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The Hellinger square-integrability of matrix-valued measures with respect to a non-negative hermitian measure

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Introduction

Let $\mathcal{B}$ be a σ-algebra of subsets of a given space Ω, and let $\mathbf{F}$ be a fixed non-negative hermitian measure on $\mathcal{B}$. For matrix-valued measures $\mathbf{M}$ and $\mathbf{N}$ the Hellinger integral $\int_\Omega (d\mathbf{M}\,d\mathbf{N}^*/d\mathbf{F})$ (* = conjugate) is defined in such a way that the space of all matrix-valued measures $\mathbf{M}$ for which $\int_\Omega (d\mathbf{M}\,d\mathbf{M}^*/d\mathbf{F})$ exist becomes a Hilbert space under the inner product $\tau\int_\Omega (d\mathbf{M}\,d\mathbf{N}^*/d\mathbf{F})$ (τ = trace). It will follow that $\int_\Omega (d\mathbf{M}\,d\mathbf{M}^*/d\mathbf{F})$ exists iff there exists a $\mathcal{B}$-measurable matrix-valued function $\mathbf{\Psi}$ on Ω such that $\mathbf{\Psi}\in \mathbf{L}_{2,\mathbf{F}}$ [9, p. 295], and for each $B\in\mathcal{B}$, $\mathbf{M}(B)=\int_B \mathbf{\Psi}\,d\mathbf{F}$. These generalize the corresponding results [3, pp. 258–61] & [8, pp. 1414–18] concerning the Hellinger integrals $\int_\Omega (d\nu\, d\bar{\gamma}/d\mu)$, where ν and γ are complex-valued measures and μ is a non-vegative real-valued measure on $\mathcal{B}$.

For any matrix $\mathbf{G}$ we write $\mathbf{G}^-$ for the generalized inverse of $\mathbf{G}$ [7, p. 407]. If μ is a σ-finite non-negative real-valued measure on $\mathcal{B}$ with respect to (w.r.t.) which $\mathbf{F}$ is absolutely continuous (a.c.), then it is easy to show that $(d\mathbf{F}/d\mu)^-$ is a $\mathcal{B}$-measurable matrix-valued function on Ω.

Lemma 1. *Let* (i) $\mathbf{M}$ *and* $\mathbf{N}$ *be matrix-valued measures on* $\mathcal{B}$.

(ii) μ *and* ν *be σ-finite non-negative real-valued measures on $\mathcal{B}$ w.r.t. which* $\mathbf{M}, \mathbf{N}$ *and* $\mathbf{F}$ *are a.c.*[1] *Then*

(a)
$$\int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu \quad \textit{exists iff}$$

$$\int_\Omega (d\mathbf{M}/d\nu)\,(d\mathbf{F}/d\nu)^-\,(d\mathbf{N}/d\nu)^*\,d\nu \quad \textit{exists.}$$

(b) *If these integrals exist, they are equal.*

Proof. (a) Let $\gamma=\mu+\nu$. If $\int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu$ exists, then from the relations

[1] Each matrix-valued measure is a.c. w.r.t. the sum of the total variation measures of its components. Hence such a μ exists.

$$\int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu = \int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,(d\mu/d\gamma)\,d\gamma$$
$$= \int_\Omega (d\mathbf{M}/d\gamma)\,(d\mathbf{F}/d\gamma)^-\,(d\mathbf{N}/d\gamma)^*\,d\gamma \qquad (1)$$

follows that $\int_\Omega (d\mathbf{M}/d\gamma)\,(d\mathbf{F}/d\gamma)^-\,(d\mathbf{N}/d\gamma)^*\,d\gamma$ exists.

Conversely if $\int_\Omega (d\mathbf{M}/d\gamma)\,(d\mathbf{F}/d\gamma)^-\,(d\mathbf{N}/d\gamma)^*\,d\gamma$ exists,

again from (1) we infer that

$$\int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu \quad \text{exists.}$$

Similar argument can be used to show that

$$\int_\Omega (d\mathbf{M}/d\nu)\,(d\mathbf{F}/d\nu)^-\,(d\mathbf{N}/d\nu)^*\,d\nu \quad \text{exists iff}$$

$$\int_\Omega (d\mathbf{M}/d\gamma)\,(d\mathbf{F}/d\gamma)^-\,(d\mathbf{N}/d\gamma)^*\,d\gamma \quad \text{exists.}$$

Hence (a) is proved.

(b) From the argument used in the proof of (a) we infer (b). (Q.E.D.)

Thus the following definition makes sense.

Definition 1. *Let* $\mathbf{M}$, $\mathbf{N}$, $\mathbf{F}$ *and* μ *be as in the previous lemma. Then* (a) *we say that* $(\mathbf{M}, \mathbf{N})$ *is Hellinger integrable w.r.t.* $\mathbf{F}$ *if* $\int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu$ *exists. We write*

$$\int_\Omega \frac{d\mathbf{M}\,d\mathbf{N}^*}{d\mathbf{F}} = \int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^-\,(d\mathbf{N}/d\mu)^*\,d\mu.$$

(b) $\mathbf{H}_{2,\mathbf{F}}$ *is the class of all matrix-valued measures* $\mathbf{M}$ *on* $\mathcal{B}$ *for which* $\int_\Omega (d\mathbf{M}\,d\mathbf{M}^*/d\mathbf{F})$ *exist.*

It is easy to see that

(1) $\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}} \Rightarrow (\mathbf{M}, \mathbf{N})$ is Hellinger integrable w.r.t. $\mathbf{F}$,

$\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$ and $\mathbf{A}$ is a matrix $\Rightarrow \mathbf{A}\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$,

$\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}} \Rightarrow \mathbf{M} + \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}}$.

By (1) $\int_\Omega (d\mathbf{M}\,d\mathbf{N}^*/d\mathbf{F})$ exists for $\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}}$. This matrix-valued integral behaves like an inner product. It is therefore convenient to write

$$(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = \int_{\Omega} \frac{d\mathbf{M}\, d\mathbf{N}^*}{d\mathbf{F}}.$$

We define the ordinary inner product for $\mathbf{H}_{2,\mathbf{F}}$ by

$$((\mathbf{M}, \mathbf{N}))_{\mathbf{F}} = \tau(\mathbf{M}, \mathbf{N})_{\mathbf{F}}.$$

Thus from (1) we immediately get:

Lemma 2. $\mathbf{H}_{2,\mathbf{F}}$ *is an inner product space under* $((.,.))_{\mathbf{F}}$, *where for* $\mathbf{M}$ *and* $\mathbf{N} \in \mathbf{H}_{2,\mathbf{F}}$

$$((\mathbf{M}, \mathbf{N}))_{\mathbf{F}} = \tau(\mathbf{M}, \mathbf{N})_{\mathbf{F}}.$$

Let $\mathbf{L}_{2,\mathbf{F}}$ be the class of all matrix-valued functions $\mathbf{\Phi}$ on Ω for which $\int_{\Omega} \mathbf{\Phi}\, d\mathbf{F}\, \mathbf{\Phi}^*$ exist (A detailed discussion of integrals $\int_{\Omega} \mathbf{\Phi}\, d\mathbf{F}\, \mathbf{\Psi}^*$ and $\int_{\Omega} \mathbf{\Phi}\, d\mathbf{F}$ are given in [6] and [9]). It is known [9, p. 295] that $\mathbf{L}_{2,\mathbf{F}}$ is a Hilbert space under the inner product.

$$((\mathbf{\Phi}, \mathbf{\Psi}))_{\mathbf{F}} = \tau(\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}} = \tau \int_{\Omega} \mathbf{\Phi}\, d\mathbf{F}\, \mathbf{\Psi}^*.$$

The following lemma is needed to establish an isomorphism between $\mathbf{L}_{2,\mathbf{F}}$ and $\mathbf{H}_{2,\mathbf{F}}$.

Lemma 3. *Let* (i) $\mathbf{\Phi}$ *and* $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$.

(ii) *For each* $B \in \mathcal{B}$

$$\mathbf{M}(B) = \int_B \mathbf{\Phi}\, d\mathbf{F} \quad \textit{and} \quad \mathbf{N}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}.$$

Then $(\mathbf{M}, \mathbf{N})$ is Hellinger-integrable w.r.t. $\mathbf{F}$ and

$$(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = (\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}.$$

Proof. Let μ be any σ-finite non-negative measure w.r.t. which $\mathbf{F}$ is a.c. Then for each $B \in \mathcal{B}$, $\mathbf{M}(B) = \int_B \mathbf{\Phi}(d\mathbf{F}/d\mu)\, d\mu$ and $\mathbf{N}(B) = \int_B \mathbf{\Psi}(d\mathbf{F}/d\mu)\, d\mu$. Hence

$$(d\mathbf{M}/d\mu) = \mathbf{\Phi}(d\mathbf{F}/d\mu), \quad (d\mathbf{N}/d\mu) = \mathbf{\Psi}(d\mathbf{F}/d\mu).$$

Therefore
$$\begin{aligned}\int_{\Omega} \frac{d\mathbf{M}\, d\mathbf{N}^*}{d\mathbf{F}} &= \int_{\Omega} (d\mathbf{M}/d\mu)\, (d\mathbf{F}/d\mu)^{-}\, (d\mathbf{N}/d\mu)^*\, d\mu \\ &= \int_{\Omega} \mathbf{\Phi}(d\mathbf{F}/d\mu)\, (d\mathbf{F}/d\mu)^{-}\, (d\mathbf{F}/d\mu)\, \mathbf{\Psi}^*\, d\mu \\ &= \int_{\Omega} \mathbf{\Phi}(d\mathbf{F}/d\mu)\, \mathbf{\Psi}^*\, d\mu = (\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}. \qquad (1)\end{aligned}$$

Since for $\mathbf{\Phi}$ and $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$, $(\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}$ exists, from (1) it follows that $(\mathbf{M}, \mathbf{N})_{\mathbf{F}}$ is Hellinger integrable w.r.t. $\mathbf{F}$. Moreover $(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = (\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}$. (Q.E.D.)

If $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$, Then $\mathbf{M}_{\mathbf{\Psi}}$ will denote the matrix-valued measure in $\mathbf{H}_{2,\mathbf{F}}$ such that for each $B \in \mathcal{B}$, $\mathbf{M}_{\mathbf{\Psi}}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}$. Hence the following definition makes sense.

Definition 2. *Let the transformation* $\mathbf{T}$ *be defined on* $\mathbf{L}_{2,\mathbf{F}}$ *into* $\mathbf{H}_{2,\mathbf{F}}$ *as follows:*

$$\mathbf{T\Psi} = \mathbf{M}_{\mathbf{\Psi}}.$$

The important properties of $\mathbf{T}$ are given in the following theorem:

Theorem 1. (a) $\mathbf{T}$ *is a linear operator on* $\mathbf{L}_{2,\mathbf{F}}$ *into* $\mathbf{H}_{2,\mathbf{F}}$, *i.e., if* $\mathbf{A}$ *and* $\mathbf{B}$ *are matrices and* $\mathbf{\Phi}$ *and* $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$, *then*

$$\mathbf{T}(\mathbf{A\Phi} + \mathbf{B\Psi}) = \mathbf{AT\Phi} + \mathbf{BT\Psi}.$$

(b) $\mathbf{T}$ *is an isometry on* $\mathbf{L}_{2,\mathbf{F}}$ *into* $\mathbf{H}_{2,\mathbf{F}}$. *In fact*

$$(\mathbf{T\Phi}, \mathbf{T\Psi})_{\mathbf{F}} = (\mathbf{\Phi}, \mathbf{\Psi})_{\mathbf{F}}.$$

(c) $\mathbf{T}$ *is onto* $\mathbf{H}_{2,\mathbf{F}}$, *i.e., for each* $\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$, *there exists a* $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$ *such that* $\mathbf{M} = \mathbf{T\Psi}$. *In fact we can take* $\mathbf{\Psi}$ *to be* $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-}$, *where* μ *is any* σ*-finite non-negative real-valued measure w.r.t.* $\mathbf{M}$ *and* $\mathbf{F}$ *are a.c.*

Proof. (a) and (b) follow from Lemma 3 and Definition 2.

(c) Let $\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$. If μ is any σ-finite non-negative real-valued measure on $\mathcal{B}$ w.r.t. which $\mathbf{M}$ and $\mathbf{F}$ are a.c., then

$$\begin{aligned}(\mathbf{M}, \mathbf{N})_{\mathbf{F}} &= \int_\Omega (d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^{-}\,(d\mathbf{M}/d\mu)^{*}\,d\mu \\ &= \int_\Omega [(d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^{-}]\,(d\mathbf{F}/d\mu)\,[(d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^{-}]^{*}\,d\mu,\end{aligned}$$

where the first equality follows from the definition of $(\mathbf{M}, \mathbf{N})_{\mathbf{F}}$ and the second one is a consequence of $(d\mathbf{F}/d\mu)^{-}(d\mathbf{F}/d\mu)(d\mathbf{F}/d\mu)^{-} = (d\mathbf{F}/d\mu)^{-}$. Hence $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-}$ is in $\mathbf{L}_{2,\mathbf{F}}$. Let $\mathbf{N}(B) = \int_B (d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-}\, d\mathbf{F}$. Then $(\mathbf{M}, \mathbf{M})_{\mathbf{F}} = (\mathbf{N}, \mathbf{N})_{\mathbf{F}}$ and $(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = (\mathbf{N}, \mathbf{M})_{\mathbf{F}}$. Hence $(\mathbf{N} - \mathbf{M}, \mathbf{N} - \mathbf{M})_{\mathbf{F}} = 0$, i.e., $\mathbf{N}$ and $\mathbf{M}$ as elements of $\mathbf{H}_{2,\mathbf{F}}$ are equal. By Definition 2, $\mathbf{T}(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-} = \mathbf{N}$. Therefore $\mathbf{T}(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-} = \mathbf{M}$. (Q.E.D.)

We immediately obtain the following result.

Theorem 2. (a) $\mathbf{H}_{2,\mathbf{F}}$ *is a Hilbert space under the inner-product* $((\cdot,\cdot))_{\mathbf{F}}$.

(b) $\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$ *iff there exists a* $\mathcal{B}$*-measurable matrix-valued function* $\mathbf{\Psi}$ *on* Ω *such that* $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$ *and for each* $B \in \mathcal{B}$,

$$\mathbf{M}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}.$$

Moreover if μ *is a* σ*-finite non-negative real-valued measure w.r.t. which* $\mathbf{M}$ *and* $\mathbf{F}$ *are a.c., then* $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^{-}(d\mathbf{F}/d\mu) = (d\mathbf{M}/d\mu)$ *a.e.* μ.

Proof. (a) (b). (a) and the first part of (b) are immediate consequences of Theorem 1. For the second part of (b) we have

$$(d\mathbf{M}/d\mu)\,(d\mathbf{F}/d\mu)^{-}\,(dF/d\mu) = \mathbf{\Psi}(d\mathbf{F}/d\mu)\,(d\mathbf{F}/d\mu)^{-}\,(d\mathbf{F}/d\mu)$$
$$= \mathbf{\Psi}(d\mathbf{F}/d\mu) = (d\mathbf{M}/d\mu) \text{ a.e. } \mu,$$

where the first and the third equalities are consequences of $\mathbf{M}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}$ and the second one is a consequence of $(d\mathbf{F}/d\mu) = (d\mathbf{F}/d\mu)\,(d\mathbf{F}/d\mu)^{-}\,(d\mathbf{F}/d\mu)$ a.e. μ. (Q.E.D.)

Remark. The significance of the Hellinger integrals $\int_\Omega (d\nu\, d\gamma^{-}/d\mu)$, where ν and γ are complex-valued measures and μ is a non-negative real-valued measure, in univariate stochastic processes has bee pointed out by H. Cramér [1, p. 487] and U. Grenander [2, p. 207]. Our Hellinger integrals play an important role in q-variate stochastic processes. In particular, they give rise to an extension of P. Masani's work on q-variate full-rank minimal processes [5, pp. 145–150] which in turn is a generalization of a well-known result of A. N. Kolmogorov on univariate minimal sequences [4, Thm. 24]. These and other results will be published separately.

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