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Pure submodules

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1. Introduction

The notion of purity is extremely important in abelian group theory. One reason for this is that there are enough pure-injective (and also enough pure-projective) groups, which makes it possible to apply the methods of relative homological algebra. When extending the purity concept to modules over arbitrary rings, it is therefore natural to look for two types of generalizations, one giving enough pure-projectives and one giving enough pure-injectives. The most useful notion of purity of the first type seems to be the one introduced by Butler and Horrocks [2]. We will give a rather detailed treatment of it in sec. 6–9. By dualization we obtain a theory of copurity, with enough copure-injectives (sec. 10). The usual examples of copurity in module categories are related to exactness properties of the tensor product (sec. 11 and 13).

C. L. Walker [15] has proposed another way of defining purity, which also generalizes the traditional notions of purity for modules over Dedekind rings, but which seems somewhat less natural for modules over more general rings (in particular it does not include purity in the sense of Cohn [4] and Bourbaki ([1], ch. 1, § 2, exercise 24).

Our theory of purity and copurity probably covers all reasonable notions of purity for abelian groups and modules that have been used in the literature. Most of the time we will work in an abelian category with a projective generator, thus being rather near to a category of modules. Actually, our examples (collected in sec. 9 and 13) deal only with modules.

We also study the theories of torsion and divisibility which are associated to the concepts of purity and copurity, and are "torsion theories" in the technical sense of Dickson [5].

The first five sections are of a preliminary nature. Sec. 2 and 3 contain some general remarks on proper classes, in sec. 4 Maranda's theory of pure-essential extensions is generalized, and sec. 5 contains some remarks on the relative homological algebra associated to a torsion theory, related to the results of Walker [15].

2. Proper classes

Let $\mathcal{A}$ be an abelian category. Consider a class $\mathcal{E}$ of short exact sequences of $\mathcal{A}$, such that every sequence isomorphic to a sequence in $\mathcal{E}$ also is in $\mathcal{E}$. The corresponding class of monomorphisms (epimorphisms) is written $\mathcal{E}_m(\mathcal{E}_e)$. $\mathcal{E}$ is called a *proper class* if it satisfies:

P1. Every split short exact sequence is in $\mathcal{E}$.

P2. If $\alpha, \beta \in \mathcal{E}_m$, then $\beta\alpha \in \mathcal{E}_m$ if defined.

P2*. If $\alpha, \beta \in \mathcal{E}_e$, then $\beta\alpha \in \mathcal{E}_e$ if defined.

P3. If $\beta\alpha \in \mathcal{E}_m$ and β is a monomorphism, then $\alpha \in \mathcal{E}_m$.

P3*. If $\beta\alpha \in \mathcal{E}_e$ and α is an epimorphism, then $\beta \in \mathcal{E}_e$.

A subobject L of M is then called *$\mathcal{E}$-proper* if the inclusion morphism $L \to M$ belongs to $\mathcal{E}_m$. It is well-known that $\mathcal{E}_m(\mathcal{E}_e)$ is closed under push-outs (pull-backs). Using this fact one sees that it is unneccessary to assume that β is a monomorphism in P3 and α an epimorphism in P3*.

Suppose we have a functor $T: \mathcal{A} \to \mathcal{B}$ to another abelian category $\mathcal{B}$. Given any class $\mathcal{J}$ of short exact sequences of $\mathcal{B}$, let $T^{-1}(\mathcal{J})$ be the class of those short exact sequences of $\mathcal{A}$ which are carried into $\mathcal{J}$ by T. Using standard diagram lemmas one proves:

Proposition 2.1. *Suppose T is either left or right exact. $T^{-1}(\mathcal{J})$ is then a proper class.*

For a proper class $\mathcal{E}$ of $\mathcal{A}$ we denote the class of $\mathcal{E}$-projective objects by $\pi(\mathcal{E})$. On the other hand, if O is a class of objects in $\mathcal{A}$, we let $\pi^{-1}(O)$ denote the class of all those short exact sequences of $\mathcal{A}$ for which the objects in O are relative projectives. $\mathcal{E}$ is *projectively closed* if $\mathcal{E} = \pi^{-1}(\pi(\mathcal{E}))$.

Proposition 2.2. *$\pi^{-1}(O)$ is a proper class and is projectively closed.*

Proof. To prove properness, apply prop. 2.1 to the functor Hom $(P, \cdot)$ for each $P \in O$, and note that any intersection of proper classes is a proper class.

A proper class $\mathcal{E}$ is said to have enough $\mathcal{E}$-projectives if for every object M there exists an epimorphism $P \to M$ in $\mathcal{E}_e$ with $P \in \pi(\mathcal{E}_e)$. No general criteria are known for a proper class $\mathcal{E}$ to have enough $\mathcal{E}$-projectives. However, there are two important cases in which this is known.

Proposition 2.3. *Assume that $\mathcal{A}$ has infinite direct sums. Let O be a set of objects of $\mathcal{A}$, including a family of generators for $\mathcal{A}$. Then:*

(i) *There are enough $\pi^{-1}(O)$-projectives.*

(ii) $\pi(\pi^{-1}(O)) = \{$*direct summands of direct sums of objects in* $O\}$.

Proof. ([12], sec. 2) For each M we put $I = \{\varphi : P_\varphi \to M \mid P_\varphi \in O \text{ and } \varphi \neq 0\}$ and consider the naturally defined morphism $\oplus_{\varphi \in I} P_\varphi \to M$. The sum exists because O is a set, and the morphism is evidently epimorphic and $\pi^{-1}(O)$-proper.

In the situation described by the above proposition we can find an object F which is a direct sum of copies of the generators included in O, so that every object in O is a quotient object of F. This remark leads to the notion of purity, as introduced by Butler and Horrocks, which will be studied in sec. 6.

The other case with enough relative projectives is the following one, which has been treated by C. L. Walker in [15].

Proposition 2.4. *Assume that $\mathcal{A}$ is locally small, has infinite direct sums and has enough projectives. Let $\mathcal{O}$ be a class of objects of $\mathcal{A}$, closed under quotients. Then:*

(i) *A subobject L of M is $\pi^{-1}(\mathcal{O})$-proper if and only if L is a direct summand of every subobject K of M such that $L \subset K$ and $K/L \in \mathcal{O}$.*
(ii) *There are enough $\pi^{-1}(\mathcal{O})$-projectives.*
(iii) *$\pi(\pi^{-1}(\mathcal{O})) = \{$direct summands of direct sums of projectives and objects in $\mathcal{O}\}$.*

Proof. (i) is easy to check. For (ii), let M be any object and choose an epimorphism $P \to M$ with P projective. If $(L_i)_{i\in I}$ is the set of subobjects of M, the naturally defined epimorphism $P \oplus (\oplus_I L_i) \to M$ is $\pi^{-1}(\mathcal{O})$-proper. (iii) is then easily verified.

All definitions and results of this section may be dualized to the injective case, where the notations $\iota(\mathcal{E})$ and $\iota^{-1}(\mathcal{O})$ are used corresponding to $\pi(\mathcal{E})$ and $\pi^{-1}(\mathcal{O})$.

3. Flatly generated proper classes

Let now $\mathcal{A}$ be the category of right modules over a ring A. If $\mathcal{E}$ is any proper class of short exact sequences of $\mathcal{A}$, then a left A-module P is called *$\mathcal{E}$-flat* if for every $L \to M$ in $\mathcal{E}_m$ also $L \otimes_A P \to M \otimes_A P$ is a monomorphism. The class of $\mathcal{E}$-flat left modules is denoted by $\tau(\mathcal{E})$. Conversely, for any given class $\mathcal{O}$ of left A-modules we let $\tau^{-1}(\mathcal{O})$ be the class of those short exact sequences of $\mathcal{A}$ for which the modules in $\mathcal{O}$ act as relatively flat modules. It follows from prop. 2.1 that $\tau^{-1}(\mathcal{O})$ is a proper class.

Proposition 3.1. *$\tau^{-1}(\mathcal{O})$ is closed under direct limits.*

Proof. Clear, since direct limits are exact and commute with tensor products.

More information about $\tau^{-1}(\mathcal{O})$ may be obtained by employing duality theory. For every right A-module M we denote by M^* its dual $\mathrm{Hom}_Z(M, Q/Z)$, where Z stands for the integers and Q for the rationals. M^* may be considered as a left A-module by defining $(a\varphi)(x) = \varphi(xa)$ for $x \in M$, $a \in A$ and $\varphi \in M^*$. We do analogously for left A-modules P and then have the duality formula

$$\mathrm{Hom}_A(M, P^*) \cong \mathrm{Hom}_Z(M \otimes_A P, Q/Z) \cong \mathrm{Hom}_A(P, M^*).$$

Since Q/Z is an injective cogenerator for the category of abelian groups, the functors $T: M \to M^*$ and $S: P \to P^*$ are faithful and exact. Let $\mathcal{O}^*$ be the class of dual modules P^* for $P \in \mathcal{O}$. The duality formula then gives:

Proposition 3.2. $\tau^{-1}(\mathcal{O}) = \iota^{-1}(\mathcal{O}^*) = T^{-1}(\pi^{-1}(\mathcal{O}))$.

The functors S and T are "adjoint on the right" and hence we may apply

The adjoint theorem. Let $T: \mathcal{A} \to \mathcal{B}$ and $S: \mathcal{B} \to \mathcal{A}$ be contravariant functors which are adjoint on the right, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. Suppose T is faithful. If $\mathfrak{J}$ is a projectively closed proper class of $\mathcal{B}$ with enough $\mathfrak{J}$-projectives, then $T^{-1}(\mathfrak{J})$ is injectively closed with enough $T^{-1}(\mathfrak{J})$-injectives. M is $T^{-1}(\mathfrak{J})$-injective if and only if M is a direct summand of $S(P)$ for some $\mathfrak{J}$-projective P.

This theorem is proved in [14] (p. 136). Applied to our actual situation it gives:

Proposition 3.3. *Suppose there are enough $\pi^{-1}(O)$-projectives. Then there are enough $\tau^{-1}(O)$-injectives. A right module M is $\tau^{-1}(O)$-injective if and only if it is a direct summand of P^* for some $\pi^{-1}(O)$-projective module P.*

4. $\mathcal{E}$-essential extensions

Maranda [11] has developed a theory of pure-essential extensions of abelian groups which here will be extended to the case of a proper class $\mathcal{E}$ of an abelian category $\mathcal{A}$.

Definitions. *A monomorphism $\alpha: L \to M$ in $\mathcal{E}_m$ is $\mathcal{E}$-essential if every $\varphi: M \to N$, such that $\varphi\alpha$ is a monomorphism belonging to $\mathcal{E}_m$, is a monomorphism. An $\mathcal{E}$-essential monomorphism $\alpha: L \to M$ is maximal if every monomorphism $\varphi: M \to N$, with $\varphi\alpha$ $\mathcal{E}$-essential, is an isomorphism. An $\mathcal{E}$-essential monomorphism $\alpha: L \to M$ with M $\mathcal{E}$-injective is called an $\mathcal{E}$-injective envelope.*

The following three propositions have straight-forward proofs (cf. [11]).

Proposition 4.1. *If $\alpha: L \to M$ is $\mathcal{E}$-essential and $\beta: L \to Q$ is in $\mathcal{E}_m$ with Q $\mathcal{E}$-injective, then there exists a monomorphism $\varphi: M \to Q$ such that $\varphi\alpha = \beta$.*

Proposition 4.2. *If M is $\mathcal{E}$-injective, then it is a maximal $\mathcal{E}$-essential extension of itself.*

Proposition 4.3. *Any two $\mathcal{E}$-injective envelopes of L are equivalent.*

For the rest of this section we assume:

(1) $\mathcal{A}$ is locally small and has exact direct limits.

(2) $\mathcal{E}$ is closed under direct limits.

(3) There are enough $\mathcal{E}$-injectives.

Proposition 4.4. *Let $\alpha: L \to M$ be a maximal $\mathcal{E}$-essential monomorphism. If $\beta: M \to N$ is a monomorphism and $\beta\alpha$ is in $\mathcal{E}_m$, then β is a coretraction.*

Proof. Consider the set $\mathcal{K}$ of those quotient objects $\lambda: N \to K$ of N for which $\lambda\beta\alpha$ is a monomorphism belonging to $\mathcal{E}_m$. Note that also $\lambda\beta$ will be a monomorphism. Using assumptions (1) and (2) above together with Zorn's lemma, we find that $\mathcal{K}$ has a maximal member $\lambda': N \to K'$. One easily verifies that $\lambda'\beta\alpha: L \to K'$ is $\mathcal{E}$-essential, because of the maximality of λ'. But the maximality of α then implies that $\lambda'\beta$ is an isomorphism, and hence β is a coretraction.

Corollary. *M is $\mathcal{E}$-injective if and only if there exists a maximal $\mathcal{E}$-essential monomorphism $\alpha: L \to M$ for some L.*

Proposition 4.5. *Every M has an $\mathcal{E}$-injective envelope $\alpha: M \to E$, and α is a maximal $\mathcal{E}$-essential monomorphism.*

Proof. One only has to show the existence of maximal $\mathcal{E}$-essential extensions of M. By assumption (3) there exists a monomorphism $M \to N$ in $\mathcal{E}_m$ with N $\mathcal{E}$-injective. It is easily seen that the set of subobjects of N which are $\mathcal{E}$-essential extensions of M contains a maximal element E. Using prop. 4.1, one verifies that E is a maximal $\mathcal{E}$-essential extension of M.

5. Subfunctors of thc identity

Let $\mathcal{A}$ be an abelian category with enough projectives. Let S be a subfunctor of the identity functor of $\mathcal{A}$, and suppose S is idempotent and radical, i.e. $S \circ S = S$ and $S(M/S(M))=0$ for every M. To S there corresponds a torsion theory (in the sense of Dickson [5]), where M is a torsion object if $S(M)=M$ and M is torsion-free if $S(M)=0$. Conversely, every torsion theory defines such a functor S. We denote by $\mathcal{D}$ the class of all short exact sequences which split under S. Walker [15] has shown that $\mathcal{D}=\pi^{-1}(\mathcal{J})$, where $\mathcal{J}$ is the class of torsion objects. $\mathcal{J}$ is closed under quotients, so similarly to Prop. 2.4 we have ([15], th. 3.11):

Proposition 5.1. *There are enough $\mathcal{D}$-projectives, and every $\mathcal{D}$-projective object is a direct summand of the direct sum of a projective and a torsion object.*

We write Dext^n for the relative extension functor defined by the proper class $\mathcal{D}$.

Proposition 5.2. $\mathrm{Dext}^1(L, M) = \mathrm{Im}(\mathrm{Ext}^1(L/S(L), M) \to \mathrm{Ext}^1(L, M))$.

Proof. [15], Corollary 2.9.

The following condition is usually satisfied: (α) For every torsion-free M there exists an epimorphism $P \to M$ with P torsion-free projective.

Proposition 5.3. *Suppose* (α) *is satisfied. Then*

$$\mathrm{Dext}^n(L, M) = \mathrm{Ext}^n(L/S(L), M) \text{ for } n > 1.$$

Proof. The sequence $0 \to S(L) \to L \to L/S(L) \to 0$ belongs to $\mathcal{D}$ and gives $\mathrm{Dext}^n(L, M) \simeq \mathrm{Dext}^n(L/S(L), M)$ for $n>1$. This reduces the proof to the case of a torsion-free L. But then (α) guarantees that L has a projective resolution which also is a $\mathcal{D}$-projective resolution.

Note that if (α) is satisfied, then M is $\mathcal{D}$-projective if and only if $M/S(M)$ is projective. Not so much is known about the $\mathcal{D}$-injectives in general, and we only note the following two facts (cf. [9], where Prop. 5.5 and its dual are given in a special case).

Proposition 5.4. *M is $\mathcal{D}$-injective if and only if* $\mathrm{Ext}^1(L, M)=0$ *for every torsion-free L.*

Proof. Follows immediately from Prop. 5.2.

Proposition 5.5. *Suppose projectives are torsion-free. Then every $\mathcal{D}$-injective object has injective dimension* $\leqslant 1$.

Proof. If M is $\mathcal{D}$-injective and L is arbitrary, choose an exact sequence $0 \to K \to P \to L \to 0$ with P torsion-free projective. We obtain

$$\mathrm{Ext}^1(K, M) \to \mathrm{Ext}^2(L, M) \to \mathrm{Ext}^2(P, M)$$

with both end terms zero.

All results in this section may be dualized to the proper class $\mathcal{D}^*$ of short exact sequences which split under the quotient functor $M \rightsquigarrow M/S(M)$ of the identity functor. The explicit formulation of the dualized prop. 5.1*–5.5* may be left to the reader.

6. Purity

$\mathcal{A}$ is now assumed to have sums, products and a projective generator F. Let $\mathcal{O}$ be a set of quotient objects of F, with $F \in \mathcal{O}$.

Definition. *An exact sequence $0 \to L \to M \to N \to 0$ is $\mathcal{O}$-pure if it belongs to $\pi^{-1}(\mathcal{O})$. L is then an $\mathcal{O}$-pure subject of M.*

We know, by Prop. 2.3, what the $\mathcal{O}$-pure-projectives look like and we know that there are enough of them. A useful characterization of pure sequences is given by

Proposition 6.1. *An exact sequence $0 \to L \to M \to N \to 0$ is $\mathcal{O}$-pure if and only if for every commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\gamma} & F & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \xrightarrow[\mu]{} & N & \longrightarrow & 0 \end{array}$$

with $H \in \mathcal{O}$, the following equivalent statements are true:

(*a*) *there exists $\tau: H \to M$ such that $\mu\tau = \psi$*
(*b*) *there exists $\sigma: F \to L$ such that $\sigma\gamma = \varphi$.*

Proof. $\mathcal{O}$-purity is equivalent to (*a*) since F is projective. The equivalence of (*a*) and (*b*) is quite obvious.

We will denote by $\mathcal{O}'$ the set obtained from $\mathcal{O}$ by excluding F. The set $\mathcal{O}'$ generates a torsion theory in the manner described in sec. 3 of [5], so that M is $\mathcal{O}$-torsion-free if and only if $\mathrm{Hom}(P, M) = 0$ for every $P \in \mathcal{O}'$. The corresponding idempotent and radical subfunctor of the identity will be denoted by T. $T(M)$ may be described as the smallest $\mathcal{O}$-pure subobject of M such that every $P \to M$, with $P \in \mathcal{O}'$, factors through it; in other words:

Proposition 6.2. *The following statements are equivalent:*

(*a*) *L is $\mathcal{O}$-pure in M and contains $T(M)$.*
(*b*) *M/L is $\mathcal{O}$-torsion-free.*

Proposition 6.3. *Suppose that direct sums of F are $\mathcal{O}$-torsion-free. The following statements are then equivalent for N:*

(*a*) *N is $\mathcal{O}$-torsion-free.*
(*b*) *Every exact sequence $0 \to L \to M \to N \to 0$ is $\mathcal{O}$-pure.*
(*c*) *There exists an $\mathcal{O}$-pure sequence $0 \to L \to M \to N \to 0$ with M $\mathcal{O}$-torsion-free.*

Proof. Easy verification.

There is another subfunctor of the identity functor which also is of some interest. Let us say that M is of *$\mathcal{O}$-type* if there exists an epimorphism $\oplus P_i \to M$ with $P_i \in \mathcal{O}'$. Every M contains a unique maximal subobject $t(M)$ of $\mathcal{O}$-type, and this defines an idempotent subfunctor t of the identity. M is $\mathcal{O}$-torsion-free if and only if $t(M)=0$, and it follows that T may be described as the smallest radical containing t (cf. Maranda [12], p. 109–110). If $\mathcal{O}'$ is closed under quotients, then an object is of $\mathcal{O}$-type if and only if it is the sum of its subobjects isomorphic to objects in $\mathcal{O}'$.

7. $\mathcal{O}$-injective objects

In the case of usual purity for abelian groups it is clear that the union of any ascending chain of pure subgroups of a group G is a pure subgroup of G. The analogous result for general purity is related to Bass' well-known characterization of noetherian rings ([3], Prop. 4.1). We now assume that $\mathcal{A}$ has exact direct limits and a projective generator F. Let $\mathcal{O}$ be a set of quotient objects of F and Γ the corresponding set of subobjects of F. An object M is called *$\mathcal{O}$-injective* if $\mathrm{Ext}^1(P, M)=0$ for every $P \in \mathcal{O}$.

Proposition 7.1. *The following statements are equivalent for L:*

(*a*) *L is $\mathcal{O}$-injective.*

(*b*) *Every exact sequence $0 \to L \to M \to N \to 0$ is $\mathcal{O}$-pure.*

(*c*) *There exists an $\mathcal{O}$-pure sequence $0 \to L \to M \to N \to 0$ with M $\mathcal{O}$-injective.*

Proof. Easy verification.

Proposition 7.2. *The following statements are equivalent:*

(*a*) *Every quotient object of an $\mathcal{O}$-injective object is $\mathcal{O}$-injective.*

(*b*) *Every object in Γ is projective.*

Proof. Cf. [8], Prop. 7.

Proposition 7.3. *Consider the following properties of $\mathcal{O}$:*

(*a*) *Every object in Γ is of finite type.*

(*b*) *For every directed family $(L_j)_J$ of $\mathcal{O}$-pure subobjects of an object M, also $\Sigma_J L_j$ is $\mathcal{O}$-pure in M.*

(*c*) *For every directed family $(L_j)_J$ of $\mathcal{O}$-injective subobjects of an object M, also $\Sigma_J L_j$ is $\mathcal{O}$-injective.*

(*d*) *The direct sum of any family of $\mathcal{O}$-injective objects is $\mathcal{O}$-injective.*

The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) always hold.

Proof. (*a*) $\Rightarrow$ (*b*): We use Prop. 6.1 and consider any diagram

$$\begin{array}{ccc} G & \longrightarrow & F \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \Sigma L_j & \longrightarrow & M \end{array}$$

with G in Γ. Since G is of finite type, φ carries it into some L_j. The existence of $\sigma_j: F \to L_j$ gives $\sigma: F \to \Sigma L_j$ as desired.

$(b) \Rightarrow (c)$: It follows from Prop. 7.1 that every L_j is $\mathcal{O}$-pure in the injective envelope $E(M)$ of M. By hypothesis, ΣL_j is then $\mathcal{O}$-pure in $E(M)$ and therefore $\mathcal{O}$-injective, by Prop. 7.1.

$(c) \Rightarrow (d)$ is clear.

When proving the converse of this proposition we require F to be a small projective, in which case $\mathcal{A}$ is equivalent to the category of modules over Hom(F, F) ([14], ch. IV.4).

Proposition 7.4. *Let $\mathcal{A}$ be the category of left modules over a ring A whose left ideals are countably generated. If F is finitely generated, then the four conditions in* Prop. 7.3 *are equivalent.*

Proof. It remains to show $(d) \Rightarrow (a)$. Let G be any module in Γ and consider any ascending chain $G_1 \subset G_2 \subset \ldots$ with $\bigcup_1^\infty G_n = G$. There is a well-defined morphism $G \to \oplus_1^\infty E(F/G_n)$, which may be extended to F since $\mathrm{Ext}^1(F/G, \oplus_1^\infty E(F/G_n)) = 0$ by hypothesis. But F is finitely generated and hence mapped into $\oplus_1^m E(F/G_n)$ for some $m < \infty$. The same is then true for G, so the chain must be stationary.

8. Pext

The notations of sec. 6 are retained. We also assume that projectives are $\mathcal{O}$-torsion-free. The relative extension functor corresponding to the proper class of $\mathcal{O}$-pure sequences will be written Pext^n. We may also consider the proper class $\mathcal{D}$ of short exact sequences which split under the $\mathcal{O}$-torsion functor T, and its relative extension functor Dext^n. Since every object in $\mathcal{O}$ is $\mathcal{D}$-projective, it follows that

$$\mathrm{Dext}^1 (L, M) \subset \mathrm{Pext}^1 (L, M)$$

for all L and M. We recall that M is $\mathcal{D}$-projective if and only if $M/T(M)$ is projective, while $\mathcal{D}$-injectives are called *$\mathcal{O}$-cotorsion* objects in the terminology of Fuchs [7].

Whereas Dext may be described explicitly in terms of Ext (Prop. 5.2, 5.3), it is more difficult to compute Pext.

Proposition 8.1. ([15], Th. 2.8.). *Suppose $\mathcal{O}'$ is closed under quotients, and let $\{L_j\}_J$ be the set of subobjects of L which belong to $\mathcal{O}'$. Then* $\mathrm{Pext}^1 (L, M) = \bigcap_J \mathrm{Im}\, f_j$, *where* $f_j : \mathrm{Ext}^1 (L/L_j, M) \to \mathrm{Ext}^1 (L, M)$ *naturally.*

Proposition 8.2. *If L is $\mathcal{O}$-torsion-free, then*

$$\mathrm{Dext}^n (L, M) = \mathrm{Pext}^n (L, M) = \mathrm{Ext}^n (L, M) \quad \textit{for all } n \textit{ and } M.$$

Proof. L has a projective resolution which also is a $\mathcal{D}$-projective and pure-projective resolution.

The following condition is satisfied in many applications: (β) For every torsion object L there exists an $\mathcal{O}$-pure sequence $0 \to K \to P \to L \to 0$ where P is an $\mathcal{O}$-pure-projective torsion object and also K is a torsion object.

Proposition 8.3. *If (β) is satisfied, and M is $\mathcal{O}$-torsion-free, then*

$$\mathrm{Dext}^n (L, M) = \mathrm{Pext}^n (L, M) = \mathrm{Ext}^n (L/T(L), M) \quad \textit{for all } n.$$

Proof. We first suppose that L is a torsion object. Using the sequence given above by (β) we get

$$\mathrm{Pext}^n (K, M) \to \mathrm{Pext}^n (L, M) \to \mathrm{Pext}^n (P, M) = 0,$$

and the result follows by induction on n. In the general case, the $\mathcal{O}$-pure sequence $0 \to T(L) \to L \to L/T(L) \to 0$ gives $\mathrm{Pext}^n(L, M) = \mathrm{Pext}^n(L/T(L), M) \simeq \mathrm{Ext}^n(L/T(L), M)$, using Prop. 8.2.

Concerning the pure-injectives, we notice that every $\mathcal{O}$-pure-injective object of course is an $\mathcal{O}$-cotorsion object and hence has injective dimension $\leqslant 1$ (Prop. 5.5). Conversely it follows from Prop. 8.3 that if M is $\mathcal{O}$-torsion-free and $\mathcal{O}$-cotorsion, then M is $\mathcal{O}$-pure-injective (assuming (β)).

9. Examples of purity

$\mathcal{A}$ is now the category of left modules over a ring A. We will study $\mathcal{O}$-purity for different choices of the set $\mathcal{O}$ of quotient modules of a free module F.

9.1. F is a free module on an infinite basis, and $\mathcal{O}$ is the set of modules F/G for all finitely generated submodules G of F.

Proposition 9.1. *The following statements are equivalent for an exact sequence* $0 \to L \to M \to N \to 0$:

(*a*) *It is* $\mathcal{O}$*-pure.*

(*b*) *The sequence* $0 \to V \otimes_A L \to V \otimes_A M \to V \otimes_A N \to 0$ *is exact for every right* A*-module* V.

(*c*) *The sequence* $0 \to N^* \to M^* \to L^* \to 0$ *of dual modules splits.*

Proof. $(a) \Leftrightarrow (b)$ has been proved by P. M. Cohn [4]. $(b) \Leftrightarrow (c)$ follows from Prop. 3.2. In this case there are not only enough $\mathcal{O}$-pure-projectives, but also enough $\mathcal{O}$-pure-injectives.

Proposition 9.2. *The following properties of a left module* M *are equivalent:*

(*a*) M *is* $\mathcal{O}$*-pure-injective.*

(*b*) M *is a direct summand of* P^* *for some right module* P.

(*c*) *Every system of equations* $\sum_j a_{ij} X_j = y_i$ *(where* $i \in I$, $j \in J$, $y_i \in M$; I *and* J *are arbitrary but summation is finite) is solvable in* M *whenever every finite subsystem is solvable in* M.

Proof. $(a) \Leftrightarrow (b)$ follows from Prop. 3.3. $(a) \Leftrightarrow (c)$ follows by a rather direct generalization of the proof given in [6] for the case of abelian groups.

9.2. F is a free module on an infinite basis of cardinality $\mathfrak{m}$, and $\mathcal{O}$ consists of F and all modules generated by $< \mathfrak{m}$ elements (considered as quotients of F). This gives $\mathfrak{m}$-*purity*, which is studied in detail by Walker in [15].

In the remaining examples we take $F = A$ and put $\Gamma = \{$left ideals $I \neq 0$ such that $A/I \in \mathcal{O}\}$. $t(M)$ is now the submodule generated by the set $\{x \in M \mid Ix = 0$ for some $I \in \Gamma\}$. We note:

(i) If Γ consists of two-sided ideals, then T is left-exact (use Th. 2.9 of [5]).

(ii) If Γ consists of two-sided ideals and for every pair I_1, I_2 of ideals in Γ there exists an ideal $I \in \Gamma$ such that $I \subset I_1 \cap I_2$, then $t(M) = \{x \in M \mid Ix = 0$ for some $I \in \Gamma\}$.

(iii) Suppose T is left exact. Condition (β) in sec. 8 is then satisfied if and only if $t = T$.

9.3. $\Gamma = \{\text{all left ideals} \neq 0\}$. L is $\mathcal{O}$-pure (we simply say *pure*) in M if and only if for every $u \in M/L$ there exists $x \in M$, mapping canonically on u, with $\text{Ann}(u) = \text{Ann}(x)$. Other descriptions of purity may be obtained by means of Prop. 6.1. A module is pure-projective just when it is a direct summand of a direct sum of cyclic modules. The $\mathcal{O}$-injective modules are just the ordinary injective modules. The associated torsion theory is the same one as in the next example (and is trivial if A has zero-divisors).

9.4. $\Gamma = \{\text{all principal left ideals} \neq 0\}$. L is $\mathcal{O}$-pure (we say *weakly pure*) in M if and only if $aM \cap L = aL$ for every $a \in A$ (prop. 6.1). M is $\mathcal{O}$-injective if and only if it satisfies: if $a \in A$, $x \in M$ and $ba = 0$ implies $bx = 0$ for all $b \in A$, then $x \in aM$ ([8], where such modules are called "divisible").

9.5. $\Gamma = \{\text{all principal left ideals generated by non-zero-divisors}\}$. L is $\mathcal{O}$-pure in M when $aM \cap L = aL$ holds for every non-zero-divisor a. M is $\mathcal{O}$-injective (we say *divisible*, cf. sec. 13.3 and [10]) if and only if $M = aM$ for every non-zero-divisor a. The $\mathcal{O}$-torsion-free modules coincide with those called torsion-free by Levy [10]. Levy shows that $t(M) = \{x \in M \mid ax = 0 \text{ for some non-zero-divisor } a\}$ exactly when A has a left ring of fractions, and in that case t and T obviously coincide and are left exact.

9.6. $\Gamma = \{\text{maximal left ideals}\}$. $\mathcal{O}$-pure submodules are called *neat*. A module is neat-projective if and only if it is a direct summand of the direct sum of a free module and a semi-simple module. The modules of type $\mathcal{O}$ are just the semi-simple modules. $t(M)$ is the socle of M, and $T(M)$ is the minimal neat submodule of M containing the socle of M.

Proposition 9.3. *$T(M)$ is the sum of all artinian submodules of M.*

Proof. Put $S(M) =$ sum of all artinian submodules of M. S is an idempotent and radical subfunctor of the identity functor and therefore defines a torsion theory. But this torsion theory coincides with the one determined by T, since the torsion-free modules obviously are the same in both cases.

Both t and T are left exact functors in this case. This torsion theory has also been studied by Dickson ([5], sec. 4). When A is a commutative noetherian ring, T coincides with the functor X used by Matlis in [13].

10. Copurity

A dualization of the purity theory introduced in sec. 6 gives a theory of copurity, which also is a generalization of the usual notion of purity for abelian groups. The abelian category $\mathcal{A}$ is assumed to have sums, products and an injective cogenerator K. Let $\mathfrak{J}$ be a set of subobjects of K, with $K \in \mathfrak{J}$.

Definition. *A short exact sequence is $\mathfrak{J}$-copure if it belongs to $\iota^{-1}(\mathfrak{J})$.*

By prop. 2.3* we know that there are enough $\mathfrak{J}$-copure-injectives. Let $\mathfrak{J}'$ be the set obtained from $\mathfrak{J}$ by excluding K. The set $\mathfrak{J}'$ cogenerates a torsion theory ([5], sec. 3; instead of the terms "torsion" and "torsion-free" we use the terms "divisible" and "reduced") with L $\mathfrak{J}$-divisible if and only if $\text{Hom}(L, Q) = 0$ for every $Q \in \mathfrak{J}'$.

The corresponding idempotent and radical subfunctor of the identity functor will be denoted by D. $D(M)$ is the largest $\mathfrak{J}$-copure subobject of M such that every $M \to Q$, with $Q \in \mathfrak{J}'$, factors through $M/D(M)$. Dually to Prop. 6.2 and 6.3 we have:

Proposition 10.1. *The following statements are equivalent:*

(*a*) *L is $\mathfrak{J}$-copure in M and $L \subset D(M)$.*
(*b*) *L is $\mathfrak{J}$-divisible.*

Proposition 10.2. *Suppose the products of K are $\mathfrak{J}$-divisible. The following statements are then equivalent for L:*

(*a*) *L is $\mathfrak{J}$-divisible.*
(*b*) *Every exact sequence $0 \to L \to M \to N \to 0$ is $\mathfrak{J}$-copure.*
(*c*) *There exists an $\mathfrak{J}$-copure sequence $0 \to L \to M \to N \to 0$ with M $\mathfrak{J}$-divisible.*

11. Copurity in module catcgories

The general notions of the preceding section will now be applied to the study of copurity in the category $\mathcal{A}^*$ of right A-modules. Let F be a projective generator for the category $\mathcal{A}$ of left A-modules and let $\mathcal{O}$ be a set of quotient objects of F, $F \in \mathcal{O}$. $F^* = \mathrm{Hom}_Z(F, Q/Z)$ is then an injective cogenerator for A^* and to $\mathcal{O}$ there corresponds a set $\mathcal{O}^*$ of subobjects of F^*. Prop. 3.2 gives:

Proposition 11.1. *The following statements are equivalent for an exact sequence $0 \to L \to M \to N \to 0$ in A^*:*

(*a*) *It is $\mathcal{O}^*$-copure.*
(*b*) *$0 \to N^* \to M^* \to L^* \to 0$ is $\mathcal{O}$-pure.*
(*c*) *$0 \to L \otimes_A P \to M \otimes_A P \to N \otimes_A P \to 0$ is exact for $P \in \mathcal{O}$.*

M is $\mathcal{O}^*$-copure-injective if and only if M is a direct summand of a direct product of modules P^* with $P \in \mathcal{O}$. Also note that $\iota^{-1}(\mathcal{O}^*)$ is closed under direct limits (prop. 3.1), so the theory of relative essential extensions (sec. 4) works in the case of $\mathcal{O}^*$-copurity.

There are some relations between $\mathcal{O}$-torsion and $\mathcal{O}^*$-divisibility, which are described in the following propositions. The proofs are obvious.

Proposition 11.2. *The following properties of a right module M are equivalent:*

(*a*) *M is $\mathcal{O}^*$-divisible.*
(*b*) *$M \otimes_A P = 0$ for every $P \neq F$ in $\mathcal{O}$.*
(*c*) *M^* is $\mathcal{O}$-torsion-free.*

Proposition 11.3. *If M is an $\mathcal{O}$-torsion left module, then M^* is $\mathcal{O}^*$-reduced.*

Proposition 11.4. *If L is an $\mathcal{O}^*$-divisible right module and M is an $\mathcal{O}$-torsion left module, then $L \otimes_A M = 0$.*

Proposition 11.5. *Assume A is commutative. Then:*

(i) *If L or M is $\mathcal{O}^*$-divisible, then $L \otimes_A M$ is $\mathcal{O}^*$-divisible.*
(ii) *If L is $\mathcal{O}^*$-divisible or M is $\mathcal{O}$-torsion-free, then $\mathrm{Hom}_A(L, M)$ is $\mathcal{O}$-torsion-free.*

12. $\mathcal{O}$-flat modules

We keep the notations of the preceding section. The theory of $\mathcal{O}$-flat modules is obtained by a dualization of the theory of $\mathcal{O}$-injective modules (sec. 7).

Definition. *A right A-module M is $\mathcal{O}$-flat if* $\operatorname{Tor}_1^A(P, M)=0$ *for every* $P \in \mathcal{O}$.

The duality formula $\operatorname{Ext}_A^1(L, M^*) \cong \operatorname{Hom}_Z(\operatorname{Tor}_1^A(L, M), Q/Z)$ gives:

Proposition 12.1. *M is $\mathcal{O}$-flat if and only if M^* is $\mathcal{O}$-injective.*

The proofs of the following two results are dual to the proofs of Prop. 7.1 and 7.2. As in sec. 7 we write Γ for the set of submodules of F corresponding to $\mathcal{O}$.

Proposition 12.2. *The following statements are equivalent for a given right module N:*

(*a*) *N is $\mathcal{O}$-flat.*
(*b*) *Every exact sequence $0 \to L \to M \to N \to 0$ is $\mathcal{O}^*$-copure.*
(*c*) *There exists an $\mathcal{O}^*$-copure sequence $0 \to L \to M \to N \to 0$ with M $\mathcal{O}$-flat.*

Proposition 12.3. *The following statements are equivalent:*

(*a*) *Every submodule of an $\mathcal{O}$-flat module is $\mathcal{O}$-flat.*
(*b*) *All modules in Γ are flat.*

13. Examples of copurity

In the following examples of $\mathcal{O}^*$-copurity we choose $F=A$ and put $\Gamma=\{$left ideals $I \neq 0$ such that $A/I \in \mathcal{O}\}$. A submodule L of M is $\mathcal{O}^*$-copure in M if and only if $L \cap MI = LI$ for every $I \in \Gamma$ (Prop. 11.1). M is $\mathcal{O}^*$-divisible if and only if $M=MI$ for every $I \in \Gamma$.

13.1. $\Gamma=\{$all left ideals $\neq 0\}$, or equivalently $\{$all finitely generated left ideals $\neq 0\}$. $\mathcal{O}^*$-copure sequences will simply be called *copure*. The $\mathcal{O}$-flat modules are just the ordinary flat modules.

13.2. $\Gamma=\{$all principal left ideals $\neq 0\}$. The $\mathcal{O}^*$-copure sequences are just the weakly pure sequences (9.4). So in the case of weak purity there are both enough relative projectives and enough relative injectives. M is $\mathcal{O}$-flat if and only if it satisfies: if $a \in A$, $x \in M$ and $xa=0$, then $x \in Mb$ for some $b \in A$ with $xa=0$ ([8], where such modules are called "torsion-free").

13.3. $\Gamma=\{$all principal left ideals generated by non-zero-divisors$\}$. $\mathcal{O}^*$-copurity coincides with the kind of purity introduced in sec. 9.5. The $\mathcal{O}^*$-divisible modules coincide with those called divisible in sec. 9.5 and [10]. M is $\mathcal{O}$-flat if and only if it satisfies: if $a \in A$, $x \in M$ and $xa=0$, then $x=0$ or a is a zero-divisor.

14. Purity over Prüfer rings

We will compare the various notions of purity for right modules over a ring A. A sequence which is pure in the sense of Cohn (sec. 9.1) is necessarily $\mathcal{O}^*$-copure, for any choice of $\mathcal{O}$, and it is $\mathcal{O}_1$-pure for $\mathcal{O}_1=\{A/I \mid I$ finitely generated left ideal$\}$.

Proposition 14.1. *Let A be a Prüfer ring. The following properties of a short exact sequence are equivalent:*

(*a*) *It is weakly pure.*
(*b*) *It is pure in the sense of Cohn.*
(*c*) *It is copure.*
If A is a Dedekind ring, they are also equivalent to:
(*d*) *It is pure.*

Proof. It only remains to show $(a) \Rightarrow (b)$. After localization it suffices to do the proof over a valuation ring, where it is easy (and well-known, cf. [1], ch. 6, § 5, exercice 6).

Proposition 14.2. *Let A be an integral domain. A is a Prüfer ring if and only if every pure short exact sequence of A-modules is copure.*

Proof. Necessity follows from prop. 14.1. Conversely, if every pure sequence is copure, then every torsion-free module is flat, by Prop. 6.2 and 12.2. But in such a case A must be a Prüfer ring by [3], Th. 4.2.

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