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On stochastic stationarity of renewal processes

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ABSTRACT

We shall consider point systems in R_1 which are stationary renewal distributed. We let the points undergo random translations which are assumed to be independent identically distributed random variables with a non-degenerate distribution function. The translations are also independent of the starting positions. It is shown in theorem 3.1 that the only distribution of the points which is conserved after the random translations is the Poisson one. Finally in section 4 we give a characterization of renewal processes on the positive semiaxis in terms of conditional mean values.

1. Introduction

In a paper on point systems in R_1 under independent random motion (see T. Thedéen [8]) we proved that the only time-invariant spatial distributions for the point system are the weighted Poisson ones. We, however, had to impose certain independence conditions to hold for all time $t>0$. We shall in this paper assume that the point system initially is stationary renewal distributed and that the points are subject to independent identically distributed random translations (see theorem 3.1). This result has some implications for the theory of road traffic flow. In the stochastic model for low density traffic the cars are usually considered as points which move independently of each other. Then it follows from theorem 3.1 that the only renewal distribution for the points which is conserved in time is the Poisson one (cf. F. Haight [5] Ch. 4).

In our treatment in the following section we shall however not use the notion of point systems under random motion, which was the origin to our interest in this field. Lastly we shall in section 4 consider a characterization of renewal processes on $(0, \infty)$.

2. Preliminaries

Let $\{X_n, n=\pm 1, \pm 2, \ldots\}$ be an ordered sequence of random variables (r.v.'s) such that almost surely (a.s.)

$$\ldots X_{-2}<X_{-1}<0<X_1<X_2<\ldots$$

Put $X_0=0$ and $Y_n=X_n-X_{n-1}$. We shall assume that $\{X_n\}$ is *stationary renewal distributed* with the distribution function (d.f.) $F(y)$, i.e.

(i) $\{(Y_0, Y_1), Y_n, n\neq 0, 1\}$ is a set of independent positive r.v.'s and

(ii) $\{Y_n, n\neq 0, 1\}$ is a set of independent identically distributed (i.i.d.) r.v.'s with $P(Y_n\leqslant y)=F(y)$, $F(0)=0$ and $EY_n=1/m<\infty$ and

(iii)[1]
$$P(Y_0 > y_0,\ Y_1 > y_1) = \int_{y_0+y_1}^{\infty} m(1-F(y))\,dy. \tag{2.1}$$

Let for any finite interval I

$$N(I) = \text{no. of } X_n \in I, \quad n \neq 0.$$

Then it is well-known that
$$EN(I) = m|I|. \tag{2.2}$$

It follows from the theory of renewal processes on $(0, \infty)$ that the distribution of $\{X_n\}$ is determined by the so-called *renewal function*

$$H(x) = \sum_{k=1}^{\infty} F^{k*}(x)$$

where $*$ stands for convolution. We shall define $H(x)$ for negative x as

$$H(x) = -H(-x-0), \quad x<0 \tag{2.3}$$

and this equation (2.3) should be used to define any renewal function for a negative argument. Then any renewal function $H(x)$ is a right-continuous non-decreasing function on $(-\infty, +\infty)$. Let us note that if $H(x)=mx$ then $\{X_n\}$ has the same distribution as the set of discontinuity points of a Poisson process with intensity m, shortly $\{X_n\}$ is Poisson distributed with the parameter m. In the case when there exists a $d>0$ such that

$$\sum_{k=1}^{\infty} P(Y_n = kd) = 1, \quad n \neq 0, 1 \tag{2.4}$$

we will say that $\{X_n\}$ is *discrete*. The largest d for which (2.4) is fulfilled is called the *span*. If $P(Y_n=d)=1$ we shall say that $\{X_n\}$ is deterministic. If $\{X_n\}$ is not discrete it will be called *continuous*.

Let Z_n, $n=\pm 1, \pm 2, \dots$ be i.i.d. r.v.'s with $P(Z_n \leqslant z) = G(z)$. Let us further assume that $\{X_n\}$ and $\{Z_n\}$ are independent. Define $\{X_{1n}, n=\pm 1, \pm 2, \dots\}$ by

$$X_{1n} = X_n + Z_n.$$

We shall say that $\{X_{1n}\}$ is stationary renewal distributed with the d.f. $F(y)$ if the sequence obtained by ordering $\{X_{1n}\}$ is stationary renewal distributed with the d.f. $F(y)$. (It can be shown that with $N_1(I)=$ no. of $X_{1n} \in I$, I finite interval, we always have $EN_1(I)=m|I|$. Thus $\{X_{1n}\}$ can, irrespective of its distribution, almost surely (a.s.) be ordered.) It is seen at once that if $G(z)$ is degenerated then $\{X_{1n}\}$ is stationary renewal distributed with the d.f. $F(y)$. Further for any d.f. $G(z)$ if $\{X_n\}$ is Poisson distributed with the parameter m, then $\{X_{1n}\}$ has the same distribution (see Doob [2] pp. 404–407). The Poisson distribution is a stationary renewal distribution which is, what we shall call, stochastic stationary. Using the notation introduced above we have

Definition 2.1. *Let $G(z)$ be a non-degenerated d.f. The stationary renewal distribution of $\{X_n\}$ is stochastic stationary with respect to $G(z)$ if $\{X_{1n}\}$ is stationary renewal distributed with the same d.f. as $\{X_n\}$.*

[1] Cf. Feller [3] p. 371, problem 3.

In the following section we shall prove that the only stationary renewal distribution which is stochastic stationary with respect to a non-degenerated d.f. $G(z)$ is the Poisson one.

3. Stochastic stationarity

We shall need the following lemma in the proof of theorem 3.1.

Lemma 3.1. *Let $K_i(x)$, $i=1,2$ and $F(x)$ be d.f.'s on $(0,\infty)$ and let*

$$H_{K_i}(x)=\begin{cases} K_i(x)+\sum_{k=1}^{\infty} K_i(x)\ast F^{k*}(x), & 0\leqslant x<\infty \\ -H_{K_i}(-x-0), \ -\infty<x<0, & i=1,2. \end{cases}$$

Then if $\quad K_1(x)\leqslant K_2(x),\quad 0\leqslant x<\infty$ *we have*

$$H_{K_1}(x)\leqslant H_{K_2}(x),\quad 0\leqslant x<\infty$$

$$H_{K_1}(x)\geqslant H_{K_2}(x),\quad -\infty<x<0.$$

The *proof* follows at once from the given definition of $H_{K_i}(x)$, $i=1,2$.

Theorem 3.1. *Let $\{X_n\}$ be stationary renewal distributed. Then the distribution of $\{X_n\}$ is stochastic stationary with respect to a non-degenerated d.f. $G(z)$ if and only if $\{X_n\}$ is Poisson distributed.*

Proof. The sufficiency is well-known (see Doob [2] pp. 404–407).

Necessity. The idea of the proof is the following. We shall in point 1–7 of the proof deduce the integral equation

$$H(x)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(H(x-z_1+z_2)-H(z_2-z_1))\,dG(z_1)\,dG(z_2). \tag{3.1}$$

Using known results we shall in point 8 show that the only possible solutions of (3.1) are $H(x)=a_1x$ and in the case when $G(z)$ is d-lattice $H(x)=[a_2x]$ where a_1 and a_2 are constants. Lastly we shall rule out $H(x)=[a_2x]$. Thus

$$H(x)=a_1\cdot x$$

which corresponds to $\{X_n\}$ being Poisson distributed.

1. We shall use the following notation:

$$M(B)=\text{no. of } (X_n, Z_n)\in B,\ B \text{ Borel set in } R_2,$$

$$M_1(B)=\text{no. of } (X_{1n}, Z_n)\in B,\ B \text{ Borel set in } R_2.$$

Let $\mathcal{B}_i$ be the σ-algebra of Borel sets in the z_i-axis L_i and let μ_i be the probability measure on $\mathcal{B}_i$ corresponding to the d.f. $G(z_i)$, $i=1,2$.

Let further $\mu=\mu_1\times\mu_2$ be the product measure on $(L,\mathcal{B})=(L_1\times L_2, \mathcal{B}_1\times\mathcal{B}_2)$. Let $I_1=(-h,0]$ and $I_2=(0,x]$.

Then $\nu_1(B_1) = M_1(I_1 \times B_1)$ and $\nu_2(B_2) = M_1(I_2 \times B_2)$ are a.s. finite measures on $(L_1, \mathcal{B}_1)$ and $(L_2, \mathcal{B}_2)$ respectively. They generate an a.s. finite product measure $\nu = \nu_1 \times \nu_2$ on $(L, \mathcal{B})$. Since $E\nu(L) < \infty$ (cf. point 4 of the proof) the set function $\varkappa$ defined by

$$\varkappa(B) = E\nu(B),\ B \in \mathcal{B}$$

is a finite measure on $(L, \mathcal{B})$.

Let now for $B \in \mathcal{B}$

$$X_{jk}(B) = \begin{cases} 1, X_{1j} \in I_1, X_{1k} \in I_2, (Z_j, Z_k) \in B \\ 0 \text{ otherwise.} \end{cases}$$

Then
$$\nu(B) = \sum_{\text{all } j,\, k} \chi_{jk}(B).$$

But $\mu(B) = 0$ implies that $E\chi_{jk}(B) = 0$. Since $\varkappa(B) = E\nu(B)$ we conclude that $\varkappa$ is absolutely continuous with respect to μ.

By the Radon–Nikodym theorem there is a function $f(z_1, z_2)$ such that

$$\varkappa(B) = \iint_B f(z_1, z_2)\, dG(z_1)\, dG(z_2). \tag{3.2}$$

The results of the following points 2–4 will make it possible to estimate $f(z_1, z_2)$. We shall return to equation (3.2) in point 5.

2. Put $Y = Y_0 + Y_1$. Using the definition of a stationary renewal distribution we get the conditional d.f. of Y given Y_0

$$F(y \mid y_0) = \begin{cases} 0, & y < y_0 \\ \dfrac{F(y) - F(y_0)}{1 - F(y_0)}, & y \geqslant y_0 \end{cases}$$

for $y_0 < \sup\{y;\ F(y) < 1\}$ (which we assume to hold in the following).

For $0 \leqslant h \leqslant h_n$ we have

$$F(y) \geqslant F(y \mid h) \geqslant F(y \mid h_n), \quad 0 \leqslant y < \infty. \tag{3.3}$$

Further it is seen that

$$\lim_{h \downarrow 0} F(y \mid h) = F(y); \quad \lim_{h_n \downarrow h} F(y \mid h_n) = F(y \mid h). \tag{3.4}$$

Using the notation of lemma 3.1 we define

$$H_{F(y|h)}(x) = F(x \mid h) + \sum_{k=1}^{\infty} F(x \mid h) * F^{k*}(x), \quad x \geqslant 0$$

and $H_{F(y|h_n)}(x)$ in the same way. Let us put $H_{h_n}(x)$ for $H_{F(y|h_n)}(x)$ and $H_h(x)$ for $H_{F(y|h)}(x)$. Then we get from lemma 3.1

$$H_{h_n}(x) \leqslant H_h(x) \leqslant H(x), \quad 0 \leqslant x < \infty, \quad 0 < h \leqslant h_n. \tag{3.5}$$

By the definitions of $H(x)$, $H_{h_n}(x)$ and $H_h(x)$ also for negative x (see p. 2) and (3.4) it is easily shown that

$$\lim_{h_n \downarrow h} H_{h_n}(x) = H_h(x); \quad \lim_{h \downarrow 0} H_h(x) = H(x), \quad -\infty < x < \infty. \tag{3.6}$$

Let $N(I)$ = no. of $X_n \in I$, where I is a finite interval. Then

$$N((0, x]) = N((X_{-1}, x])$$

and hence
$$EN((0, x]) \mid X_{-1} = EN((X_{-1}, x]) \mid X_{-1} \text{ a.s.} \tag{3.7}$$

Consider the case when $X_{-1} > -h$ or since $Y_0 = -X_{-1}$ equivalently $Y_0 < h$. Using (3.5) and (3.7) we get for $x \geq 0$

$$EN((0, x]) \mid X_{-1} = EN((-Y_0, x]) \mid Y_0 = H_{F(y \mid Y_0)}(x + Y_0) \leq H_{F(y \mid Y_0)}(x+h) \leq H(x+h) \text{ a.s.}$$

In the same way it is seen that

$$EN((0, x]) \mid X_{-1} \geq H_h(x) \text{ a.s.} \quad x \geq 0.$$

Thus for $x \geq 0$

$$H_h(x) \leq EN((0, x]) \mid X_{-1} \leq H(x+h), \; X_{-1} > -h, \text{ a.s.} \tag{3.8}$$

In the same way it can be proved that for $x < -h$

$$-H_h(x+h) \leq EN((x, -h]) \mid X_{-1} \leq -H(x), \; X_{-1} > -h, \text{ a.s.} \tag{3.9}$$

3. Let J_1 and J_2 be two finite semi-closed intervals closed to the right and let B_1 and B_2 be Borel sets in R_1 with $G(B_i) > 0$, $i = 1, 2$ (Here $G(B_i)$ stands for $\int_{B_i} dG(z)$.)

We shall in this point give upper and lower bounds for $EM(J_1 \times B_1) M(J_2 \times B_2)$ in the cases (i) $J_1 \cap J_2 = \varnothing$ and (ii) $J_1 \subset J_2$, $B_1 \cap B_2 = \varnothing$. From the stationarity of the renewal distribution of $\{X_n\}$ we conclude that for any finite number c

$$EM((J_1 + c) \times B_1) M((J_2 + c) \times B_2) = EM(J_1 \times B_1) M(J_2 \times B_2).$$

Then we can always choose the right endpoint of J_1 as our origin without changing the value of $EM(J_1 \times B_1) M(J_2 \times B_2)$. Put $J_1 = (-a, 0]$ and $J_2 = (b, d]$

(i) $J_1 \cap J_2 = \varnothing$

For any disjoint finite intervals I and J

$$EM(I \times B_1) M(J \times B_2) = G(B_1) G(B_2) EN(I) N(J). \tag{3.10}$$

Now
$$EN(J_1) N((0, x]) = E_{\{X_{-1} > -a\}} EN(J_1) \mid X_{-1} EN((0, x]) \mid X_{-1}. \tag{3.11}$$

Further
$$E_{\{X_{-1} > -a\}} EN(J_1) \mid X_{-1} = EN(J_1) = ma. \tag{3.12}$$

Using (3.8) and (3.12) in (3.11) we get for $x \geq 0$

$$H_a(x) ma \leq EN(J_1) N((0, x]) \leq H(x+a) ma. \tag{3.13}$$

In the same way we get, using (3.9), that for $x < -a$

$$-H_a(x+a) ma \leq EN(J_1) N((x, -a]) \leq -H(x) ma. \tag{3.14}$$

Now if $b>0$ $\quad EN(J_1)N(J_2)=EN(J_1)N((0,d])-EN(J_1)N((0,b]).$ (3.15)

From (3.10), (3.13) and (3.15) we get

$$H_a(d)-H(b+a)\leqslant\frac{EM(J_1\times B_1)\,M(J_2\times B_2)}{maG(B_1)\,G(B_2)}\leqslant H(d+a)-H_a(b),\quad b>0. \tag{3.16}$$

For $b=0$ we get from (3.10) and (3.13)

$$H_a(d)\leqslant\frac{EM(J_1\times B_1)\,M(J_2\times B_2)}{maG(B_1)\,G(B_2)}\leqslant H(d+a),\quad b=0. \tag{3.17}$$

For $d<-a$ we have

$$EN(J_1)N(J_2)=EN(J_1)N((b,\ -a])-EN(J_1)N((d,\ -a]). \tag{3.18}$$

By (3.10), (3.14) and (3.18) we have

$$H(d)-H_a(b+a)\leqslant\frac{EM(J_1\times B_1)\,M(J_2\times B_2)}{maG(B_1)\,G(B_2)}\leqslant H_a(d+a)-H(b),\quad d<-a. \tag{3.19}$$

For $d=-a$ we get from (3.10) and (3.14)

$$-H_a(b+a)\leqslant\frac{EM(J_1\times B_1)\,M(J_2\times B_2)}{maG(B_1)\,G(B_2)}\leqslant -H(b),\quad d=-a. \tag{3.20}$$

(ii) $J_1\subset J_2$, $B_1\cap B_2=\varnothing$

We have

$$\begin{aligned}EM(J_1\times B_1)M(J_2\times B_2)&\\&=EM(J_1\times B_1)M((b,\ -a]\times B_2)+EM(J_1\times B_1)M(J_1\times B_2)\\&\qquad+EM(J_1\times B_1)M((0,d]\times B_2).\end{aligned} \tag{3.21}$$

The first and last terms in the right member of (3.21) can be estimated by means of (3.20) and (3.17). We get by considering the generating function of $(M(J_1\times B_1),\ M(J_1\times B_2))$ that for $B_1\cap B_2=\varnothing$

$$EM(J_1\times B_1)M(J_1\times B_2)=G(B_1)G(B_2)\,EN(J_1)\,(N(J_1)-1).$$

But (see e.g. Cox [1] p. 56)

$$EN(J_1)\,(N(J_1)-1)=2m\int_0^a H(y)\,dy.$$

Now
$$0\leqslant\frac{1}{a}\int_0^a H(y)\,dy\leqslant H(a)$$

and thus
$$0\leqslant\frac{EM(J_1\times B_1)\,M(J_1\times B_2)}{maG(B_1)\,G(B_2)}\leqslant 2H(a). \tag{3.22}$$

Using (3.17), (3.20) and (3.22) on (3.21) we get

$$H_a(d)-H_a(b+a)\leqslant\frac{EM(J_1\times B_1)\,M(J_2\times B_2)}{maG(B_1)\,G(B_2)}\leqslant H(d+a)-H(b)+2H(a). \quad (3.23)$$

4. Let I and J be finite intervals. We shall here show that

$$\frac{EN(I)\,N(J)}{|I|}\leqslant c_1H(|I|)+c_2H(|J|)+c_3, \quad (3.24)$$

where c_1, c_2 and c_3 are finite constants. It is easily seen that it is no restriction to choose I and J semi-closed, closed to the right. We shall deal with two cases (i) and (ii) separately.

(i) $I\cap J=\emptyset$. Suppose that J is situated to the right of I. It is no restriction to choose the left endpoint of J as our origin. Then it is easily seen that

$$E(N(J)\,|\,X_{-1},X_{-2},\dots)\leqslant E(N([X_1,X_1+|J|]\,|\,X_{-1},X_{-2},\dots)\leqslant H(|J|)+1 \text{ a.s.} \quad (3.25)$$

Since $EN(I)=m|I|$ we get from (3.25)

$$\frac{EN(I)\,N(J)}{|I|}\leqslant m(H(|J|)+1). \quad (3.26)$$

The same result holds when J lies to the left of I, which can be proved in the same way.

(ii) $I\cap J\neq\emptyset$ By the stationarity of the distribution of $\{X_n\}$ it is no restriction to choose the origin such that $I=(-|I|,0]$. Then $J\subset J'=(-|I|-|J|,\ |J|]$ and $I\subset J'$.

$$EN(I)N(J)\leqslant EN(I)N((-|J|-|I|,\ -|I|])+E(N(I))^2+EN(I)N((0,\ |J|]). \quad (3.27)$$

Using (3.26), (3.22) and (3.25) we get from (3.27) that also in this case (ii) the inequality (3.24) holds.

5. Let us consider the left member of (3.2), $\varkappa(B)$ for $B=L_1\times L_2$.
Now by (3.10) with $B_1=L_1$, $B_2=L_2$ and the stochastic stationarity

$$\varkappa(L_1\times L_2)=EN(I_1)N(I_2).$$

By (3.13) we get
$$H_h(x)\leqslant\frac{\varkappa(L_1\times L_2)}{mh}\leqslant H(x+h) \quad (3.28)$$

and from (3.6)
$$\lim_{h\downarrow 0}\frac{\varkappa(L_1\times L_2)}{mh}=H(x). \quad (3.29)$$

6. Let now $D_G=\{u_1,u_2,\dots\}$ be the discontinuity set of $G(z)$ with the corresponding jumps $p_1,p_2,\dots$ and put $D=D_G\times D_G$. Then (3.2) can be written

$$\varkappa(B) = \sum_{B \cap D} p_j p_k f_{jk} + \iint_{B \cap \bar{D}} f(z_1, z_2)\, dG(z_1)\, dG(z_2) \tag{3.30}$$

where $f_{jk} = f(u_j, u_k)$ and $\sum_{B \cap D}$ denotes the sum over all j, k such that $(u_j, u_k) \in B \cap D$.
In this point we shall consider f_{jk}. From (3.30) follows that

$$f_{jk} = \frac{EM_1(I_1 \times \{u_j\})\, M_1(I_2 \times \{u_k\})}{p_j p_k}.$$

Now

$$M_1(I_1 \times \{u_j\}) = M((I_1 - u_j) \times \{u_j\})$$
$$M_1(I_2 \times \{u_k\}) = M((I_2 - u_k) \times \{u_k\})$$

and thus

$$f_{jk} = \frac{EM((I_1 - u_j) \times \{u_j\})\, M((I_2 - u_k) \times \{u_k\})}{p_j p_k}. \tag{3.31}$$

We shall deal with three cases (i), (ii) and (iii) separately.

(i) $u_j \geqslant u_k$. From (3.16) we get

$$H_h(x + u_j - u_k) - H(u_j - u_k + h) \leqslant \frac{f_{jk}}{mh} \leqslant H(x + u_j - u_k + h) - H_h(u_j - u_k). \tag{3.32}$$

(ii) $u_j - u_k < -x$. For sufficiently small h we have $u_j - u_k < -x - h$ and hence $(I_1 - u_j) \cap (I_2 - u_k) = \varnothing$. From (3.19) we get

$$H(x + u_j - u_k) - H_h(u_j - u_k + h) \leqslant \frac{f_{jk}}{mh} \leqslant H_h(x + u_j - u_k + h) - H(u_j - u_k). \tag{3.33}$$

(iii) $-x \leqslant u_j - u_k < 0$. For sufficiently small h we have $u_j - u_k < -h$ and $(I_1 - u_j) \subset (I_2 - u_k)$. Using (3.23) we get

$$H_h(x + u_j - u_k) - H_h(u_j - u_k + h) \leqslant \frac{f_{jk}}{mh} \leqslant H(x + u_j - u_k + h) - H(u_j - u_k) + 2H(h). \tag{3.34}$$

Using (3.6) and the definition of renewal functions for a negative argument we get from (3.32), (3.33) and (3.34) that

$$\lim_{h \downarrow 0} \frac{f_{jk}}{mh} = H(x + u_j - u_k) - H(u_j - u_k). \tag{3.35}$$

7. Let

$$\mathfrak{M}_n = \{A_{jk}^{(n)};\ A_{jk}^{(n)} = [j \cdot 2^{-n}, (j+1)\, 2^{-n}) \times [k \cdot 2^{-n}, (k+1)\, 2^{-n}), \quad j, k = 0, \pm 1, \pm 2, \ldots\}.$$

In the sense of Saks [7] p. 153, $\mathfrak{M}_n$ is a net in $L_1 \times L_2$ with the meshes $A_{jk}^{(n)}$ and $\{\mathfrak{M}_n, n = 1, 2, \ldots\}$ is a regular sequence of nets. The support set of a d.f. $F(x)$ is

$$S_F = \{x;\ F(x+h) - F(x-h) > 0, \quad \text{all} \quad h > 0\}.$$

Let now $z_1, z_2 \in S_G$. For any n there is a mesh $J_{1n} \times J_{2n} \in \mathfrak{M}_n$ with $z_1 \in J_{1n}$, $z_2 \in J_{2n}$. From a theorem by Saks [7] p. 155 and the definition of $\varkappa(B)$ we have

$$\lim_{n\to\infty} \frac{EM_1(I_1 \times J_{1n})\, M_1(I_2 \times J_{2n})}{G(J_{1n})\, G(J_{2n})} = f(z_1, z_2) \tag{3.36}$$

for a.s. all (z_1, z_2). Let now

$$\begin{aligned} I_{in}^{+} &= (I_i - z_i - 2^{-n}) \cup (I_i - z_i + 2^{-n}) \\ I_{in}^{-} &= (I_i - z_i + 2^{-n}) \cap (I_i - z_i - 2^{-n}) \quad n = 1, 2, \ldots;\ i = 1, 2. \end{aligned} \tag{3.37}$$

For sufficiently large n (we shall in the following just consider such n) both I_{in}^{+} and I_{in}^{-} are non-degenerated intervals such that

$$\begin{gathered} I_{in}^{+} \supset I_i - z_i \supset I_{in}^{-}, \quad i = 1, 2 \\ M(I_{in}^{-} \times J_{in}) \leqslant M_1(I_i \times J_{in}) \leqslant M(I_{in}^{+} \times J_{in}), \quad i = 1, 2. \end{gathered} \tag{3.38}$$

By (3.38) we get

$$\begin{aligned} EM(I_{1n}^{-} \times J_{1n}) M(I_{2n}^{-} \times J_{2n}) &\leqslant EM_1(I_1 \times J_{1n}) M_1(I_2 \times J_{2n}) \\ &\leqslant EM(I_{1n}^{+} \times J_{1n}) M(I_{2n}^{+} \times J_{2n}). \end{aligned} \tag{3.39}$$

Now $|I_{in}^{+}| = |I_i| + 2^{-n+1}$. Using this fact, (3.10), (3.24) and (3.36) on (3.39) we see that for fixed x there is a $h_0 < \infty$ and a finite constant C, such that for sufficiently large n

$$\frac{f(z_1, z_2)}{h} \leqslant C \text{ a.s.}, \quad h \leqslant h_0. \tag{3.40}$$

Let D_H be the (countable) discontinuity set of $H(x)$. Let us now consider the case when $(z_1, z_2) \in \bar{D} \cap (S_G \times S_G)$ and $z_1 \neq z_2$, $z_1 \neq z_2 - x$ and further $z_1 - z_2 \in \bar{D}_H$. We shall deal with three cases separately.

(i) $z_1 < z_2 - x$. Choose $h < z_2 - z_1 - x$. Then for sufficiently large n we have $I_{1n}^{+} \cap I_{2n}^{+} = \emptyset$ so that we can use (3.19) in the estimation of the first and last member of (3.39). We get

$$\begin{aligned} (H(z_1 - z_2 + x) - H_{h-2^{-n+1}}(z_1 - z_2 + h)) \frac{h - 2^{-n+1}}{h} &\leqslant \frac{EM_1(I_1 \times J_{1n})\, M_1(I_2 \times J_{2n})}{mhG(J_{1n})\, G(J_{2n})} \\ &\leqslant \frac{h + 2^{-n+1}}{h} (H_{h+2^{-n+1}}(z_1 - z_2 + x + h + 2^{-n+1}) - H(z_1 - z_2 - 2^{-n+1})). \end{aligned} \tag{3.41}$$

Then if first $n \to \infty$ and then $h \downarrow 0$ in (3.41) we get using (3.6)

$$\lim_{h\downarrow 0} \lim_{n\to\infty} \frac{EM_1(I_1 \times J_{1n})\, M_1(I_2 \times J_{2n})}{mhG(J_{1n})\, G(J_{2n})} = H(x + z_1 - z_2) - H(z_1 - z_2). \tag{3.42}$$

(ii) $z_1 > z_2$. For sufficiently large n we have $I_{1n}^+ \cap I_{2n}^+ = \emptyset$. Using (3.16) we get

$$(H_{h-2^{-n+1}}(z_1 - z_2 + x) - H(z_1 - z_2 + h)) \frac{h - 2^{-n+1}}{h} \leqslant \frac{EM_1(I_1 \times J_{1n})\, M_1(I_2 \times J_{2n})}{mh\, G(J_{1n})\, G(J_{2n})}$$

$$\leqslant \frac{h + 2^{-n+1}}{h} (H(z_1 - z_2 + x + h + 2^{-n+1}) - H_{h+2^{-n+1}}(z_1 - z_2 - 2^{-n+1})). \quad (3.43)$$

Letting $n \to \infty$ and then $h \downarrow 0$ in (3.43) we get again (3.42).

(iii) $z_2 - x < z_1 < z_2$. Choose $h < z_2 - z_1$. For sufficiently large n we have

$$I_{1n}^+ \subset I_{2n}^+,\ I_{1n}^- \subset I_{2n}^- \quad \text{and} \quad J_{1n} \cap J_{2n} = \emptyset.$$

Then from (3.23)

$$(H_{h-2^{-n+1}}(z_1 - z_2 + x) - H_{h-2^{-n+1}}(z_1 - z_2 + h)) \frac{h - 2^{-n+1}}{h} \leqslant \frac{EM_1(I_1 \times J_{1n})\, M_1(I_2 \times J_{2n})}{mh\, G(J_{1n})\, G(J_{2n})}$$

$$\leqslant \frac{h + 2^{-n+1}}{h} (H(z_1 - z_2 + x + h + 2^{-n+1}) - H(z_1 - z_2 - 2^{-n+1}) + 2H(h + 2^{-n+1})). \quad (3.44)$$

Letting first $n \to \infty$ and then $h \downarrow 0$ we obtain again (3.42). For fixed $h > 0$ we see from (3.36), (3.32), (3.33), (3.34), and the inequalities (3.41), (3.43) and (3.44) that there exist functions $f_h^+(z_1, z_2)$ and $f_h^-(z_1, z_2)$ such that

$$f_h^-(z_1, z_2) \leqslant \frac{f(z_1, z_2)}{mh} \leqslant f_h^+(z_1, z_2) \quad (3.45)$$

for $(z_1, z_2) \in \bar{A}_h$ where $\mu(A_h) = 0$. Put

$$\frac{f(z_1, z_2)}{mh} = \tfrac{1}{2}(f_h^+(z_1, z_2) + f_h^-(z_1, z_2))$$

for $(z_1, z_2) \in A_h$. This will not change the value of the integral in (3.2). Then from (3.35) and (3.42) we have

$$\lim_{h \downarrow 0} \frac{f(z_1, z_2)}{mh} = H(x + z_1 - z_2) - H(z_1 - z_2) \quad (3.46)$$

in the set $A = A_1 \cup (A_2 \cap A_3)$ where

$$A_1 = D_G \times D_G$$

$$A_2 = \bar{D} \cap (S_G \times S_G)$$

$$A_3 = \{(z_1, z_2);\ z_1 \neq z_2,\ z_1 \neq z_2 - x,\ z_1 - z_2 \in \bar{D}_H\}.$$

Since D_H is countable we have $\mu(A) = 1$ and thus (3.46) holds a.s. By (3.40) we have $f(z_1, z_2)/(mh) \leqslant C_1$, except for a fixed (independent of h) μ-null set. Putting $B = L_1 \times L_2$ in (3.2) and using (3.29) and (3.46) we get from the Lebesgue bounded convergence theorem that

$$H(x)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(H(x+z_1-z_2)-H(z_1-z_2))\,dG(z_1)\,dG(z_2) \tag{3.47}$$

for $x\geqslant 0$.

8. Define the d.f. $K(z)$ as

$$K(z)=\int_{-\infty}^{+\infty}G(z+y)\,dG(y).$$

Using that $H(x)=-H(-x-0)$ we can write (3.47)

$$H(x)=\int_{-\infty}^{+\infty}(H(x-z)+H(z-0))\,dK(z),\quad x\geqslant 0. \tag{3.47'}$$

Further for $x<0$ we get from (3.47')

$$H(x)=\int_{-\infty}^{+\infty}(H(x-z)+H(z))\,dK(z),\quad x<0. \tag{3.48}$$

Let $y>0$ be a fixed number. Then by (3.25)

$$\varphi_y(x)=H(x+y)-H(x)\leqslant H(y)+1.$$

In the cases when $x\geqslant 0$ or $x+y<0$ we get from (3.47') and (3.48)

$$\varphi_y(x)=\int_{-\infty}^{+\infty}\varphi_y(x-z)\,dK(z). \tag{3.49}$$

For $x<0$, $x+y\geqslant 0$ we get

$$\varphi_y(x)=\int_{-\infty}^{+\infty}(\varphi_y(x-z)-(H(z)-H(z-0))\,dK(z).$$

Now $$\int_{-\infty}^{+\infty}(H(z)-H(z-0))\,dK(z)=2\sum\Delta H(z_\nu)\,\Delta K(z_\nu)$$

where the sum is over all $z_\nu>0$ with $z_\nu\in D_H$ and $\Delta H(z_\nu)$ and $\Delta K(z_\nu)$ are the jumps of $H(z)$ and $K(z)$ in z_ν. In order that this sum should be larger than zero we must have $D_H\cap D_K\neq\emptyset$. Suppose that e.g. $z_1\in D_H\cap D_K$. From $z_1\in D_H$ we see that there is a n such that $F^{n*}(x)$ also has a discontinuity point at z_1. Then

$$P(X_{n+1}-X_1=z_1)>0$$

and since $z_1\in D_K$ also $$P(Z_1-Z_{n+1}=z_1)>0.$$

Thus we see that $$P(X_{11}=X_{1,\,n+1})>0$$

which contradicts the assumption of $F(0)=0$ (i.e. no X_n's or X_{1n}'s can coincide). Thus (3.49) holds for all x.

It is well-known (see e.g. Feller [4]) that the only bounded solutions of (3.49) are

(i) $\varphi_y(x) = \text{constant}$, when $K(z)$ is nonlattice,

(ii) $\varphi_y(nd) = \text{constant}$, $n = 0, \pm 1, \pm 2, \ldots$ when $K(z)$ is d-lattice.

We shall deal with the cases (i) and (ii) separately.

(i) We have $\quad H(x+y) - H(x) = H(y),\ K(z)\quad$ non-lattice.

But $H(x)$ is bounded in e.g. $[0, 1]$ and then (see e.g. Parzen [6] p. 123, problem 10)

$$H(x) = \text{constant} \cdot x$$

It follows from the so-called renewal theorem that

$$\lim_{x \to \pm\infty} \frac{H(|x|)}{|x|} = m \tag{3.50}$$

and thus
$$H(x) = mx \tag{3.51}$$

which corresponds to $\{X_n\}$ being Poisson distributed with the parameter m.

(ii) We have
$$H(nd+y) - H(nd) = H(y) \tag{3.52}$$

Consider first the case when $\{X_n\}$ is continuous (see p. 2). Then we get by Blackwell's theorem (see e.g. Feller [3] p. 347) that

$$\frac{H(nd+x) - H(nd)}{x} \to m,\ n \to \infty$$

where the left member by (3.52) is independent of n. This implies that

$$H(x) = mx$$

and $\{X_n\}$ must be Poisson distributed.

Consider lastly the case when $\{X_n\}$ is discrete with the span d_0. From Blackwell's theorem

$$\lim_{x \to \infty} \frac{H(x+y) - H(x)}{y} = m,\ y = nd_0,\ n \text{ positive integer.} \tag{3.53}$$

Let now $x_n = nd$. Then by (3.53)

$$\frac{H(nd + kd_0) - H(nd)}{kd_0} \to m,\ n \to \infty.$$

But by (3.52) the left member is independent of n. Thus

$$H(nd + kd_0) - H(nd) = mkd_0. \tag{3.54}$$

Putting $n = 0$ in (3.54) we get

$$H(kd_0) = mkd_0,\ k = 0, 1, 2, \ldots \tag{3.55}$$

If $\{X_n\}$ is discrete with the span d_0, $H(x)$ is a non-decreasing step-function with the jumps in kd_0. (3.55) implies that all the jumps have the size md_0 and this in turn gives that $\{X_n\}$ is deterministic.

Further by (3.50) and (3.52) we have

$$H(nd) = mnd$$

and thus d must be a multiple of d_0. Then it is easily shown that with positive probability some of the X_{1n}'s coincide, which contradicts the assumption that $F(0)=0$, i.e. $\{X_n\}$ can have no multiple points. The theorem is proved.

4. A characterization of renewal processes on $(0, \infty)$

Let $\{X_n, n=1, 2, ...\}$ be an ordered sequence of r.v.'s such that a.s.

$$0 < X_1 < X_2 < \dots$$

Put formally $X_0=0$ and let $Y_n = X_n - X_{n-1}$, $n=1, 2, \dots$. If $\{X_n, n=1, 2, \dots\}$ is (ordinary) renewal distributed, i.e. Y_n, $n=1, 2, \dots$ are i.i.d. positive r.v.'s with the d.f. $F(y)$, then the distribution of $\{X_n\}$ is given by the renewal function

$$H(x) = EN((X_n, X_n+x]) = \sum_{k=1}^{\infty} F^{k*}(x),$$

where $N(I)=$no. of $X_n \in I$, I finite interval.

Further we have in this case

$$EN((X_n, X_n+x]) \mid X_0, \dots, X_n = H(x) \text{ a.s.}, \quad n=0, 1, 2, \dots \tag{4.1}$$

Note that for $n=0$, (4.1) can be written

$$EN((0, x]) = H(x).$$

We shall here show that if (4.1) holds then $\{X_n\}$ must be ordinary renewal distributed with the renewal function $H(x)$.

Theorem 4.1. *Let $\{X_n, n=0, 1, 2, \dots\}$ be an ordered sequence of r.v.'s with $X_0 = 0 < X_1 < X_2 < \dots$ a.s. and such that $EN(I) < \infty$ for I finite interval and*

$$EN((X_n, X_n+x]) \mid X_0, \dots, X_n = H(x) \text{ a.s.}, \ n=0, 1, 2, \dots \tag{4.1}$$

Then $\{X_n\}$ is ordinary renewal distributed with the renewal function $H(x)$.

Proof. From (4.1) we see that

$$H(x) = E(N((X_{n-1}, X_{n-1}+x]) \mid Y_1, \dots, Y_{n-1}) \text{ a.s.} \tag{4.2}$$

Put
$$x_n(y) = \begin{cases} 1, & Y_n \leqslant y \\ 0 & \text{otherwise.} \end{cases}$$

Then
$$E(N((X_{n-1}, X_{n-1}+x])\,|\,Y_1, \ldots, Y_n)$$
$$= E((\chi_n(x) + N((X_{n-1}+Y_n, X_{n-1}+x]))\,|\,Y_1, \ldots, Y_n)$$
$$= \chi_n(x) + E(N((X_n, X_n - Y_n + x])\,|\,Y_1, \ldots, Y_n) \text{ a.s.} \tag{4.3}$$

since
$$X_{n-1} + Y_n = X_n,\ X_{n-1} + x = X_n - Y_n + x.$$

From (4.2) and (4.3) we get

$$H(x) = E(E((\chi_n(x) + N((X_n, X_n - Y_n + x]))\,|\,Y_1, \ldots, Y_n)\,|\,Y_1, \ldots, Y_{n-1})$$
$$= E(\chi_n(x)\,|\,Y_1, \ldots, Y_{n-1}) + E(H(x - Y_n)\,|\,Y_1, \ldots, Y_{n-1}) \text{ a.s.} \tag{4.4}$$

where $H(x) = 0$, $x \leqslant 0$.

Let now $F^{\mathcal{B}_{n-1}}(x)$ be the conditional d.f. of Y_n given the sub-σ-algebra of Borel sets generated by $Y_1, \ldots, Y_{n-1}$ in the sample space of $\{Y_n\}$. Then the first and last member of (4.4) can be written.

$$H(x) = F^{\mathcal{B}_{n-1}}(x) + \int_0^x H(x-y)\,dF^{\mathcal{B}_{n-1}}(y). \tag{4.5}$$

Denote for a moment $F^{\mathcal{B}_{n-1}}$ by F. Outside a set of probability zero we then get

$$H = F + (F + H * F) * F = \ldots = \sum_{k=1}^{n_0} F^{k*} + H * F^{n_0 *}.$$

Put $F_{n_0}(x) = F^{n_0 *}(x)$. Now $H(0) = 0$ and hence

$$H * F^{n_0 *} = \int_0^\infty H(x-y)\,dF_{n_0}(y) = \int_0^x H(x-y)\,dF_{n_0}(y).$$

Since $F(0) = 0 < 1$ it is easily seen that $F_{n_0}(x) \to 0$, $n_0 \to \infty$ and thus

$$\int_0^x H(x-y)\,dF_{n_0}(y) \to 0,\ n_0 \to \infty.$$

Returning to the original notation we have

$$H(x) = \sum_{k=1}^{\infty} (F^{\mathcal{B}_{n-1}}(x))^{k*}. \tag{4.6}$$

But (4.6) holds for any $n \geqslant 1$ and thus we can put

$$F^{\mathcal{B}_{n-1}}(x) = F(x), \quad n = 1, 2, \ldots \text{ a.s.} \tag{4.7}$$

where $F(x)$ is a d.f. on $(0, \infty)$.

From (4.7) we get by induction

$$P(Y_k \leqslant y_k,\ k = 1, \ldots, n) = \prod_{k=1}^{n} F(y_k)$$

for any $n \geqslant 1$ which proves the theorem.

Corollary 4.1. *Let* $\{X_n, n=0, 1, 2, \ldots\}$ *be an ordered sequence of r.v.'s with* $X_0 = 0 < X_1 < X_2 < \ldots$ *a.s. with* $EN(I) < \infty$, *I finite interval and such that*

$$EN((X_n, X_n + x]) \mid X_0, \ldots, X_n = \lambda x \text{ a.s.}, \quad n = 0, 1, 2, \ldots \tag{4.8}$$

Then $\{X_n\}$ *is Poisson distributed (on* $(0, \infty)$*) with the parameter* λ.

It should be remarked that from (4.8) it is possible by elementary methods to deduce a differential equation for the conditional frequency function of Y_n given $Y_1, \ldots, Y_{n-1}$ and hence directly prove the corollary without using theorem 4.1.

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