}$. Then $C_\nu^{(n^*)}$ is rectilinear and γ_n is the analytic image of $C_\nu^{(n^*)}$ under the mapping $z_n(z_{n-1}(\ldots(z_{n^*})\ldots))$.

We need some notation. We say that n^* is *the first predecessor* of n and that n is *the first successor* of n^*. Generically speaking n ($<m$) is *a predecessor* of m if there exists a *chain* of indices $n=n_1<n_2<\ldots<n_p=m$ such that n_ν is the first predecessor of $n_{\nu+1}$, $\nu=1, 2, \ldots, p-1$. Such a chain is said to have *length* p. The relation "n is a predecessor of m or a successor of m or equals m" is an equivalence relation. It is easily seen that the number of equivalence classes is at most $k-1$. We denote them by $E_1, E_2, \ldots, E_{k-1}$. We say that Δ_n satisfies a *condition of quotient* or a *Q-condition* if

$$\Delta_{n^*}<Q\Delta_n,$$

where Q is an arbitrarily fixed positive number. To be concrete we shall in the following let Q equal 2^{k-1}. For convenience we suppose that the slits $\{C_\nu^{(0)}\}_1^k$ (one of which is rectilinear) are analytic and that Δ_n, $n=0, 1, 2, \ldots$, are small, which is guaranteed if Δ_0 is sufficiently small. In the following the letter C denotes a positive constant not necessarily the same each time it occurs but in any case independent of n.

Lemma 5.1. 1°. *If Δ_n satisfies the Q-condition then*

$$\mu_n \geqslant C\Delta_n^2,\ n=0, 1, 2, \ldots.$$

2°. *In any case it is true that*

$$\sum_{\nu=n}^{\infty} \mu_\nu \leqslant C(\Delta_n \log \Delta_n)^2, \quad n=0,1,2,\ldots.$$

Proof. We observe that $z_{n+1}(z_n)$, $n>n_0$ can be continued over $\gamma_n=C_\nu^{(n)}$ into a double-sheeted Riemann surface branched at the endpoints of γ_n. The conintuation is given by

$$\tilde{z}_{n+1}(\tilde{z}_{n^*}(z_n))=z_n+r_{n+1}^*(z_n),$$

where $z_{n^*}(z_n)$ is the mapping of $D^{(n)}$ onto $D^{(n^*)}$ etc., and $\tilde{}$ means reflection in $C_\nu^{(n+1)}$ and $C_\nu^{(n^*)}$ respectively. By an argument similar to that of Remark 1 on Theorem 3.1 (p. 106) we conclude that

$$|r_{n+1}^*(z_n)| \leqslant C(\Delta_{n^*}+\Delta_{n+1}),\ z_n \in L,$$

where L is the curve $d(z_n, \gamma_n)=d>0$ thought of as lying in the second sheet. Here $d(z_n, \gamma_n)$ is the distance between z_n and γ_n and d can be chosen independently of n as a consequence of the compactness of $\Sigma'(D)$. If, besides, Δ_n satisfies the Q-condition then also using $\Delta_{n+1}<2\Delta_n$ (which is always true) we have

$$|r_{n+1}^*(z_n)| \leqslant C\Delta_n,\ z_n \in L.$$

We shall study the inverse of $z_{n+1}(z_n)$ and for simplicity we write it as $\omega=f(z)=z+a_1/z+\ldots$, where $a_1=-a_1^{(n)}$. Since the $z_n(z)$ belong to the compact family $\Sigma'(D)$ there is no loss of generality in supposing that the rectilinear slit $C_\nu^{(n+1)}$ equals $\{z_{n+1} \mid |x| \leqslant 2, y=0\}$, $z_{n+1}=x+iy$. At worst this will only cause simple modifications of the constants involved. Further we write Δ instead of Δ_n.

According to the above the function

$$g(\zeta)=f(\zeta+\zeta^{-1})=\zeta^{-1}+\zeta+h(\zeta)=\zeta^{-1}+(1+a_1)\zeta+a_2\zeta^2+\dots$$

is meromorphic (singular part ζ^{-1}) in a fixed domain containing some closed disc $|\zeta|\leqslant R$ ($R>1$ and is independent of Δ, i.e. of n) and it is, moreover, univalent in $|\zeta|<1$. Further we have

$$|\operatorname{Im} h(\zeta)|\leqslant\Delta,\ |\zeta|=1, \tag{5.1}$$

and there exists some point ζ_0, $|\zeta_0|=1$, such that

$$|\operatorname{Im} h(\zeta_0)|\geqslant\Delta/2. \tag{5.2}$$

If, moreover, $\Delta_n=\Delta$ satisfies the Q-condition then

$$|h(\zeta)|\leqslant C\Delta,\ |\zeta|=R. \tag{5.3}$$

The function $g(\zeta)$ maps $|\zeta|<1$ onto a domain which has a complement of zero area. Hence we obtain from the area theorem that

$$-2\operatorname{Re} a_1=\sum_{\nu=1}^{\infty}\nu|a_\nu|^2. \tag{5.4}$$

We now prove 1°. It follows from (5.2) and (5.3) that there exists an arc σ of length $\pi\alpha$ (α independent of Δ) on the unit circle such that

$$|\operatorname{Im} h(\zeta)|\geqslant\Delta/3,\ \zeta\in\sigma. \tag{5.5}$$

We have

$$\sum_{\nu=1}^{\infty}\nu|a_\nu|^2\geqslant\sum_{\nu=1}^{\infty}|a_\nu|^2=\frac{1}{2\pi}\int_0^{2\pi}|h(e^{i\varphi})|^2\,d\varphi=\frac{1}{\pi}\int_0^{2\pi}(\operatorname{Im} h(e^{i\varphi}))^2\,d\varphi$$

and hence from (5.4) and (5.5)

$$-2\operatorname{Re} a_1\geqslant\frac{\alpha}{9}\Delta^2.$$

This proves 1°.

We now turn to 2°, which we prove by induction on k. If $k=1$ then we use the observations made above about the function $f(\zeta)$ (here regarded as the inverse of $z_1(z)$). It follows from (5.1) that $|a_\nu|\leqslant 2\Delta$ and further we have $|a_\nu|\leqslant CR^{-\nu}$, $\nu=1, 2, \dots$. Choosing $N=[\log 2\Delta C^{-1}/\log R^{-1}]$ we obtain

$$\sum_{\nu=1}^{\infty}\nu|a_\nu|^2=\sum_{1}^{N}+\sum_{N+1}^{\infty}\leqslant 2N(N+1)\,\Delta^2+C^2\sum_{N+1}^{\infty}\nu R^{-2\nu}.$$

Hence from (5.4) we deduce that

$$-2\operatorname{Re} a_1\leqslant C(\Delta\log\Delta)^2.$$

The constant C depends of course on the parameters determining D but C is uniformly bounded as soon as the parameters are suitably bounded.

Thus 2° is true if $k=1$. Using (4.1), Section 4 (p. 114) we can now prove it true for k if it is true for $k-1$. It is essential here that the widths in the modified iterative process decrease exponentially and that $\Sigma'(D)$ is compact. The details of the proof are then elementary. Lemma 5.1 is now proved.

Lemma 5.2. 1°. *We have* $\sup_{m\geqslant n}\Delta_m\leqslant C\Delta_n|\log\Delta_n|$, $n=0, 1, 2, \ldots$.

2°. *Let n' be the smallest index $>n$ such that $\Delta_{n'}<\frac{1}{2}\Delta_n$. Then*

$$n'-n\leqslant C\,(\log\Delta_n)^2,\; n=0, 1, 2, \ldots.$$

Proof. From the convergence of the process it follows that $\sup_{m\geqslant n}\Delta_m$ is attained for some (smallest) m, and we may suppose that $m\geqslant n+k$ since otherwise $\Delta_m<2^k\Delta_n$. Further $\Delta_{m-\nu}\geqslant 2^{-\nu}\Delta_m$, $\nu=0, 1, 2, \ldots$. If any one of the widths $\Delta_{m-\nu}$, $\nu=0, 1, \ldots, k-1$, satisfies the Q-condition it follows from Lemma 5.1 that

$$C(2^{1-k}\Delta_m)^2\leqslant\sum_{\nu=n}^{\infty}\mu_\nu\leqslant C(\Delta_n\log\Delta_n)^2, \tag{5.6}$$

and 1° follows. Suppose that none of these widths satisfies the Q-condition. Then the first predecessors ($n>n_0$) of m, $m-1, \ldots, m-k+1$ are all $<n$ since $\Delta_{(m-\nu)*}\geqslant Q\Delta_m 2^{1-k}=\Delta_m$ for $\nu=0, 1, \ldots, k-1$, $(Q=2^{k-1})$. Thus the k consecutive indices m, $m-1, \ldots, m-k+1>n$ all have predecessors $<n$. But this is impossible. Thus (5.6) is true and we have proved 1°.

To prove 2° we make two observations. First suppose that we have a chain $n\leqslant n_1<n_2<\ldots<n_p<n'$ such that Δ_{n_1} satisfies the Q-condition. Then we have

$$\sum_{\nu=1}^{p}\mu_{n_\nu}\geqslant pC\Delta_n^2. \tag{5.7}$$

We prove this by induction. If $p=1$ then (5.7) follows from Lemma 5.1, 1° ($\Delta_m\geqslant\Delta_n/2$, $n\leqslant m<n'$). We suppose that (5.7) is true for $p<p'$. Let $p''\geqslant 1$ be the largest number $\leqslant p'$ such that $\Delta_{n_{p''}}$ satisfies the Q-condition. Then we have $\Delta_{n_{p''}}\geqslant Q^{p'-p''}\Delta_{n_{p'}}$ and thus $\mu_{n_{p''}}\geqslant(p'-p''+1)C\Delta_n^2$. Using the induction hypothesis and $\sum_{\nu=1}^{p'}\mu_{n_\nu}\geqslant\sum_{\nu=1}^{p''-1}\mu_{n_\nu}+\mu_{n_{p''}}$ we deduce (5.7) for chains of length p'.

Secondly suppose that we have a chain $n\leqslant n_1<n_2<\ldots<n_p<n'$ such that no one of the widths Δ_{n_ν}, $1\leqslant\nu\leqslant p$, satisfies the Q-condition. Then we have

$$p\leqslant C\log|\log\Delta_n| \tag{5.8}$$

since in this case it follows that $\Delta_{n_1}\geqslant Q^{p-1}\Delta_{n_p}\geqslant\frac{1}{2}Q^{p-1}\Delta_n$. Now according to Lemma 5.2, 1° we have

$$\tfrac{1}{2}Q^{p-1}\Delta_n\leqslant C\Delta_n|\log\Delta_n|$$

which proves (5.8).

Now we prove 2°. Write $N_n=n'-n$ and $M_n=C\log|\log\Delta_n|$ (see (5.8)). We may suppose that M_n/N_n is small. Let E_i' be the chain $\{\nu\,|\,\nu\in E_i,\; n\leqslant\nu<n'\}$ and denote its length by N_i', $i=1, 2, \ldots, k-1$. We suppose that $N_i'>M_n$, $1\leqslant i\leqslant j$, and $N_i'\leqslant M_n$,

$j < i \leqslant k-1$. It follows from (5.8) that at least one width Δ_m has to satisfy the Q-condition where m is one of the M_n+1 first numbers of E'_i $(i \leqslant j)$. Applying (5.7) we obtain

$$\sum_{\nu \in E'_i} \mu_\nu \geqslant (N'_i - M_n)\, C \Delta_n^2. \tag{5.9}$$

Observing that
$$\sum_{i=1}^{j} N'_i - jM_n \geqslant N_n - (k-1)\, M_n$$

we deduce from (5.9) that

$$\sum_{\nu=n}^{\infty} \mu_\nu > \sum_{i=1}^{j} \sum_{\nu \in E'_i} \mu_\nu \geqslant C N_n \Delta_n^2.$$

On the other hand, we know from Lemma 5.1, 2° that

$$\sum_{\nu=n}^{\infty} \mu_\nu \leqslant C(\Delta_n \log \Delta_n)^2.$$

Thus
$$N_n \leqslant C\,(\log \Delta_n)^2.$$

Lemma 5.2 is now proved.

We now return to the problem posed in the beginning of this section: Given $\varepsilon > 0$ we let $\varepsilon' = \varepsilon^2$ (ε small). Then if $\Delta_N < \varepsilon'$ for some index N it follows from Lemma 5.2, 1°, that $\Delta_n < \varepsilon$, $n \geqslant N$. Hence this N would be a solution of our problem. To find such an index we successively determine indices n_ν such that $\Delta_{n_{\nu+1}} < \frac{1}{2}\Delta_{n_\nu}$, where $n_{\nu+1}$ is the smallest index $> n_\nu$ with this property, and $n_0 = 0$. Suppose that $\Delta_{n_\nu} \geqslant \varepsilon'$, $\nu \leqslant p$ and that $\Delta_{n_{p+1}} < \varepsilon'$. Applying Lemma 5.2, 2° we obtain

$$N = n_{p+1} = \sum_{\nu=0}^{p} (n_{\nu+1} - n_\nu) \leqslant C \sum_{\nu=0}^{p} (\log \Delta_{n_\nu})^2 \leqslant C \sum_{\nu=0}^{p} (\log (\varepsilon' 2^\nu))^2.$$

But obviously $p \leqslant C\,|\log \varepsilon'|$ and inserting $\varepsilon' = \varepsilon^2$ we have

$$N \leqslant C \sum_{\nu=0}^{p} (\log (\varepsilon' 2^\nu))^2 \leqslant C (\log \varepsilon^{-1})^3.$$

Of course the same estimate is true if the parallel slit domain is θ-angled. Let $\varphi_\theta(z)$ be the function of Theorem 3.1. The following theorem follows at once from the above (for Grötzsch's method, see p. 115).

Theorem 5.1. *Let $\varphi_\theta(z) = \lim_{n\to\infty} z_n(z)$, where the $z_n(z)$ are determined by Grötzsch's method. Then there exist constants C and q, $0 < q < 1$, independent of n such that*

$$|\varphi_\theta(z) - z_n(z)| \leqslant C q^{n^{\frac{1}{3}}}$$

in any fixed closed subset of D.

Remark 1. According to Theorem 3.1, the modified iterative process shows that all widths are $< \varepsilon$ after $N_\varepsilon = O\,(\log \varepsilon^{-1})$ mappings. Since the number N_ε here means a number of mappings of $(k-1)$-connected domains this estimate and the one above concerning Grötzsch's process are not comparable from a practical point of view $(k \geqslant 3)$. However, the modified process can be approximated as follows. For example, if $k=3$ we first perform N_1 mappings of simply connected domains involving slits which are numbered 1 and 2. If N_1 is large then these slits are "nearly" rectilinear. Then we perform N_2 mappings involving slits which are numbered for example 2 and 3 so that they become "nearly" rectilinear etc. Given $\varepsilon > 0$ we may seek a number $N_k(\varepsilon)$ such that all widths are $< \varepsilon$ after $N_k(\varepsilon)$ mappings (in the indicated manner) of simply connected domains. We indicate the proof of the following estimate:

$$N_k(\varepsilon) = O\left((\log \varepsilon^{-1})^{k-1}\right). \tag{5.10}$$

This estimate is true for $k=2$ (the proof of Theorem 3.1) and we suppose that it is true for $k-1$. Let D be a given k-connected domain and let $f(z) \in \Sigma'(D)$, $\Omega = f(D)$. By excluding one arbitrarily chosen boundary component of $\partial\Omega$ we obtain a $(k-1)$-connected domain Ω'. Consider the set of all such domains Ω' obtained under all mappings $f(z) \in \Sigma'(D)$. From the induction hypothesis and the compactness of $\Sigma'(D)$ it is obvious that it is possible to choose a number $N = N_{k-1}(\varepsilon^2) = O\left((\log \varepsilon^{-1})^{k-2}\right)$ such that the approximate iterative process described above gives slits of widths $< \varepsilon^2$ after N mappings starting from any one domain Ω'.

We write the approximate mapping of D as $\omega = f_n(f_{n-1}(\dots(z)\dots)) = F_n(z)$ where the f_ν refer to mappings of $(k-1)$-connected domains and where these mappings are composed of N mappings of simply connected domains. Let $D_\nu = F_\nu(D)$ and let Δ_ν be the maximal width of the slits constituting ∂D_ν. From the compactness of $\Sigma'(D)$ and the exponential rate of convergence of the modified process it follows that if $\Delta_\nu \geqslant \varepsilon$ for $\nu \leqslant n$ then $\Delta_n \leqslant Cq^n$, $0 < q < 1$ (ε is supposed to be small). Thus necessarily $n \leqslant C \log \varepsilon^{-1}$ and $N_k(\varepsilon) = nN_{k-1}(\varepsilon^2) = O\left((\log \varepsilon^{-1})^{k-1}\right)$.

Remark 2. We conclude this section with a description of an iterative technique which is intermediate between Grötsch's process and the modified iterative process.

Suppose for example that we want to map the k-connected domain D conformally onto a zero-angled parallel slit domain. Using induction we suppose that this mapping is possible for any domain of connectivity $\leqslant k-1$ and that these mappings have the extremal property of Corollary 4.1. With the notation of Section 2 we let $z_{n+1}(z_n) = z_n + a_1^{(n)}/z_n + \dots \in \Sigma'(D_*^{(n)})$ where $D_*^{(n)}$ is a domain of connectivity k_n, $1 \leqslant k_n \leqslant k-1$, bounded by k_n of the continua $C_1^{(n)}, C_2^{(n)}, \dots, C_k^{(n)}$. The choices of these can be made arbitrarily with the restriction that each index $1, 2, \dots, k$ must occur infinitely often. Let the width of $C_\nu^{(n)}$ be $\Delta_{\nu n}$, $\nu = 1, 2, \dots, k$ and let $\Delta_n = \operatorname{Max}_\nu \Delta_{\nu n}$. Suppose that $\overline{\lim}_{n\to\infty} \Delta_n > 0$. Then there exists a sequence $\{n_i\}$ such that $D_*^{(n_i)}$ is bounded by $C_{m_1}^{(n_i)}, C_{m_2}^{(n_i)}, \dots, C_{m_p}^{(n_i)}$, $(1 \leqslant p \leqslant k-1)$, (where the indices are fixed), and such that the maximal width of these continua is at least $\Delta > 0$. Then there exists $\Delta' > 0$ such that $\operatorname{Re} a_1^{(n_i)} \geqslant \Delta'$ because otherwise $\operatorname{Re} a_1^{(n_i)} \to 0$, $D_*^{(n_i)} \to D_*$ (at least for a subsequence which we suppose already chosen). Thus there exists $f(\zeta) = \zeta + a_1/\zeta + \dots \in \Sigma'(D_*)$ mapping D_*, which is not a parallel slit domain, onto a zero-angled parallel slit domain with $\operatorname{Re} a_1 = 0$, which contradicts the assumption that Corollary 4.1 is true.

We have
$$z_n(z) = z + \frac{\sum_{\mu=0}^{n} a_1^{(\mu)}}{z} + \dots$$

and it follows that $\operatorname{Re} \sum_{\mu=0}^{n} a_1^{(\mu)} \to \infty$ while, on the other hand, by the compactness of $\Sigma'(D)$ we have $\left|\sum_{\mu=0}^{n} a_1^{(n)}\right| \leq C$. This is a contradiction. Thus $\lim_{n\to\infty} z_n(z)$ converges and the extremal property for connectivity k follows at once.

II. On conformal mappings onto domains of general type

6. *Notation and definitions*

Let $L_\nu : \omega = \omega_\nu(t)$, $0 \leq t \leq 1$, $\nu = 1, 2, \ldots, k$ be simple closed, analytic curves in the ω-plane. A curve $L_\nu^* : \omega = a_\nu \omega_\nu(t) + b_\nu$, $0 \leq t \leq 1$, where $a_\nu > 0$ and b_ν are constants, will for short be referred to as a curve of *type* L_ν.

Let $C_\nu^* : z = z_\nu(t)$, $0 \leq t \leq 1$, $\nu = 1, 2, \ldots, k$ be k fixed and bounded continua in the z-plane. Let $C_\nu : z = z_\nu(t) + c_\nu$, where c_ν is a constant, $\nu = 1, 2, \ldots, k$. Writing $p = (c_1, c_2, \ldots, c_k)$ we make the

Definition 6.1. *We say that p is admissible if $\{C_\nu\}_1^k$ are the boundary components of a k-connected domain $D = D(p)$ such that the point at infinity belongs to D.*

Let p be admissible and let $d_{ij}(p) = d(C_i, C_j)$ be the distance between C_i and C_j $(i \neq j)$. Let A_i (independent of p) be the diameter of C_i. Then we write

$$d(p) = \min_{1 \leq i < j \leq k} d_{ij}(p), \quad A = \max_{1 \leq i \leq k} A_i, \quad a = \min_{1 \leq i \leq k} A_i.$$

Definition 6.2. *$D(p)$ is (d, B)-bounded if*

1° *p is admissible,*

2° *$d(p) \geq dA$, $(d > 0)$,*

3° *there exists b such that $z \in \bigcup_{\nu=1}^{k} C_\nu \Rightarrow |z - b| \leq B\, d(p)$.*

The aim of this chapter is to apply the iterative process to domains $D(p)$ suitably bounded in the sense of Definition 6.2 and prove the existence of $\omega = F(z) = \lim_{n\to\infty} z_n(z) \in \Sigma'(D(p))$ mapping $D(p)$ onto a domain bounded by k curves L_ν^* of type L_ν in such a way that L_ν^* corresponds to C_ν, $\nu = 1, 2, \ldots, k$.

7. *The iterative process and the main theorem*

We will here use the iterative process described in the introduction, (**B**, p. 101) in the following way. Let $z_{n+1}(z_n) \in \Sigma'(D_{\nu_n}^{(n)})$ map the *simply* connected domain $D_{\nu_n}^{(n)}$ bounded by $C_{\nu_n}^{(n)}$ $(\infty \in D_{\nu_n}^{(n)})$ onto a domain bounded by a curve $C_{\nu_n}^{(n+1)}$ of type L_{ν_n}, $1 \leq \nu_n \leq k$ (Riemann Mapping Theorem). As before $z_{n+1}(C_\nu^{(n)}) = C_\nu^{(n+1)}$, $\nu \neq \nu_n$ and $D^{(n+1)} = z_{n+1}(D^{(n)})$, $D^{(0)} = D(p)$, $z_0 = z$, $C_\nu^{(0)} = C_\nu$, $\nu = 1, 2, \ldots, k$. Further we choose ν_n so that $\nu_n - 1 \equiv n \pmod{k}$. This particular choice is not necessary but it provides some formal simplification. We observe that $z_n(z_m) = z_n(z_{n-1}(\ldots(z_m)\ldots)) \in \Sigma'(D^{(m)})$, $(n > m)$. If $F(z) = \lim_{n\to\infty} z_n(z)$ exists then necessarily $F(z) \in \Sigma'(D(p))$ maps $D(p)$ onto a domain bounded by $\bigcup_{\nu=1}^{k} L_\nu^*$ where L_ν^* corresponds to C_ν, $\nu = 1, 2, \ldots, k$.

Theorem 7.1. *There exists a number $d > 0$ depending only on $\{C_\nu^*\}_1^k$, $\{L_\nu\}_1^k$ and B such that $\lim_{n\to\infty} z_n(z)$ exists for any (d, B)-bounded domain $D(p)$.*

8. Lemmas

For the proof of Theorem 7.1 we need three lemmas.

Lemma 8.1. *Let $D(p)$ be (d, B)-bounded and let $\omega = F(z; p) \in \Sigma'(D(p))$ map $D(p)$ onto $D'(p)$. Let $d'(p)$, $A'(p)$ and $a'(p)$ have the same meaning with respect to $D'(p)$ as $d(p)$, A and a have with respect to $D(p)$. Then there exist constants $N_i > 0$, $i = 1, 2, 3$, independent of p and F such that*

$$d'(p) > N_1 d(p), \tag{8.1}$$

$$A'(p) < N_2 A, \tag{8.2}$$

$$a'(p) > N_3 a. \tag{8.3}$$

Proof. We may suppose that $A = 1$ and $b = 0$ (Definition 6.2). Suppose that (8.1) is false. Then there exist sequences $\{p_\nu\}_1^\infty$ and $\{F_\nu(z; p_\nu)\}_1^\infty$, $F_\nu \in \Sigma'(D(p_\nu))$ such that $d'(p_\nu)/d(p_\nu) \to 0$ when $\nu \to \infty$. Either we can select a subsequence $\{p_{\nu_\mu}\}_1^\infty$ such that $p_{\nu_\mu} \to p_0$ or $d(p_\nu) \to \infty$ when $\nu \to \infty$.

In the former case we may suppose $\{p_{\nu_\mu}\}_1^\infty$ selected so that $F_{\nu_\mu}(z; p_{\nu_\mu}) \to F_0(z; p_0) \in \Sigma'(D(p_0))$ when $\mu \to \infty$. Then $d'(p_0) > 0$ which contradicts $d'(p_0)/d(p_0) = 0$.

In the latter case $G_\nu(\zeta) = d(p_\nu)^{-1} F_\nu(\zeta d(p_\nu)) \in \Sigma'(S_\nu)$ where $\zeta \in S_\nu \Leftrightarrow \zeta d(p_\nu) \in D(p_\nu)$. Thus S_ν is a domain such that $\zeta \in \partial S_\nu \Rightarrow |\zeta| \leqslant B$ and such that the boundary components have distances $\geqslant 1$ from one another. As above this gives a contradiction. Thus (8.1) is true.

We prove (8.2) and (8.3) in a similar way. If $p_{\nu_\mu} \to p_0$ the argument is the same. In the case $d(p_\nu) \to \infty$ there is no loss of generality in supposing C_i to be the circle $|z| = r_i$ and C_i' corresponding to C_i to be $|\omega - \omega_\nu| = r_{i\nu}'$. Also $F_\nu(z; p_\nu)$ is univalent and analytic in $r_i \leqslant |z| \leqslant d(p_\nu)$ and as in the proof of (8.1) it is clear that there exist constants $m_1 > 0$ and m_2 independent of p and F such that

$$m_1 d(p_\nu) \leqslant |F_\nu(z; p_\nu) - \omega_\nu| \leqslant m_2 d(p_\nu), \quad |z| = d(p_\nu).$$

Thus according to a well-known property

$$\frac{r_{i\nu}'}{m_2 d(p_\nu)} \leqslant \frac{r_i}{d(p_\nu)} \leqslant \frac{r_{i\nu}'}{m_1 d(p_\nu)}.$$

Thus
$$m_1 r_i \leqslant r_{i\nu}' \leqslant m_2 r_i$$

and this proves (8.2) and (8.3). Lemma 8.1 is proved.

Lemma 8.2. *Let D be a simply connected domain on the z-sphere, $\infty \in D$. Let the diameter of its boundary ∂D be $< R$ and let $f(z) \in \Sigma'(D)$. Then*

$$|f(z) - z| \leqslant \frac{3R^2}{\operatorname*{Max}_{a \in \partial D} |z - a|}, \quad z \in D.$$

Proof. Choose any $a \in \partial D$. Then $\partial D \subset D_R$ where D_R is the disc $|z-a|<R$. It is a well-known fact that $|f(z)-a| \leqslant 2R$, $z \in D \cap D_R$. Also $g(z, a)=(z-a)(f(z)-z)$ is analytic in D and we have

$$|g(z, a)| \leqslant 3R^2, \quad z \in D \cap D_R.$$

From the maximum principle it follows that this holds for $z \in D$. Since a is arbitrary this means that

$$|f(z)-z| \leqslant \frac{3R^2}{\operatorname*{Max}_{a \in \partial D} |z-a|}.$$

Lemma 8.2 is proved.

Lemma 8.3. 1°. *Let Λ be a family of simple closed analytic curves of type L such that $0<m \leqslant l(\Gamma) \leqslant M<\infty$, where $l(\Gamma)$ is the length of $\Gamma \in \Lambda$.*

2°. *Let $U_\Gamma=\{z \mid d(z, \Gamma) \leqslant \varrho\}$ where $\Gamma \in \Lambda$, $\varrho>0$ and $d(z, \Gamma)$ is the distance between z and Γ.*

3°. *Let $\mathfrak{F}_\delta(\Gamma)=\mathfrak{F}_\delta$ be the family of functions $f(z)=z+r(z)$, analytic and univalent in U_Γ and such that $|r(z)| \leqslant \delta$, $z \in U_\Gamma$.*

4°. *Let D^* be the domain bounded by $\Gamma^*=f(\Gamma)$, $\infty \in D^*$ and let $g_{\Gamma^*}(z)=g(z) \in \Sigma'(D^*)$ map D^* onto Ω which is bounded by a curve Γ' of type L.*

Then there exist $\delta_0>0$ and K depending only on L, m, M and ϱ such that for all $\Gamma \in \Lambda$ and for all $f \in \mathfrak{F}_\delta$; $\delta \leqslant \delta_0$ we have

$$|g(z)-z| \leqslant K\delta, \quad z \in \Gamma^*.$$

Proof. We may without loss of generality suppose that $z \in \Gamma \Rightarrow |z| \leqslant M$ for any $\Gamma \in \Lambda$. If δ_0 is chosen sufficiently small then $\mathfrak{F}_\delta$, $\delta \leqslant \delta_0$, is necessarily compact. Suppose that Lemma 8.3 is false. Then it is possible to select a sequence $\{\Gamma_\nu, \delta_\nu, f_\nu, g_\nu, z_\nu\}_1^\infty$ such that $\Gamma_\nu \in \Lambda$, $\delta_\nu \to 0$ $(\delta_\nu \leqslant \delta_0)$, $f_\nu \in \mathfrak{F}_{\delta_\nu}$, $f_\nu(\Gamma_\nu)=\Gamma_\nu^*$, $\omega=g_\nu(z)=g_{\Gamma_\nu^*}(z)$, $z_\nu \in \Gamma_\nu^*$ and

$$\delta_\nu^{-1}|g_\nu(z_\nu)-z_\nu| \to \infty. \tag{8.4}$$

Let D_ν and D_ν^* be the domains bounded by Γ_ν and Γ_ν^* respectively $(\infty \in D_\nu, D_\nu^*)$ and let $g(D_\nu^*)=\Omega_\nu$, $\partial\Omega_\nu=L_\nu$, where L_ν is of type L.

Let $W=h_\nu(\omega)$ be the analytic function conformally mapping Ω_ν onto the interior of the unit circle such that $h_\nu(\omega)=c_{1\nu}/\omega+c_{2\nu}/\omega^2+\ldots$, $c_{1\nu}>0$ near the point at infinity. Then $\zeta=h_\nu^*(z)=h_\nu(\alpha_\nu z+\beta_\nu)$ with suitable constants $\alpha_\nu>0$ and β_ν is the corresponding function with respect to the domain D_ν.

Since Γ_ν is of type L (analytic) and $0<m \leqslant l(\Gamma_\nu) \leqslant M$ there exists a number $\Delta>0$ independent of ν such that $h_\nu^*(z)$ is analytic and univalent in a domain containing D_ν and in particular all points interior to Γ_ν and distant at most Δ from Γ_ν. For $\nu \geqslant \nu_0$ we have $\operatorname{Max}_{z \in \Gamma_\nu^*} d(z, \Gamma_\nu)<\frac{1}{2}\Delta$. For $\nu \geqslant \nu_1 \geqslant \nu_0$, $h_\nu^*(\Gamma_\nu^*)$ is star-shaped with respect to the origin (since $|r_\nu'(z)|$, $z \in \Gamma_\nu$, becomes arbitrarily small). Thus $h_\nu^*(L_\nu^*)$ may be represented in polar coordinates

$$R_\nu(\varphi)e^{i\varphi} = h_\nu^*(f_\nu(h_\nu^{*-1}(e^{i\varphi}))), \quad -\pi < \varphi \leqslant \pi,$$

where $R_\nu(\varphi)$ is absolutely continuous in $-\pi < \varphi \leqslant \pi$, $|R_\nu'(\varphi)/R_\nu(\varphi)| \leqslant K_1\delta_\nu$ and $|R_\nu(\varphi)-1| \leqslant K_2\delta_\nu$ where K_1 and K_2 are independent of ν. Then an elementary modification of Satz 3, ([3], p. 261) gives

$$|h_\nu^*(g_\nu^{-1}(h_\nu^{-1}(W))) - W| \leqslant K_3\delta_\nu, \; |W| \leqslant 1 \tag{8.5}$$

where K_3 is independent of ν. This is equivalent to

$$|h_\nu(\alpha_\nu g_\nu^{-1}(\omega) + \beta_\nu) - h_\nu(\omega)| \leqslant K_3\delta_\nu, \, \omega \in L_\nu. \tag{8.6}$$

From (8.5) it follows in particular that for $\nu \geqslant \nu_2 \geqslant \nu_1$

$$|\alpha_\nu - 1| \leqslant K_4\delta_\nu, \; |\beta_\nu| \leqslant K_5\delta_\nu. \tag{8.7}$$

For $\nu \geqslant \nu_3 \geqslant \nu_2$ the straight line segment connecting $h_\nu(\omega)$ and $h_\nu(\alpha_\nu g_\nu^{-1}(\omega)+\beta_\nu)$, $\omega \in L_\nu$, is the image of an analytic curve s_ν connecting ω and $\omega_1 = \alpha_\nu g_\nu^{-1}(\omega)+\beta_\nu$, and $|h_\nu'(\omega)| \geqslant K_6 > 0$, $\omega \in s_\nu$, where K_6 is independent of ν. Thus using (8.6) we obtain

$$K_3\delta_\nu \geqslant |h_\nu(\omega_1) - h_\nu(\omega)| = \int_{s_\nu} |h_\nu'(\omega)| \, |d\omega| \geqslant K_6 |\omega_1 - \omega|. \tag{8.8}$$

Using $|z| \leqslant M + 2\delta_0$ on Γ_ν^* we finally deduce from (8.7) and (8.8) that

$$|g_\nu(z) - z| \leqslant K_7\delta_\nu, \, z \in \Gamma_\nu^*, \tag{8.9}$$

where K_7 is independent of ν. However (8.9) contradicts (8.4). Lemma 8.3 is proved.

Corollary 8.1. *With the hypothesis and notation of Lemma* 8.3

$$|g(z) - z| \leqslant \frac{MK\delta}{\operatorname*{Max}_{a \in \Gamma^*} |z-a|}, \quad z \in D^*, \quad \delta \leqslant \delta_0 \left(\leqslant \frac{M}{4} \right).$$

Proof. Choose any $a \in \Gamma^*$. The function $F(z, a) = (z-a)(g(z)-z)$ is analytic in D^* and according to Lemma 8.3 we have

$$|F(z, a)| \leqslant MK\delta, \quad z \in \Gamma^*.$$

The statement follows as in the proof of Lemma 8.2.

9. *Proof of the main theorem*

We choose $d' > 0$ and $B > 0$ and consider all (d', B)-bounded domains $D(p)$ (B large). Without loss of generality we may assume that $A = 1$ and $b = 0$ (see Definition 6.2). In the iterative process the curves $C_{\nu_n}^{(n+1)}$, $n = 0, 1, 2, \ldots$, are of type L_{ν_n}.

Then according to Lemma 8.1 it is possible to determine constants m_{ν_n}, M_{ν_n} such that $0<m_{\nu_n}\leqslant l(C_{\nu_n}^{(n+1)})\leqslant M_{\nu_n}$ for all (d', B)-bounded domains $D(p)$. Then we apply Lemma 8.3 with some arbitrarily fixed $\varrho>0$ (see 2°) to the family Λ_ν of curves Γ_ν of type L_ν for which $m_\nu\leqslant l(\Gamma_\nu)\leqslant M_\nu$. We obtain constants K_ν and $\delta_{0\nu}$ $(\nu=1, 2, ..., k)$ corresponding to K and δ_0 of the lemma. Write $M=\operatorname{Max}_\nu M_\nu$, $K=\operatorname{Max}_\nu K_\nu$, $\delta_0'=\operatorname{Min}_\nu \delta_{0\nu}$ and $\delta_0=\min(\delta_0', M/4, \varrho)$.

Let $d_n(p)=\operatorname{Min}_{1\leqslant\nu<\mu\leqslant k} d(C_\nu^{(n)}, C_\mu^{(n)})$ and

$$U_n=\{\zeta \mid d(\zeta, C_{\nu_{n-1}}^{(n)})\leqslant\varrho\},\quad n=1,2,\dots,$$

and take any q, $0<q<1$.

The condition

$$\inf_n d_n(p)\geqslant MKq^{-1}(k-1)+3\varrho \tag{9.1}$$

can be realized according to Lemma 8.1 for all (d'', B)-bounded $D(p)$ if $d''\geqslant d'$ is sufficiently large.

Writing $z_k=z_k(z_{k-1}(\dots(z_i)\dots))$ for short we can satisfy the conditions

$$|z_k-z_i|\leqslant\frac{k-i}{k-1}\delta_0,\ z_i\in U_i,\quad i=1,2,\dots,k-1, \tag{9.2}$$

according to Lemma 8.2 for all (d, B)-bounded $D(p)$ if $d\geqslant d''$ is sufficiently large.

Now according to Lemma 8.3 we have

$$|z_{k+1}-z_k|\leqslant K\delta_0,\ z_k\in C_{\nu_k}^{(k)}.$$

We observe that the distance between $z_k(U_i)$ and $C_{\nu_k}^{(k)}$ is at least $d_k(p)-p-2(k-i)\,\delta_0/(k-1)\geqslant MKq^{-1}(k-1)$, $i=2, 3, ..., k-1$, and thus according to Corollary 8.1 and (9.2) we have

$$|z_{k+1}-z_i|\leqslant\frac{k+1-i}{k-1}\delta_0,\ z_i\in U_i,\quad i=2,3,\dots,k-1,$$

$$|z_{k+1}-z_k|\leqslant\frac{1}{k-1}q\delta_0,\ z_k\in U_k,$$

and by repetition

$$|z_{2k}-z_{i+k}|\leqslant\frac{k-i}{k-1}q\delta_0,\ z_{i+k}\in U_{i+k},\quad i=1,2,\dots,k-1. \tag{9.3}$$

The conditions for a repetition of the argument from (9.2) leading to (9.3) are *a fortiori* satisfied, now with (9.3) as starting point and $\delta_0 q$ replacing δ_0. Thus

$$|z_{3k}-z_{i+2k}|\leqslant\frac{k-i}{k-1}q^2\delta_0,\ z_{i+2k}\in U_{i+2k},\quad i=1,2,\dots,k-1,$$

and inductively

$$|z_{(m+1)k}-z_{i+mk}|\leqslant\frac{k-i}{k-1}q^m\delta_0,\ z_{i+mk}\in U_{i+mk},\quad i=1,2,\dots,k-1, \tag{9.4}$$

It follows from (9.4) and Lemma 8.3 that

$$|z_{n'}(z)-z_n(z)|\leqslant q^{[n/k]-1}\delta_0(K+1)(1-q)^{-1},\ n'>n\geqslant k,\ z\in\partial D(p).$$

Since $0<q<1$ it finally follows that $\lim_{n\to\infty} z_n(z)$ exists for all (d, B)-bounded domains $D(p)$. Theorem 7.1 is now proved.

III. Two eigenvalue problems

10. Statement of the problems

Let Γ_1 and Γ_2 be two disjoint and bounded rectilinear slits considered on the z-sphere making the angles α_1 and α_2 respectively with the positive real axis $(0\leqslant\alpha_\nu<\pi,\ \nu=1,2)$ and bounding a doubly connected domain D. Let D_ν be the simply connected domain bounded by Γ_ν, $\nu=1,2$.

The first problem is:

A. *Find $f_\nu(z)$, non-constant, bounded and analytic in D_ν, $\nu=1,2$ and a number μ such that*

$$\begin{aligned}&\operatorname{Im} e^{-i\alpha_1}f_1(z)=\operatorname{Im} e^{-i\alpha_1}f_2(z),\quad z\in\Gamma_1,\\ &\operatorname{Im} e^{-i\alpha_2}f_2(z)=\mu\operatorname{Im} e^{-i\alpha_2}f_1(z),\ z\in\Gamma_2.\end{aligned}\tag{10.1}$$

It is to be understood that, for example, $\operatorname{Im} e^{-i\alpha_1}f_1(z)$ takes equal values on the edges of Γ_1, $\operatorname{Im} e^{-i\alpha_1}f_1(z^+)=\operatorname{Im} e^{-i\alpha_1}f_1(z^-)=\operatorname{Im} e^{-i\alpha_1}f_2(z)$. With the aid of a linear transformation $kz+l\to z$ we may in various ways normalize D. Thus there are 4 real parameters $p=(p_1,p_2,p_3,p_4)$ which are essential for the problem. We sometimes write $D(p)$ instead of D and do similarly with the other notation.

If $$\Gamma_1=\Gamma=\{re^{i\theta}\,|\,0<a\leqslant r\leqslant b,\ \theta=0\}$$

and $$\Gamma_2=e^{i\alpha}\Gamma=\{re^{i\theta}\,|\,a\leqslant r\leqslant b,\ \theta=\alpha\},\ 0<\alpha\leqslant\pi,\ z=re^{i\theta},$$

then we also take into consideration the following problem which is closely related to Problem **A**:

B. *Find $f(z)$, non-constant, bounded and analytic outside Γ and a number μ such that*

$$\operatorname{Im} f(r)=\mu\operatorname{Im} e^{-i\alpha}f(re^{i\alpha}),\ a\leqslant r\leqslant b.\tag{10.2}$$

Letting Γ_ν be a slit on the straight line L_ν, $\nu=1,2$, we may in **A** suppose that

$$\operatorname{Re} e^{-i\alpha_\nu}f_\nu(z)=0,\ z\in L_\nu-\Gamma_\nu,\ \nu=1,2,\tag{10.3}$$

unless $\mu=\cos^{-2}(\alpha_2-\alpha_1)>1$, $|\alpha_2-\alpha_1|\neq 0,\pi/2$. This is a consequence of the easily verified fact that **A** has trivial solutions $f_\nu(z)=e^{i\alpha_\nu}(a_\nu+ib_\nu)$, $\nu=1,2$, where a_1 and a_2 can be arbitrarily chosen with the above exception. In order not to complicate the discussion we accept (10.3) as a restriction in the exceptional case. By a similar argument

$$\operatorname{Re} f(r)=0,\ r<a,\ r>b \tag{10.4}$$

in **B**.

We do not solve the problems **A** and **B** in general but simply draw some conclusions of general character and discuss these problems in certain special cases.

11. Collinear slits

Theorem 11.1. *Let Γ_1 and Γ_2 be slits on the real axis, and let R be the modulus of D. Then there exist solutions of* **A** *with $\mu=\mu_j=((R^j+R^{-j})/2)^2$, $j=1, 2, \ldots$. There are no other solutions.*

Proof. If we set $\alpha_1, \alpha_2=0$ then (10.1) and (10.3) become

$$\begin{aligned} &\operatorname{Im} f_1(z)=\operatorname{Im} f_2(z),\ z\in\Gamma_1,\\ &\operatorname{Im} f_2(z)=\mu \operatorname{Im} f_1(z),\ z\in\Gamma_2, \end{aligned} \tag{11.1}$$

$$\bar{f}_\nu(\bar{z})=-f_\nu(z). \tag{11.2}$$

Let S_0 be the z-sphere slit along Γ_1 and Γ_2. We construct a Riemann surface **R** by taking an infinity of such spheres each joined to two others, one crosswise along Γ_1 and one crosswise along Γ_2. We denote by $P=P(z)$ any point $\in \mathbf{R}$ lying over $z\in S_0$. Now every function $F(z)=a_1f_1(z)+a_2f_2(z)$ (where a_1 and a_2 are complex constants) originally defined on S_0 can be analytically continued to the whole of **R**. In fact, according to (11.1), $f_1(z)$ can be continued over Γ_1 so that on the sphere S_1 joined to S_0 along Γ_1 we have

$$f_1(P(z))=\bar{f}_1(\bar{z})+f_2(z)-\bar{f}_2(\bar{z}),\ P\in S_1$$

or according to (11.2)

$$f_1(P(z))=-f_1(z)+2f_2(z),\ P\in S_1. \tag{11.3}$$

In the same way we deduce that

$$f_2(P(z))=-f_2(z)+2\mu f_1(z),\ P\in S_2, \tag{11.4}$$

where S_2 is the sphere joined to S_0 along Γ_2. Then (11.3) and (11.4) imply that the continuation is possible to the whole of **R**.

Let $F(z)=(\tau+1)f_1(z)-2f_2(z)$ where τ, $|\tau|\geqslant 1$ satisfies the equation

$$\tau^2+2\tau(1-2\mu)+1=0. \tag{11.5}$$

Then the following is easily verified. Continuing $F(z)$ first over Γ_1 to S_1 then over Γ_2 to S_{12} gives

$$F(P(z))=\tau F(z),\ P\in S_{12}. \tag{11.6}$$

Continuing $F(z)$ first over Γ_2 to S_2 then over Γ_1 to S_{21} gives

$$F(P(z))=\tau^{-1}F(z),\ P\in S_{21}. \tag{11.7}$$

R can be conformally mapped onto the ω-sphere punctured at the origin and infinity so that Γ_1 corresponds to $|\omega|=R>1$ and Γ_2 to $|\omega|=1$ (see e.g. [16] pp. 424–425).

Let $P = g(\omega)$ be the inverse mapping. Then $f(\omega) = F(g(\omega))$ is analytic in $0 < |\omega| < \infty$. From (11.6) and (11.7) it follows that

$$f(R^{2\nu}\omega) = \tau^{\nu} f(\omega), \; \nu = 0, \pm 1, \pm 2, \ldots. \tag{11.8}$$

Suppose that $0 \leqslant \mu \leqslant 1$. Then from (11.5), $|\tau| = 1$ and according to (11.8), $f(\omega)$ is bounded in $0 < |\omega| < \infty$ and this implies that $f(\omega) \equiv$ const. It follows that $f_1(z) \equiv$ const. and $f_2(z) \equiv$ const. This is of course merely a verification of what can be directly concluded from the maximum principle for harmonic functions.

Suppose that $\mu < 0$ or $\mu > 1$. Then $|\tau| > 1$. By (11.8), $f(\omega)$ is bounded in $0 < |\omega| < 1$ and is thus analytic at $\omega = 0$. As regards the behaviour near infinity, (11.8) means that $f(\omega)$ grows at most polynomially when $\omega \to \infty$ and is thus necessarily a polynomial. It easily follows that the only non-trivial possibilities are

$$f(\omega) = C_j \omega^j, \; \tau_j = R^{2j}, \; C_j \neq 0, \; j = 1, 2, \ldots, \tag{11.9}$$

and so from (11.5) we have

$$\mu_j = \left(\frac{R^j + R^{-j}}{2}\right)^2 > 1, \quad j = 1, 2, \ldots. \tag{11.10}$$

Thus $\mu < 0$ is not possible.

Let $\omega(z)$ be the function that maps S_0 conformally onto $1 < |\omega| < R$ (Γ_2 corresponds to $|\omega| = 1$) so that $\operatorname{Im} \omega(z) = 0$, z real and $z \notin \Gamma_1 \cup \Gamma_2$. For any fixed $j = 1, 2, \ldots$, the two functions

$$\begin{aligned} f_{1j}(z) &= \frac{i}{1 + R^{2j}} \left(\omega(z)^j + \omega(z)^{-j}\right), \\ f_{2j}(z) &= \frac{i}{2} \left(R^{-2j} \omega(z)^j + \omega(z)^{-j}\right) \end{aligned} \tag{11.11}$$

satisfy the relations (11.1) with $\mu = \mu_j$. This is easily verified. In particular

$$\operatorname{Im} f_{\nu j}(z^+) = \operatorname{Im} f_{\nu j}(z^-), \; z^+, z^- \in \Gamma_{\tilde{\nu}}, \; \nu = 1, 2, \tag{11.12}$$

where z^+ and z^- are opposite points on the edges of $\Gamma_{\tilde{\nu}}$ and $\tilde{\nu} \neq \nu$, $\tilde{\nu} = 1$ or 2. Further it is readily verified that

$$\operatorname{Re} f_{\nu j}(z) = 0, \; z \in L - \Gamma_{\nu}, \; \nu = 1, 2, \tag{11.13}$$

where L is the real axis. From (11.12) and (11.13) it follows in particular that $f_{\nu j}(z)$ is analytic in D_ν, $\nu = 1, 2$. Thus the functions (11.11) are a solution of **A** with $\mu = \mu_j$. Theorem 11.1 is proved.

12. Orthogonal slits

Theorem 12.1. *Let* $\Gamma_1 = \{z \mid 0 < a_1 \leqslant x \leqslant b_1, \; y = 0\}$ *and* $\Gamma_2 = \{z \mid x = 0, \; 0 \leqslant a_2 \leqslant y \leqslant b_2\}$, $z = x + iy$. *If* **A** *has solutions then necessarily* $|\mu| > 2$.

Proof. Writing $f_\nu(z)=u_\nu(z)+iv_\nu(z)$, $\nu=1,2$, we obtain (10.1) and (10.3) in the form

$$\left.\begin{aligned} v_1(z)&=v_2(z), \quad z\in\Gamma_1,\\ u_2(z)&=\mu u_1(z), \quad z\in\Gamma_2,\\ u_1(\infty)&=v_2(\infty)=0.\end{aligned}\right\} \tag{12.1}$$

Let
$$\gamma_\nu(t)=[(t-a_\nu)(b_\nu-t)]^{-\frac{1}{2}}, \quad \nu=1,2,$$

and
$$\|h\|_\nu^2=\frac{1}{\pi}\int_{a_\nu}^{b_\nu} h^2\gamma_\nu\,dt, \quad \nu=1,2,$$

for h real-valued and continuous on Γ_ν. Since u_1 and v_2 are the normalized conjugates of v_1 and u_2 respectively (i.e. $u_1(\infty)=v_2(\infty)=0$) we have

$$\left.\begin{aligned}\|u_1\|_1&\leqslant\|v_1\|_1\\ \|v_2\|_2&\leqslant\|u_2\|_2.\end{aligned}\right\} \tag{12.2}$$

By Schwarz's inequality ($y\geqslant 0$; $u_1(x)=u_1(x+i0)$) we have

$$u_1(iy)^2=\left[\operatorname{Im}\frac{1}{\pi}\int_{a_1}^{b_1}\frac{u_1(x)}{x-iy}\,dx\right]^2\leqslant\frac{1}{\pi}\int_{a_1}^{b_1}\frac{y}{x^2+y^2}\,dx\,\frac{1}{\pi}\int_{a_1}^{b_1}\frac{y}{x^2+y^2}\,u_1(x)^2\,dx$$
$$\leqslant\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{y}{x^2+y^2}\,u_1(x)^2\,dx.$$

Thus
$$\|u_1\|_2^2\leqslant\frac{1}{2\pi}\int_{a_1}^{b_1}u_1(x)^2\left[\frac{1}{\pi}\int_{a_2}^{b_2}\frac{y}{x^2+x^2}\,\gamma_2(y)\,dy\right]dx\leqslant\tfrac{1}{2}M_1^2\,\|u_1\|_1^2, \tag{12.3}$$

where
$$M_1^2=\operatorname*{Max}_{a_1\leqslant x\leqslant b_1}\frac{\gamma_1(x)^{-1}}{\pi}\int_{a_2}^{b_2}\frac{y}{x^2+y^2}\,\gamma_2(y)\,dy.$$

In the same way we obtain
$$\|v_2\|_1^2\leqslant\tfrac{1}{2}M_2^2\,\|v_2\|_2^2, \tag{12.4}$$

where
$$M_2^2=\operatorname*{Max}_{a_2\leqslant y\leqslant b_2}\frac{\gamma_2(y)^{-1}}{\pi}\int_{a_1}^{b_1}\frac{x}{y^2+x^2}\,\gamma_1(x)\,dx.$$

Observing that the function $(t-a)\,(t^2+b^2)^{-\frac{1}{2}}$, $a\geqslant 0$, is increasing for $t\geqslant 0$ we have

$$M_1^4=\operatorname*{Max}_{a_1\leqslant x\leqslant b_1}\left(\operatorname{Im}\frac{\gamma_2(-ix)}{\gamma_1(x)}\right)^2\leqslant\operatorname*{Max}_{a_1\leqslant x\leqslant b_1}\frac{x-a_1}{(x^2+a_2^2)^{\frac{1}{2}}}\,\frac{b_1-x}{(x^2+b_2^2)^{\frac{1}{2}}}\leqslant\frac{(b_1-a_1)^2}{(b_1^2+a_2^2)^{\frac{1}{2}}\,(a_1^2+b_2^2)^{\frac{1}{2}}}$$

and in the same way
$$M_2^4\leqslant\frac{(b_2-a_2)^2}{(b_2^2+a_1^2)^{\frac{1}{2}}\,(a_2^2+b_1^2)^{\frac{1}{2}}}$$

and thus
$$(M_1M_2)^2\leqslant\frac{b_1-a_1}{(b_1^2+a_2^2)^{\frac{1}{2}}}\,\frac{b_2-a_2}{(b_2^2+a_1^2)^{\frac{1}{2}}}<\left(1-\frac{a_1}{b_1}\right)\left(1-\frac{a_2}{b_2}\right)<1. \tag{12.5}$$

From (12.1)–(12.4) it follows that

$$\|v_1\|_1 = \|v_2\|_1 \leqslant 2^{-\frac{1}{2}} M_2 \|v_2\|_2 \leqslant 2^{-\frac{1}{2}} M_2 \|u_2\|_2 = |\mu| 2^{-\frac{1}{2}} M_2 \|u_1\|_2$$
$$\leqslant \tfrac{1}{2}|\mu| M_1 M_2 \|u_1\|_1 \leqslant \tfrac{1}{2}|\mu| M_1 M_2 \|v_1\|_1.$$

From this and (12.5) we finally have

$$|\mu| \geqslant 2(M_1 M_2)^{-1} > 2.$$

Theorem 12.1 is proved.

13. General conclusions

In (10.1) we normalize D so that Γ_1 is a fixed slit on the real axis of length 1. There is no loss of generality in supposing the length l of Γ_2 to be $\leqslant 1$. Let d be the distance between Γ_1 and Γ_2. Let E' be the set of parameters p (see p. 126) for which **A** has solutions and E'_k the subset of E' for which $|\mu| > k$ holds for every solution. Let E'' be the set for which **A** has no solutions and let $E_k = E'_k \cup E''$. We sum up some general conclusions in a theorem:

Theorem 13.1. *For Problem* **A** *the following is true:*

1°. $p \in E' \Rightarrow \mu \neq 1$.

2°. $p \in E' \Rightarrow |\mu| \geqslant d^4 l^{-1}(d + l + \frac{1}{2})^{-1}(d + l/2 + 1)^{-1}$.

3°. $p \in E'$ *and* Γ_1 *and* Γ_2 *are parallel* $\Rightarrow |\mu| > 1$.

4°. E_1 *is open.*

Proof of 1°. Suppose that **A** has a solution with $\mu = 1$ and functions $f_1(z)$ and $f_2(z)$. Then the function $F(z) = f_1(z) - f_2(z)$ is analytic and bounded in D and $\operatorname{Im} e^{-i\alpha_\nu} F(z) = 0$ on Γ_ν, $\nu = 1, 2$. This clearly implies that $F(z) \equiv \text{const.}$ and hence $f_\nu(z) \equiv \text{const.}$, $\nu = 1, 2$, which gives a contradiction.

Proof of 2°. Let $v(z)$ be a real-valued continuous function on Γ_ν. We write

$$\|v\|_\nu = \operatorname*{Max}_{z', z'' \in \Gamma_\nu} |v(z') - v(z'')|, \quad \nu = 1, 2. \tag{13.1}$$

Further we write $\gamma(z) = [(z-a)(b-z)]^{-\frac{1}{2}}$ where a and $b(= a+1)$ are the end-points of Γ_1. Since

$$f_1(z) = \frac{\gamma(z)^{-1}}{\pi} \int_a^b \frac{\operatorname{Im} f_1(x)}{x - z} \gamma(x)\, dx, \quad z \in D_1, \tag{13.2}$$

it follows elementarily that

$$\|\operatorname{Im} e^{-i\alpha_2} f_1(z)\|_2 \leqslant \frac{l}{d^2}(d + l + \tfrac{1}{2}) \|\operatorname{Im} f_1(z)\|_1 \tag{13.3}$$

and in a similar way

$$\|\operatorname{Im} f_2(z)\|_1 \leq \frac{1}{d^2}\left(d+\frac{l}{2}+1\right)\|\operatorname{Im} e^{-i\alpha_2} f_2(z)\|_2. \tag{13.4}$$

From (10.1), (13.3) and (13.4) it now follows that

$$\|\operatorname{Im} f_1(z)\|_1 \leq \frac{1}{d^2}\left(d+\frac{l}{2}+1\right)\|\operatorname{Im} e^{-i\alpha_2} f_2(z)\|_2,$$

$$\|\operatorname{Im} e^{-i\alpha_2} f_2(z)\|_2 \leq |\mu| \frac{l}{d^2}\left(d+l+\tfrac{1}{2}\right)\|\operatorname{Im} f_1(z)\|_1.$$

Thus $|\mu| \geq d^4 l^{-1}(d+l+\frac{1}{2})^{-1}(d+l/2+1)^{-1}$ and 2° is proved.

Remark. In particular we obtain from this that

$$d > \tfrac{1}{2}(1+\sqrt{7}) \Rightarrow |\mu| > 1.$$

Proof of 3°. We have $\alpha_1 = \alpha_2 = 0$. Letting $\|v\|_\nu$, $\nu = 1, 2$, have the same meaning as above we deduce from the maximum principle for harmonic functions that

$$\|\operatorname{Im} f_1(z)\|_2 \leq q_1 \|\operatorname{Im} f_1(z)\|_1,$$

$$\|\operatorname{Im} f_2(z)\|_1 \leq q_2 \|\operatorname{Im} f_2(z)\|_2,$$

where $q_\nu < 1$, $\nu = 1, 2$. Thus as above we have

$$|\mu| \geq (q_1 q_2)^{-1} > 1.$$

Proof of 4°. Let $p \in E_1$. Suppose contrary to the hypothesis, that there exists a sequence $\{p_\nu\}_1^\infty$ with $p_\nu \to p$ such that **A** has for $\nu = 1, 2, \ldots$, solutions $(\mu_\nu, f_{1\nu}, f_{2\nu})$ corresponding to p_ν with $|\mu_\nu| \leq 1$. We may suppose without loss of generality that $\mu_\nu \to \mu$, $0 < |\mu| \leq 1$. We write (see (13.2))

$$\operatorname{Im} e^{-i\alpha_2} f_1(z) = \operatorname{Im} e^{-i\alpha_2} \frac{\gamma(z)^{-1}}{\pi} \int_a^b \frac{\operatorname{Im} f_1(x)}{x-z}\, \gamma(x)\, dx, \quad z \in \Gamma_2$$

in the form $$\operatorname{Im} e^{-i\alpha_2} f_1(\Gamma_2) = P_{21}(\operatorname{Im} f_1(\Gamma_1))$$

and analogously we may write

$$\operatorname{Im} f_2(\Gamma_1) = P_{12}(\operatorname{Im} e^{-i\alpha_2} f_2(\Gamma_2)).$$

Writing $v(x) = \operatorname{Im} f_1(x)$, $a \leq x \leq b$ and $T = P_{12}P_{21}$ we obtain **A** in the equivalent form

$$(I - \mu T)v = 0, \; v \not\equiv \text{const.} \tag{13.5}$$

In the present case we have

$$(I - \mu_\nu T_\nu) v_\nu = 0, \; \nu = 1, 2, \ldots, \tag{13.6}$$

$$\mu_\nu \to \mu, \ |\mu| \leq 1, \ p_\nu \to p \quad \text{and} \quad \|v_\nu\|^2 = \frac{1}{\pi}\int_a^b v_\nu^2 \gamma \, dx = 1,$$

where $\gamma = \gamma(x) = [(x-a)(b-x)]^{-\frac{1}{2}}$. According to (13.6) $v_\nu(x)$ coincides with the imaginary part of a function which is analytic in the simply connected domain $D_2(p_\nu)$ bounded by the slit $\Gamma_2(p_\nu)$. These functions ($\nu = 1, 2, ...,$) are analytic and uniformly bounded in a fixed domain containing Γ_1. Thus it is possible to select a subsequence $\{v_{\nu_i}\}_1^\infty$, which converges to a continuous function v in the norm, i.e. $\|v - v_{\nu_i}\| \to 0$ and so $\|v\| = 1$. Suppose this subsequence already selected and let T correspond to p. Obviously $\|T - T_\nu\| \to 0$. We have

$$\|(I - \mu T)v\| \leq \|I - \mu_\nu T_\nu\| \|v - v_\nu\| + |\mu - \mu_\nu| \|T\| + |\mu_\nu| \|T - T_\nu\|.$$

The right hand side becomes arbitrarily small for ν sufficiently great. The left hand side being independent of ν thus equals zero. Hence $(I - \mu T)v = 0$ and obviously $v \not\equiv$ const. contradicting the hypothesis that $p \in E_1$. Thus E_1 is open. Theorem 13.1 is proved.

In (10.2) let $b = a + 1$ (the parameters p then are a and α) and let E_k have the same meaning as before.

Theorem 13.2. *For Problem* **B** *the following is true:*

1°. $p \in E' \Rightarrow \mu < 0, \ \mu \neq -1$ *or* $\mu > 1$.

2°. $p \in E' \Rightarrow |\mu| \geq 8a^2 \sin^2 \frac{\alpha}{2} \left(4a \sin \frac{\alpha}{2} + 3\right)^{-1}$.

3°. E_1 *is open.*

Proof. The proofs of 2° and 3° are analogous to the corresponding ones in Theorem 13.1. In 1° suppose first that $|\mu| = 1$. The $F(z) = f(z) - \mu e^{i\alpha} f(ze^{-i\alpha})$ is bounded and analytic in D (the domain bounded by Γ and $e^{i\alpha}\Gamma$) and observing that $\bar{f}(\bar{z}) = -f(z)$ we have Im $F(z) = 0$ on Γ and Im $e^{-i\alpha}F(z) = 0$ on $e^{i\alpha}\Gamma$. It follows that $f(z) \equiv$ const.

Suppose now that there exists a solution with μ such that $0 < \mu < 1$. The function $v(z) = \text{Im}\,[(f(z) - f(0))/z]$ is harmonic and bounded outside Γ. From (10.2) we obtain

$$\begin{aligned} &v(r) = Ar^{-1} + \mu v(re^{i\alpha}), \ a \leq r \leq b, \\ &v(\infty) = 0, \end{aligned} \tag{13.7}$$

where $A = B(\mu \cos \alpha - 1)$ and $iB = f(0)$. According to the maximum principle, (13.7) with $0 < \mu < 1$ is possible only if $A \neq 0$. We may suppose that $A > 0$. Write $v(re^{i\alpha}) = Pv(r)$, where P is the Poisson transformation. Then (13.7) is equivalent to

$$\begin{aligned} &v(r) = A \sum_{n=0}^{\infty} \mu^n P^n r^{-1}, \\ &v(\infty) = 0. \end{aligned} \tag{13.8}$$

Since $v(r) > 0$ and $v(\infty) = 0$, (13.8) is a contradiction. Theorem 13.2 is proved.

14. Description of the set K

Before concluding the study of the eigenvalue problems of Section 10 we devote this section and the next one to a study of the properties of a certain point set K described below. In Section 16 it is shown that the properties of K may be of some importance for the eigenvalue problems. Besides K plays an important part in Koebe's mapping theorem concerning the circle domain referred to in the introduction (p. 102).

Let D_0 be a k-connected domain on the z-sphere ($k \geqslant 3$) bounded by circles $C_\nu^{(0)} = C_\nu : |z-a_\nu| = R_\nu > 0$, $\nu = 1, 2, \dots, k$, $\infty \in D_0$. Let s_ν be the reflection operator with respect to C_ν i.e. $s_\nu(z) = a_\nu + R_\nu^2/(\bar{z} - \bar{a}_\nu)$. The circles $C_\nu^{(n)} : s_{i_n} s_{i_{n-1}} \dots s_{i_1}(C_{i_0})$, $1 \leqslant i_\mu \leqslant k$, $i_{\mu+1} \neq i_\mu$, $\mu = 0, 1, 2, \dots, n-1$, constitute the boundary of a domain D_n of connectivity $\mu_n = k(k-1)^n$. Also $D_n \to D$, a domain on the z-sphere. We denote its complement by K. It is completely determined by the circles $\{C_\nu^{(0)}\}_1^k$, which we call the fundamental circles of K. A circle $C_\nu^{(n)}$ is said to be of generation n.

Let W be a domain on the z-sphere. We define $W \in P_G$, $W \in P_{AB}$ and $W \in P_{AD}$ respectively if Green's function exists in W, if there exist non-constant, bounded, analytic functions in W and if there exist non-constant analytic functions in W with finite Dirichlet's integral over W. The complements of P_G, P_{AB} and P_{AD} are denoted O_G, O_{AB} and O_{AD} respectively. If $W \in P_{AB}$ the complement of W has positive analytic capacity, otherwise the analytic capacity is zero.

15. Remarks on K

First we prove a lemma.

Lemma 15.1. *Let Λ be the class of linear transformations $s_{i_n} s_{i_{n-1}} \dots s_{i_1}(z)$ (n even) or $\bar{s}_{i_n} s_{i_{n-1}} \dots s_{i_1}(z)$ (n odd), $i_{\nu+1} \neq i_\nu$. Let $a \neq \infty$ be a fixed point $\in D_0$. Then there exist positive constants b and B such that*

$$b|l'(a)| \leqslant |l'(z)| \leqslant B|l'(a)|, \quad z \in \partial D_0 = \bigcup_{\nu=1}^{k} C_\nu^{(0)}$$

for any $l(z) \in \Lambda$.

Proof. Apart from the trivial case $l(z) \equiv z$, the $l(z)$ are uniformly bounded, $z \in D_0 \cup \partial D_0$. The statement now follows from Koebe's distortion theorem.

K has a number of well-known properties, some of which we list below. For 1°–2° see [16], p. 111. The third property should be compared with [16], p. 422.

1°. *The measure of K, $m(K)$ is zero.*

2°. $D \in P_G$ ($k \geqslant 3$).

3°. *If $f(z)$ is analytic in D and if for any $\varepsilon > 0$ there exists n_ε such that*

$$\sum_{\nu=1}^{\mu_n} \omega_{\nu n}^2 < \varepsilon \quad \textit{for} \quad n > n_\varepsilon$$

where $$\omega_{\nu n} = \max_{z_1, z_2 \in C_\nu^{(n)}} |f(z_2) - f(z_1)| \quad \textit{then} \quad f(z) \equiv \text{const.}$$

4°. $D \in O_{AD}$.

Proof. We use the criterion of Theorem X.22 ([16], p. 446). It is required to construct a regular covering of $\{\partial D_n\}$ such that $\sum_{n=1}^{\infty} N^*(n)^{-1}=\infty$ where $N^*(n)$ has the meaning of Theorem X.22. Let the discs $C_{\nu\mu}$, $\mu=1, 2, ..., N(\nu)<\infty$, be centred at $C_\nu^{(0)}$, $\nu=1, 2, ..., k$. Further we demand that $C_{\nu_1\mu_1} \cap C_{\nu_2\mu_2}=\phi$, $\nu_1 \neq \nu_2$, $C_{\nu\mu} \cap \partial D_1=\phi$ and $C_{\nu,\mu+1} \cap C_{\nu\mu} \neq \phi$ (where $C_{\nu,N(\nu)+1} \equiv C_{\nu 1}$). Then $U_0=\bigcup_{\nu,\mu} C_{\nu\mu}$ is a covering of ∂D_0. With the aid of the reflection operators we can map U_0 (directly or indirectly) conformally onto U_n to cover ∂D_n, $n=1, 2, ...$ It is readily seen that $\{U_n\}$ is a regular covering of $\{\partial D_n\}$ such that $N^*(n)=N=\mathrm{Max}_\nu\, N(\nu)$. Thus $\Sigma N^*(n)^{-1}$ diverges and hence $D \in O_{AD}$.

Remark. Koebe's proofs, mentioned in the previous section, are connected with property 3°. A modification showing the connection with property 4° can be made as follows. We temporarily adopt the notation of Theorem IX.35 ([16], pp. 424–426) and give a slightly modified form of the proof of that theorem as follows. The reflection property of C_ν means that there exists a function $f_\nu(\zeta)$ analytic and univalent in a neighbourhood of the unit circle so that $f_\nu(|\zeta|=1)=C_\nu$. A suitable covering of the unit circle, similar to the above, can then with the aid of $f_\nu(\zeta)$ and the transformations of G be mapped directly or indirectly conformally to cover $\partial\Omega_m$ (N.B. the Ω_m exhaust Ω). As above it follows that $\Omega \in O_{AD}$. Since $\bar{S}_\nu(\omega)$ is meromorphic and univalent in Ω it follows that $\bar{S}_\nu(\omega)$ is linear ([16], p. 445) and it follows in an elementary way that C_ν is a circle.

Since $O_G \subset O_{AB} \subset O_{AD}$ with strict inclusions it is a natural question to ask whether it is in general true that $D \in P_{AB}$ or O_{AB}. The following points, 5°–8°, show that the answers to both questions are negative.

5°. *$D \in O_{AB}$ if C_ν, $\nu=1, 2, ..., k$, are orthogonal to one and the same circle C.*

Proof. We may suppose that C is the real axis since the property $D \in O_{AB}$ is invariant under linear transformations of D. Then $C_\nu^{(n)}: |z-a_\nu^{(n)}|=R_\nu^{(n)}$ with $\mathrm{Im}\, a_\nu^{(n)}=0$, $\nu=1, 2, ..., \mu_n$, $n=0, 1, 2, ...$ Let the circle of generation $n+1$, $C_{\nu\mu}^{(n+1)}$, be interior to $C_\mu^{(n)}$. From the geometric positions of these circles and Lemma 15.1 it follows that

$$\sum_{\nu=1}^{k-1} R_{\nu\mu}^{(n+1)} \leqslant q R_\mu^{(n)}, \quad \mu=1, 2, \dots, \mu_n,$$

for some fixed $q<1$. Thus

$$\sum_{\nu=1}^{\mu_{n+1}} R_\nu^{(n+1)} \leqslant q \sum_{\nu=1}^{\mu_n} R_\nu^{(n)}, \quad n=0, 1, 2, \dots .$$

It follows that $\sum_{\nu=1}^{\mu_n} R_\nu^{(n)}=O(q^n) \to 0$. Thus K has linear measure zero and this implies that $D \in O_{AB}$ ([1], p. 252).

6°. $k=3 \Rightarrow D \in O_{AB}$.

Proof. 6° is an immediate corollary of 5°.

7°. $|a_i-a_j|>R(1+R(k-1)) \Rightarrow D \in O_{AB}$ *where* $R=\mathrm{Max}_\nu\, R_\nu$.

Proof. Let a circle C with radius r interior to C_i be reflected in C_j, $i \neq j$, to a circle $\tilde{C}$ with radius $\tilde{r}$. Then

$$\tilde{r} \leqslant \frac{R_j^2 r}{|a_i - a_j| - R_i} \leqslant \frac{qr}{k-1}, \quad 0 < q < 1.$$

It follows that
$$\sum_{\nu=1}^{\mu_{n+1}} R_\nu^{(n+1)} \leqslant q \sum_{\nu=1}^{\mu_n} R_\nu^{(n)}.$$

Thus K has linear measure zero and $D \in O_{AB}$.

8°. *There exists K positive analytic capacity, i.e. $D \in P_{AB}$.*

Proof. According to Theorem 4 ([2], p. 613) there exists K with Hausdorff dimension $d(K) > 1$. This implies that there exists a distribution μ of the unit mass with support on K, and constants A and $\alpha > 1$, such that $\mu(C) \leqslant AR^\alpha$ for any disc C of any radius R. Then

$$f(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}$$

is a non-constant, bounded, analytic function in D.

16. *Analytic continuation*

The method of analytic continuation used in Section 11 can be generalized and this shows a connection between the eigenvalue problems **A** and **B** and the classification problem for certain Riemann surfaces. The special case of Section 11 gives a very simple surface, which however is not so for other choices of parameters p determining Γ_1 and Γ_2. The aim of this section is to exemplify this method.

Let α in (10.2) equal πr $(0 < r < 1)$ where r is a rational number and write $\varepsilon = e^{i\alpha}$. Then there is a smallest natural number N such that $\varepsilon^N = 1$. Let $\Gamma_\nu = \varepsilon^\nu \Gamma$, $\nu = 0, \pm 1, \pm 2, \ldots$, $(\Gamma_{\nu+N} \equiv \Gamma_\nu)$ and let $f_\nu(z) = f(\varepsilon^{-\nu} z)$, $\nu = 0, \pm 1, \pm 2, \ldots$, $(f_{\nu+N}(z) \equiv f_\nu(z))$. Then $f_\nu(z)$ is analytic and bounded outside Γ_ν and from (10.2) it follows that

$$\operatorname{Im} f_\nu(z) = \mu \operatorname{Im} \bar{\varepsilon} f_{\nu-1}(z), \; z \in \Gamma_\nu. \tag{16.1}$$

Let S_0 be the z-sphere slit along Γ_ν, $\nu = 0, 1, \ldots, N-1$. We connect replicas S_ν of S_0 to S_0 crosswise along Γ_ν, $\nu = 0, 1, \ldots, N-1$. Let $P_\nu(z)$ be the point of S_ν lying over $z \in S_0$. From (16.1) it follows that $f_\nu(z)$ can be continued into S_ν and observing the property (10.3) we obtain

$$f_\nu(P_\nu(z)) = -f_\nu(z) + \mu\bar{\varepsilon} f_{\nu-1}(z) + \mu\varepsilon f_{\nu+1}(z), \; \nu = 0, \pm 1, \pm 2, \ldots. \tag{16.2}$$

Let $F(z) = \sum_{\nu=0}^{N-1} \alpha_\nu f_\nu(z)$ where $\{\alpha_\nu\}_0^{N-1}$ are complex constants. From (16.2) it follows that $F(z)$ can be continued into a Riemann surface **R** obtained by connecting to each other an infinity of replicas of S_0 crosswise along the slits according to a rule whose form depends on r. We exemplify this rule briefly as follows. If $r = 2/3$ then $N = 3$ and **R** is a surface of planar character (see [16], p. 421) such that each sphere is connected to three others. If $r \neq 2/3$ $(0 < r < 1)$ then $N > 3$ and it follows from (16.2)

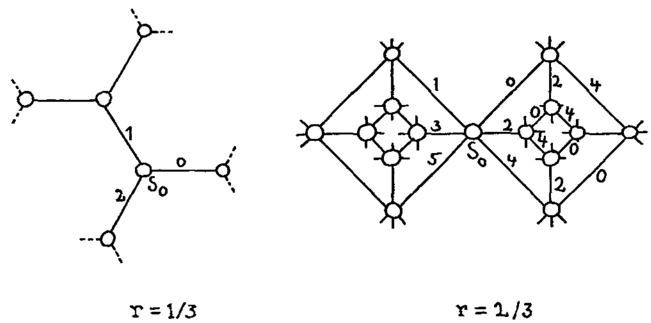

Fig. 3.

that continuations over non-consecutively numbered slits (e.g. Γ_0 and Γ_2), commute. Thus the associated Riemann surface **R** is not of planar character. (In Fig. 3 **R** is indicated in the cases $r=2/3$ and $r=1/3$.) In this case the Riemann surface of planar character, $\bar{\mathbf{R}}$, obtained by connecting to each S_0 first S_ν ($\nu=0, 1, ..., N-1$) and then to each S_ν $N-1$ new replicas and so on, is a covering surface of **R**. (Thus $\bar{\mathbf{R}}=\mathbf{R}$ in the case $r=2/3$). Also $\bar{\mathbf{R}}$ is conformally equivalent to a domain Ω on the ω-sphere bounded by the point set K of Section 14 (see [16], p. 424) which has the circles (orthogonal to the unit circle)

$$C_\nu: |\omega-\varepsilon^\nu \sqrt{1+\varrho^2}|=\varrho,\ \nu=0, 1, ..., N-1,$$

as fundamental circles. Here C_ν corresponds to $\Gamma_\nu \in S_0$ and ϱ depends on the parameters p only. Let $P=h(\omega)$ be the inverse mapping and let $g(\omega)=F(h(\omega))$. We exhaust Ω by Ω_n where $\partial\Omega_n$ is bounded by the union of the circles of generation n. Letting $l_n(\varrho)$ be the total length of these circles and writing $M_n(\mu)=\mathrm{Max}_{\omega\in\partial\Omega_n}|g(\omega)|$ we have

$$|g'(\omega_0)|=\left|\frac{1}{2\pi i}\int_{\partial\Omega_n}\frac{g(\omega)}{(\omega-\omega_0)^2}d\omega\right|\leqslant C_{\omega_0} l_n(\varrho)\, M_n(\mu),\quad \omega_0\in\Omega_0,\quad n=0, 1, 2, \ldots. \qquad (16.3)$$

Thus if $\lim_{n\to\infty} l_n(\varrho)\, M_n(\mu)=0$ then **B** lacks solutions for the μ and p concerned.

It follows from the proof of 5° that $l_n(\varrho)\leqslant A q_\varrho^n$ where $0<q_\varrho<1$ and that q_ϱ becomes arbitrarily small if ϱ is sufficiently small. Further it is possible to estimate $M_n(\mu)$ in various ways essentially in terms of μ only. For instance it follows from (16.2) that

$$M_n(\mu)\leqslant M(1+|\mu|)^n.$$

With these estimates the sufficient condition above is satisfied if

$$(1+|\mu|)q_\varrho<1.$$

IV. On conformal mappings onto rectilinear slit domains

17. *Definitions and notation*

Let D be a k-connected domain on the z-sphere bounded by k continua $\{C_\nu\}_1^k$, $\infty \in D$. Let $\{\alpha_\nu\}_1^k$ be any set of real numbers such that $0 \leqslant \alpha_\nu < \pi$, $\nu = 1, 2, \ldots, k$. A rectilinear slit Γ making the angle α $(0 \leqslant \alpha < \pi)$ with the positve real axis we call an *α-slit* for short. There exists a unique function $\omega = f(z) \in \Sigma'(D)$ mapping D onto a domain bounded by rectilinear slits $\{\Gamma_\nu\}_1^k$, where Γ_ν corresponds to C_ν, $\nu = 1, 2, \ldots, k$, such that Γ_ν is an α_ν-slit. This can for example be proved as a limiting case of more general mapping theorems. The question of whether the iterative process can be applied in general to this canonical domain may be difficult to answer. Probably we can expect to find neither a simple "measure of deviation" acting monotonically as we could for the mappings of Chapter I nor a simple functional as in [7] for the parallel slit mapping. However something would be gained if it were possible to apply the iterative process in the case when D is of "nearly" desired shape. Then the mapping functions are available for approximations with the aid of which we obtain the convergence of the process in certain cases.

We confine ourselves to the case $k = 2$. Let Ω be a mixed slit domain on the ζ-sphere bounded by an α_1-slit, Γ_1, and an α_2-slit, Γ_2. We may normalize Ω by the requirement that one of the slits be identical with a pre-assigned slit.

Then $\Omega = \Omega(p)$ is determined by a set, p, of 4 real parameters. Let $D = D(\Delta_0; p)$ on the z-sphere be conformally equivalent to $\Omega(p)$ under a mapping in $\Sigma'(\Omega(p))$. Let the boundary of D be an α_1-slit, $\Gamma_1^{(0)}$, and another slit $\Gamma_2^{(0)}$, which "deviates very little" from an α_2-slit (in some suitable definition of the term), the measure of deviation being at most Δ_0. More precisely we prescribe $\Gamma_2^{(0)}$ to be rectilinear and such that $\operatorname{Max}_{a, b \in \Gamma_2^{(0)}} |\operatorname{Im} e^{-i\alpha_2}(a-b)| \leqslant \Delta_0$. We can map Ω conformally onto such a domain as was remarked above. With D as starting-point an iterative mapping chain is constructed as follows:

1°. $z_{n+1} = z_{n+1}(z_n) = z_n + r_{n+1}(z_n)$, $n = 0, 1, 2, \ldots$, $(z_0 = z)$. $z_n(z_m) = z_n(z_{n-1}(\ldots(z_m)\ldots))$, $n > m$; $z_n(z) = z_n(z; \Delta_0, p)$.

2°. $z_n(D) = D^{(n)}$ where $D^{(n)}$ is bounded by $\Gamma_\nu^{(n)}$ corresponding to $\Gamma_\nu^{(0)}$, $\nu = 1, 2$. Let $D_\nu^{(n)}$ be the simply connected domain bounded by $\Gamma_\nu^{(n)}$.

3°. $z_{n+1}(z_n) \in \Sigma'(D_{\nu_n}^{(n)})$ and $\Gamma_{\nu_n}^{(n+1)}$ is an α_{ν_n}-slit, $n = 0, 1, 2, \ldots$, where $\nu_n = 1$ if n is odd, $\nu_n = 2$ if n is even.

The boundary components of $D^{(n)}$ i.e. $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ will also be denoted by $\bar{\Gamma}^{(n)} =$ the $\alpha_{\nu_{n+1}}$-slit and $\tilde{\Gamma}^{(n)} =$ the other slit. By the *width* of $\tilde{\Gamma}^{(n)}$ we mean the quantity $\Delta_n = \operatorname{Max}_{a, b \in \tilde{\Gamma}^{(n)}} e^{-i\alpha_{\nu_n}}(a-b)$.

Let $F_n(\zeta) = \zeta + R_n(\zeta) \in \Sigma'(\Omega(p))$ be the mapping of Ω onto $D^{(n)}$.

Let $U_\nu = \{\zeta \mid d(\zeta, \Gamma_\nu) \leqslant d_0\}$ and $L_\nu = \partial U_\nu$, $\nu = 1, 2$.

We choose $d_0 > 0$ so small that $U_1 \cap U_2 = \phi$. Let $L_\nu^{(n)} = F_n(L_\nu)$ and let $U_\nu^{(n)}$ be the point set consisting of the domain containing $\Gamma_\nu^{(n)}$ and bounded by $L_\nu^{(n)}$ together with this boundary, $\nu = 1, 2$. From the compactness of $\Sigma'(\Omega($ it follows that $d(L_1^{(n)}, L_2^{(n)}) \geqslant k_1 > 0$ and $d(\Gamma_\nu^{(n)}, L_\nu^{(n)}) \geqslant k_2 > 0$, $\nu = 1, 2$, where k_1 and k_2 are independent of n.

Let $h(z; \Gamma, v)$ be the uniquely determined function, which (1°) is bounded and analytic in the domain bounded by the rectilinear slit Γ, (2°) is zero at infinity and

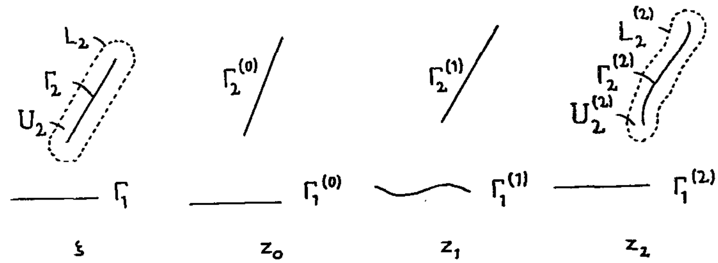

Fig. 4.

(3°) has, with respect to Γ, a symmetric imaginary part with continuous boundary values $v+C$, where C is a constant determined from (2°).

We make the

Definition 17.1. *The domain $\Omega(p)$ is an iteratively stable domain if there exists a number $\Delta(p)>0$ depending only on p such that $\lim_{n\to\infty} z_n(z;\Delta_0,p)$ exists for all $\Delta_0\leqslant\Delta(p)$.*

If $\lim_{n\to\infty} z_n(z;\Delta_0,p)$ exists then necessarily the limit function belongs to $\Sigma'(D)$ and maps D onto Ω.

18. Lemmas

Lemma 18.1. *There exists a constant K independent of n such that*

$$|r_{n+1}(z_n)|\leqslant K\Delta_{n-1},\ z_n\in\tilde{\Gamma}^{(n)}, \tag{18.1}$$

$$|R_n(\zeta)|\leqslant K\Delta_{n-1},\ \zeta\in\Gamma_1\cup\Gamma_2,\ n=1,2,\ldots, \tag{18.2}$$

and

$$|r_1(z)|\leqslant K\Delta_0,\quad z\in\Gamma_2^{(0)}, \tag{18.3}$$

$$|R_0(\zeta)|\leqslant K\Delta_0,\quad \zeta\in\Gamma_1\cup\Gamma_2. \tag{18.4}$$

Proof. From the compactness of $\Sigma'(\Omega)$ it follows that the inequalities are true if $\Delta_{n-1}\geqslant\Delta>0$. Hence we suppose that Δ_{n-1} is small. Let $\tilde{z}_n$ denote z_n reflected in $\tilde{\Gamma}^{(n)}$. The constants K_i, $i=1,2,3,4$, appearing below are all independent of n which follows from the compactness of $\Sigma'(\Omega)$.

We now prove (18.1). Since $|\operatorname{Im} e^{-i\alpha_{\nu_{n-1}}} r_n(z_{n-1})|\leqslant\Delta_{n-1}$, $z_{n-1}\in\tilde{\Gamma}^{(n-1)}$ it follows that $|r_n(z_{n-1})|\leqslant K_1\Delta_{n-1}$, $z_{n-1}\in L_{\nu_{n-1}}^{(n-1)}$ and in particular that $\Delta_n\leqslant 2K_1\Delta_{n-1}$. Thus

$$|r_{n+1}(z_n)|\leqslant K_2\Delta_{n-1},\quad z_n\in L_{\nu_n}^{(n)}. \tag{18.5}$$

The function $z_{n+1}(z_n)$ can be analytically continued over $\tilde{\Gamma}^{(n)}$ to a double-sheeted Riemann surface branched at the endpoints of $\tilde{\Gamma}^{(n)}$. The continuation is given by

$$z_{n+1}^*(z_n)=\tilde{z}_{n+1}(z_n(\tilde{z}_{n-1}(z_n)))=z_n+a_n+r_{n+1}^*(z_n)$$

where $r_{n+1}^*(\infty)=0$. It follows that $(z_{n-1}=z_{n-1}(z_n))$

$$|r_{n+1}^*(z_n)| \leqslant |r_n(z_{n-1})| + |r_n(\tilde{z}_{n-1})| + |r_{n+1}(z_n(\tilde{z}_{n-1}))|$$

and that $$|a_n| \leqslant 2K_1\Delta_{n-1}.$$

If $\Delta_{n-1} \leqslant \Delta$ (independent of n) then the continuation over $\tilde{\Gamma}^{(n)}$ is certainly possible up to and onto $L_{\nu_n}^{(n)}$, which is thought of here as lying in the second sheet and also

$$|z_{n+1}^*(z_n) - z_n| \leqslant K_3\Delta_{n-1}, \quad z_n \in L_{\nu_n}^{(n)}. \tag{18.6}$$

Applying the maximum principle we deduce (18.1) from (18.5) and (18.6).

To prove (18.2) we observe that $\operatorname{Im} e^{-i\alpha_{\nu_{n+1}}} R_n(\zeta) = C_n$, $\zeta \in \Gamma_{\nu_{n+1}}$, where C_n is a constant. Since we also have $R_n(\infty) = 0$ it necessarily follows that $|\operatorname{Im} e^{-i\alpha_{\nu_n}} R_n(\zeta)| \leqslant \Delta_n$, $\zeta \in \Gamma_{\nu_n} \cup \Omega$. Thus $|R_n(\zeta)| \leqslant K_4\Delta_n \leqslant 2K_1K_4\Delta_{n-1}$, $\zeta \in L_1 \cup L_2$. Writing $F_n(\zeta) = F_{n+1}(\zeta) - (F_{n+1}(\zeta) - F_n(\zeta))$ we realize that $|C_n| \leqslant \Delta_{n+1} + K_1\Delta_n \leqslant 3K_1\Delta_n \leqslant 6K_1^2\Delta_{n-1}$. With these observations it is possible to conclude that

$$|R_n(\zeta)| \leqslant K_4\Delta_{n-1}, \quad \zeta \in \Gamma_1 \cup \Gamma_2,$$

in the same way as in the proof of (18.1) where we now continue $F_n(\zeta)$ over Γ_1 as well as over Γ_2. The continuations are given by $\tilde{F}_n(\tilde{\zeta})$ and $z_n(\tilde{F}_{n-1}(\tilde{\zeta}))$ where the reflections are made in a slit and its image.

The function $z_1(z)$ can simply be written explicitly and (18.3) is readily verified by elementary means.

To prove (18.4) we have $|\operatorname{Im} e^{-i\alpha_2} R_0(\zeta)| \leqslant \Delta_0$, $\zeta \in \Gamma_2 \cup \Omega$ and $|R_0(\zeta)| \leqslant K_4\Delta_0$, $\zeta \in L_1 \cup L_2$ as above. Further

$$|\operatorname{Im} e^{-i\alpha_1} R_0(\zeta)| \leqslant 3K_1\Delta_0, \quad \zeta \in \Gamma_1.$$

Using the condition that $\Gamma_\nu^{(0)}$, $\nu = 1, 2$, are rectilinear, we obtain (18.4) with the aid of analytic continuation as above. Lemma 18.1 is proved.

Lemma 18.2. *There exist $\delta_1 > 0$ and A independent of n such that if $\Delta_{n-1} \leqslant \delta_1$ then*

$$r_{n+1}(z_n) = -e^{i\alpha_{\nu_n}} h(z_n; \Gamma_{\nu_n}, \operatorname{Im} e^{-\alpha_{\nu_n}} r_n) + \varrho_{n+1}(z_n), \tag{18.7}$$

where $$|\varrho_{n+1}(z_n)| \leqslant A\Delta_{n-1}^{\frac{3}{2}}, \quad z_n \in L_{\nu_{n+1}}^{(n)}, \quad n = 1, 2, \ldots .$$

Proof. To avoid complicated notation we perform the proof for the special case $n = 1$. It is readily seen from the text that we can determine δ_1 and A independently of n. This is a consequence of the compactness of $\Sigma'(D)$. The constants A_i, $i = 1, 2, 3, 4$, appearing in the present proof are of this kind. Of course we do not use here the condition that $\Gamma_2^{(0)}$ is rectilinear.

If $n = 1$ then (18.7) becomes

$$r_2(z_1) = -e^{i\alpha_1} h(z_1; \Gamma_1, \operatorname{Im} e^{-i\alpha_1} r_1) + \varrho_2(z_1).$$

We have $$\operatorname{Im} e^{-i\alpha_1} r_2(z_1) = c_2 - \operatorname{Im} e^{-i\alpha_1} z_1, \quad z_1 \in \Gamma_1^{(1)}$$

or $$\operatorname{Im} e^{-i\alpha_1} r_2(z + r_1(z)) = c_2 - c_0 - \operatorname{Im} e^{-i\alpha_1} r_1(z), \quad z \in \Gamma_1^{(0)},$$

where c_0 and c_2 are constants. Let $v(z)$ be the bounded harmonic function in $D_1^{(0)}$ which takes the values $c_2 - c_0 - \operatorname{Im} e^{-i\alpha_1} r_1(z)$ on $\Gamma_1^{(0)}$. Let a and b be the endpoints of $\Gamma_1^{(0)}$ and $z = S(\omega) = (b-a)(\omega + \omega^{-1})/4 + (b+a)/2$. Then $r_1(S(\omega))$ is analytic in a ring $\varrho^{-1} \leqslant |\omega| \leqslant \varrho$, where $\varrho > 1$ depends only on the length of $\Gamma_1^{(0)}$ and $d(\Gamma_1^{(0)}, \Gamma_2^{(0)})$. If

$$R(\omega) = \frac{1}{2\pi i} \int_{|\omega'|=\varrho} \frac{r_1(S(\omega'))}{\omega' - \omega} d\omega' \tag{18.8}$$

then
$$r_1(S(\omega)) = R(\omega) + R(\omega^{-1}) + \text{const.}$$

and hence
$$v(S(\omega)) = -\operatorname{Im} e^{-i\alpha_1}(R(\omega) + R(\bar{\omega})) + \text{const.} \tag{18.9}$$

Since $|r_1(z)| \leqslant K\Delta_0$ (Lemma 18.1) it follows from (18.8) and (18.9) that

$$|v(S(\omega_1)) - v(S(\omega_2))| \leqslant K_{\varrho'} K \Delta_0 |\omega_1 - \omega_2| \tag{18.10}$$

if $|\omega_1|, |\omega_2| \leqslant \varrho' < \varrho$. Here $K_{\varrho'}$ depends on ϱ' and ϱ only.

Let $l(z) = Bz + B'$ be such that $l^{-1}(a) = z_1(a)$ and $l^{-1}(b) = z_1(b)$. Then $l(\Gamma_1^{(1)})$ is a slit with end-points a and b.

Now we choose $\delta_1 > 0$ so small that $\Delta_0 \leqslant \delta_1$ implies the following: (1°) $l(\Gamma_1^{(1)}) \subset U_1^{(0)}$; (2°) $l(\Gamma_1^{(1)})$ is the image under $S(\omega)$ of a star-shaped (with respect to $\omega = 0$) analytic arc $\omega = R(\theta)e^{i\theta}$, $0 \leqslant \theta \leqslant \pi$ connecting $\omega = \pm 1$; (3°) $\Gamma_1, l^{-1}(\Gamma_1^{(0)}) \subset U_1^{(1)}$. In (2°) we observe that $|r_1(z)| \leqslant K\Delta_0$ implies that $|r_1'(z)| \leqslant K'\Delta_0$, $z \in \Gamma_1^{(0)}$ which ensures that (2°) can be satisfied.

Now let $S(e^{i\theta}) = z$ and $S(R(\psi)e^{i\psi}) = l(z + r_1(z))$ where $z \in \Gamma_1^{(0)}$ and $0 \leqslant \theta, \psi \leqslant \pi$. Then it is elementarily verified that

$$|Re^{i\psi} - e^{i\theta}| \leqslant A_1 \Delta_0^{\frac{1}{2}}.$$

Inserting this in (18.10) and using (1°) we obtain

$$|v(l(z + r_1(z))) - v(z)| \leqslant A_2 \Delta_0^{\frac{3}{2}}, \; z \in \Gamma_1^{(0)},$$

or according to the definition of $v(z)$

$$|v(l(z_1)) - \operatorname{Im} e^{-i\alpha_1} r_2(z_1)| \leqslant A_2 \Delta_0^{\frac{3}{2}}, \;\; z_1 \in \Gamma_1^{(1)}.$$

From the maximum principle it follows that this is true for $z_1 \in D_1^{(1)} \cup \Gamma_1^{(1)}$ and hence if $\Delta_0 \leqslant \delta_1$ that

$$r_2(z_1) = e^{i\alpha_1} h(z_1; l^{-1}(\Gamma_1^{(0)}), \; -\operatorname{Im} e^{-i\alpha_1} r_1(l(z_1)) + \varrho_2^*, \tag{18.11}$$

where $|\varrho_2^*| \leqslant A_3 \Delta_0^{\frac{3}{2}}$, $z_1 \in L_2^{(1)}$.

Using (3°) and Lemma 18.1 we have for $\Delta_0 \leqslant \delta_1$ that

$$|h(z_1; \Gamma_1, -\operatorname{Im} e^{-i\alpha_1} r_1) - h(z_1; l^{-1}(\Gamma_1^{(0)}), -\operatorname{Im} e^{-i\alpha_1} r_1(l(z_1))| \leqslant A_4 \Delta_0^2, \;\; z_1 \in L_2^{(1)}. \tag{18.12}$$

Combining (18.11) and (18.12) we obtain the statement of the lemma for $n=1$ and referring to the preliminary remark we find that Lemma 18.2 is proved.

Remark. Using the rectilinearity of $\Gamma_2^{(0)}$ we find that Lemma 18.2 is true for $n=0$ with the formal difference that $|\varrho_1(z)| \leqslant A\Delta_0^{\frac{3}{2}}$, $z \in L_1^{(0)}$. The proof is analogous to the above and it may also be carried through in a direct way since the function $z_1(z)$ can be written down explicitly.

19. The main theorem

Let E_k have the meaning of Section 13 (p. 130).

Theorem 19.1. *If $p \in E_1$ then $\Omega(p)$ is iteratively stable.*

Proof. Let v_1 be a real-valued, continuous function given on Γ_1. Then denote $\operatorname{Im} e^{i(\alpha_1-\alpha_2)}h(z;\Gamma_1,-v_1)$, $z\in\Gamma_2$, by v_2 and write $v_2=T_{21}v_1$. Further let $v_3=T_{12}v_2$ be the function which takes the values $\operatorname{Im} e^{i(\alpha_2-\alpha_1)}h(z;\Gamma_2,-v_2)$, $z\in\Gamma_1$. Hence $v_3=T_{12}T_{21}v_1$ and we pose the eigenvalue problem

$$\lambda v = T_{12}T_{21}v. \tag{19.1}$$

If Γ_2 can be obtained from Γ_1 by a rotation $T_{12}=T_{21}=T$ where v_1, v_2 and v_3 are thought of as being defined on Γ_1. In this case we pose the problem

$$\lambda v = Tv. \tag{19.2}$$

If (19.1) has a solution (λ, v), $\lambda \neq 0$, $v \not\equiv 0$, we define $f_1(z)=e^{i\alpha_1}h(z;\Gamma_1,-v)$ and $f_2(z)=e^{i\alpha_2}h(z;\Gamma_2,-T_{21}v)$. Then $f_\nu(z)$ is analytic outside Γ_ν, $\nu=1,2$. From (19.1) follows that v coincides on Γ_1 with the values of a function that is harmonic outside Γ_2. In particular this implies that $f_1(z)$ is bounded. In the same way we conclude that $f_2(z)$ is bounded. Further $f_\nu(\infty)=0$, $\nu=1,2$, and so (19.1) takes the form

$$\begin{aligned} \lambda \operatorname{Im} e^{-i\alpha_1} f_1(z) &= \lambda C_1 - \operatorname{Im} e^{-i\alpha_1} f_2(z), \quad z\in\Gamma_1,\\ \operatorname{Im} e^{-i\alpha_2} f_2(z) &= C_2 - \operatorname{Im} e^{-i\alpha_2} f_1(z), \quad z\in\Gamma_2, \end{aligned} \tag{19.3}$$

where C_1 and C_2 are real constants due to the conditions $f_\nu(\infty)=0$, $\nu=1,2$. In (19.3) $\lambda\neq 1$, which follows in the same way as in Theorem 13.1, see 1°, p. 130.

If $\lambda \neq \cos^2(\alpha_2-\alpha_1)$ then we can replace $f_1(z)$ by $\lambda^{-1}f_1(z)+ie^{i\alpha_1}A_1$ and $f_2(z)$ by $-f_2(z)+ie^{i\alpha_2}A_2$ where A_1 and A_2 are determined by

$$\lambda A_1 + A_2\cos(\alpha_2-\alpha_1) = \lambda C_1,$$

$$A_1\cos(\alpha_2-\alpha_1) + A_2 = C_2.$$

Writing $\mu=\lambda^{-1}$ we obtain

$$\left.\begin{aligned} &\operatorname{Im} e^{-i\alpha_1} f_1(z) = \operatorname{Im} e^{-i\alpha_1} f_2(z), && z\in\Gamma_1,\\ &\operatorname{Im} e^{-i\alpha_2} f_2(z) = \mu \operatorname{Im} e^{-i\alpha_2} f_1(z), && z\in\Gamma_2,\\ &\operatorname{Re} e^{-i\alpha_\nu} f_\nu(\infty) = 0, && \nu=1,2. \end{aligned}\right\} \tag{19.4}$$

Now (19.4) is identical with (10.1) in Problem A (p. 126) together with the condition (10.3). If $\lambda = \cos^2(\alpha_2 - \alpha_1)$ then $\lambda < 1$ ($\lambda = 1$ is excluded).

In a similar way (19.2) can be rewritten in the form of (10.2) together with the condition (10.4) (p. 126) for $\lambda \neq -\cos(\alpha_2 - \alpha_1)$. Here necessarily $\lambda \neq \pm 1$ and if the exceptional case $\lambda = -\cos(\alpha_2 - \alpha_1)$ occurs then $|\lambda| < 1$.

Thus if $p \in E_1$ with respect to A then necessarily $|\lambda| < 1$ in (19.1) if there are any solutions and analogously $|\lambda| < 1$ in (19.2). We perform the proof in the former case. The latter one can be handled analogously. The transformation $T_{12}T_{21}$ has a continuous and bounded kernel and it follows that there exist q, $0 < q < 1$, and m_0 depending only on p such that ($\|v\|$ see below)

$$\|(T_{12}T_{21})^m\| < q^m, \quad m \geqslant m_0. \tag{19.5}$$

We now turn to the iterative process. Suppose that Lemma 18.2 can be applied to $2m$ ($m \geqslant m_0$) consecutive functions $r_\nu(z_{\nu-1})$ starting with $r_{2n+1}(z_{2n})$, $n \geqslant 1$. Then

$$\operatorname{Im} e^{-i\alpha_1} r_{2m+2n+1}(z_{2m+2n}) = (T_{12}T_{21})^m (\operatorname{Im} e^{-i\alpha_1} r_{2n+1}) + \eta_m, \; z_{2m+2n} \in \Gamma_1, \tag{19.6}$$

where $|\eta_m| \leqslant K'_m \Delta_{2n-1}^{\frac{3}{2}}$. Here K'_m depends on p and m only. Let

$$\|v\|_\Gamma = \operatorname*{Max}_{z \in \Gamma} |v(z)|.$$

From Lemma 18.1 we obtain

$$\left| \|\operatorname{Im} e^{-i\alpha_1} r_{2m+2n+1}\|_{\Gamma_1^{(2m+2n)}} - \|\operatorname{Im} e^{-i\alpha_1} r_{2m+2n+1}\|_{\Gamma_1} \right| \leqslant K''_m \Delta_{2n-1}^2, \tag{19.7}$$

where K''_m depends on p and m only. Observing that $\|\operatorname{Im} e^{-i\alpha_1} r_{2n+1}\|_{\Gamma_1} \leqslant K\Delta_{2n-1}$ where K is the number of Lemma 18.1 we deduce from (19.5)–(19.7) that

$$\|\operatorname{Im} e^{-i\alpha_1} r_{2m+2n+1}\|_{\Gamma_1^{(2m+2n)}} \leqslant K q^m \Delta_{2n-1} + K_m \Delta_{2n-1}^{\frac{3}{2}}, \tag{19.8}$$

where K_m depends on p and m only. In particular

$$\Delta_{2m+2n+1} \leqslant 2Kq^m \Delta_{2n-1} + 2K_m \Delta_{2n-1}^{\frac{3}{2}}. \tag{19.9}$$

The same argument holds if $n = 0$. In this case Δ_{2n-1} is replaced by Δ_0, and $\Delta_{2m+2n+1}$ by Δ_{2m+1}.

Let q_1 be a number such that $0 < q_1 < 1$. Then we fix a number $m \geqslant m_0$ such that

$$q^m \leqslant \frac{q_1}{4K}.$$

It follows from Lemma 18.1 that it is possible to find a number $\delta_m > 0$ depending only on p and m such that the argument above, (19.6)–(19.9), is justified as soon as $\Delta_{2n-1} \leqslant \delta_m (\Delta_0 \leqslant \delta_m)$.

Thus with m and δ_m fixed we prescribe that $\Delta_0 \leqslant \operatorname{Min}(\delta_m, (q_1/4K_m)^2) = \Delta(p) > 0$. We now deduce from (19.9) that with Δ_0 replacing Δ_{2n-1},

$$\Delta_{2m+1} \leqslant q_1 \Delta_0 < \Delta(p).$$

The conditions for an application of the argument leading from (19.6) to (19.9) are *a fortiori* satisfied with r_{2m+3} as starting-point. We obtain

$$\Delta_{4m+3} \leqslant q_1^2 \Delta_0 < \Delta(p)$$

and inductively $$\Delta_{2N(m+1)-1} \leqslant q_1^N \Delta_0, \quad N=1, 2, \ldots,$$

which implies that $\lim_{n\to\infty} z_n(z; \Delta_0, p)$ exists if $\Delta_0 \leqslant \Delta(p)$. Thus $\Omega(p)$ is iteratively stable. We have, moreover, according to Lemma 18.1,

$$|R_n(\zeta)| = O\left(q_1^{\left[\frac{n}{2m+2}\right]}\right) = O(q_2^n), \quad \zeta \in \Gamma_1 \cup \Gamma_2, \quad 0 < q_2 < 1.$$

Theorem 19.1 is proved.

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