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Smoothness of the boundary function of a holomorphic function of bounded type

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In a recent paper [5] the author has proved the following theorem:

Theorem A. *Let $f(z)$ be holomorphic and of bounded type in $|z|<1$ and suppose its radial boundary values $f(e^{i\theta})$ coincide almost everywhere with a function $F(\theta)$ of period 2π and class C^∞ such that for some positive A,*

$$\max_\theta |F^{(n)}(\theta)| \leqslant (An)^{2n}, \quad n = 1, 2, \ldots. \tag{1}$$

Then f is bounded in $|z|<1$, and consequently f and all its derivatives are uniformly continuous in $|z|<1$. If the right side of (1) were replaced by $(An)^{pn}$ for some $p>2$, the resulting theorem would be false.

In [5] this theorem was proved by a method based on weighted polynomial approximation. It was also conjectured that theorem A is true "locally", that is, if in (1) the max is over the interval $[\theta_1, \theta_2]$ instead of $[0, 2\pi]$, a corresponding conclusion holds for the neighbourhood of the arc joining $e^{i\theta_1}$ and $e^{i\theta_2}$.

The main purpose of the present paper is to prove this conjecture. An altogether different, more direct, method is employed. Actually the relevant theorem (Theorem 1 below) is formulated for a half plane rather than a disk—the version for a disk (or indeed any domain having an analytic arc on its boundary) can easily be deduced from that for a half plane.

We also take this opportunity to settle another point from [5]. On p. 334 we stated that "it seems most plausible" an analogous theorem holds for *meromorphic* functions of bounded type, if (1) is replaced by the stronger condition that F belong to a Denjoy–Carleman quasi-analytic class (that is, under such a hypothesis on F, f is bounded, and in particular is free of poles, near the boundary). This "plausible" assertion is false, as shown in the corollary to Theorem 2 below. In fact, no majorant short of $(An)^n$, which implies analytic continuation of f across the boundary arc in question, can guarantee the absence of poles near the boundary. (But, for a plausible conjecture, see the concluding remarks.)

Before turning to Theorem 1, we wish to recall some facts concerning functions of bounded type; we formulate them for functions in a disk, the definitions and results for functions in a half plane are similar. For the notions of "inner" and "outer" function see Hoffman [2] (also [4] but in [4] these terms are not used). A *singular function* is an inner function without zeros. A complete divisibility theory exists for singular functions (see [2] pp. 84–85); and every holomorphic function f

of bounded type in the unit disk can be represented as a Blaschke product times an outer function times the quotient of two relatively prime singular functions. If the singular factor in the denominator reduces to a constant f is a *Smirnov function.* This is equivalent to saying that $f = Bg$ where B is a Blaschke product and

$$g(z) = \exp \int \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t),$$

whereby σ is a measure on the circle whose positive variation is absolutely continuous with respect to arc length. The significance of Smirnov functions is that the generalized maximum principle holds, i.e. a Smirnov function f is bounded by the essential supremum of the moduli of its boundary values. More generally, if the boundary function belongs to L^p, then $f \in H^p$. The corresponding results localized to the neighborhood of an arc on the boundary are also true (see [4], also papers of G. Ts. Tumarkin referred to there).

Theorem 1. *Let $f(z)$ be holomorphic and of bounded type in the upper half plane. Write $z = x + iy$, and suppose there exists a function $F(x)$ infinitely differentiable on an interval (x_1, x_2) and "of Gevrey class", that is for some $A > 0$*

$$|F^{(n)}(x)| \leqslant (An)^{2n}, \quad x_1 < x < x_2, \quad n = 0, 1, 2, \dots . \tag{2}$$

such that $F(x) = \lim_{y\to 0} f(x+iy)$ for almost all x in (x_1, x_2). Then $f(z)$ is bounded (and therefore $f(z)$ and all its derivatives are uniformly continuous) in a neighbourhood of each point x in (x_1, x_2). Moreover, if the right-hand side of (2) were replaced by $(An)^{pn}$ with some $p > 2$, the theorem would be false.

Lemma. *For $n \geqslant 2$, $\sum_{k=1}^{n-1} (k/n)^k < 3$.*

Proof of lemma. Let m be the least integer not less than $\frac{1}{2}n$. The sum on the left is less than

$$\sum_{k=1}^{m-1} 2^{-k} + \sum_{k=m}^{n-1} \left(\frac{k}{n}\right)^m < 1 + n \sum_{k=m}^{n-1} \left(\frac{k}{n}\right)^m \left(\frac{1}{n}\right) < 1 + n \int_{\frac{1}{2}}^{1} t^m\, dt < 1 + \frac{n}{m+1} < 3.$$

Proof of theorem. The truth of the last assertion in the theorem is established by constructing a suitable counter-example, just as in [5]. We therefore confine our attention to the sufficiency of (2). The proof will be by contradiction. If f were unbounded in the neighbourhood of some boundary point lying in (x_1, x_2), it would admit a representation $f = gU^{-1}$ where g is a Smirnov function, and U a singular function relatively prime to the singular factor of g, such that the representing measure of U has derivative equal to $+\infty$ at some point x_3 where $x_1 < x_3 < x_2$. It then follows that for every $\delta > 0$ and every $y_1 > 0$ the function $P_\delta(y) = |g(x_3+iy)| \cdot |U(x_3+iy)|^{-\delta}$ is unbounded for $0 < y < y_1$. If therefore we show that for every x_3 such that $x_1 < x_3 < x_2$ there exists a positive δ such that $P_\delta(y)$ is bounded for small y, the theorem will be proved.

By trivial changes of the independent variable we may arrange that $A = 1$ in (2), $x_3 = 0$; then (2) holds for $|x| \leqslant a$, where a is some positive number. Now,

$$F(x)=\sum_{k=0}^{n-1}\frac{F^{(k)}(0)}{k!}x^k+R_n(x)=S_{n-1}(x)+R_n(x),$$

where
$$|R_n(x)|\leqslant\frac{|x|^n n^{2n}}{n!}\leqslant(ne|x|)^n \quad\text{for}\quad |x|<a.$$

By hypothesis, we have almost everywhere on $(-a, a)$,

$$g(x)=F(x)\,U(x),\quad \text{hence a.e. on } (-a,a),$$
$$|g(x)-S_{n-1}(x)\,U(x)|=|R_n(x)\,U(x)|\leqslant(ne|x|)^n$$

Now
$$G(z)=\frac{g(z)-S_{n-1}(z)\,U(z)}{z^n}$$

is a Smirnov function in the upper half plane whose boundary values are bounded on $(-a, a)$ by $(ne)^n$, therefore in some neighbourhood of 0 we have $|G(z)|\leqslant 2(ne)^n$. Thus

$$|g(iy)-S_{n-1}(iy)\,U(iy)|\leqslant 2(ney)^n,\quad 0<y\leqslant y_1. \tag{3}$$

Now, for some positive number B, $|U(iy)|\geqslant e^{-(B/y)}$. Writing $\delta=1/8B$ we have from (3),

$$\frac{|g(iy)|}{|U(iy)|^\delta}\leqslant|S_{n-1}(iy)|+2(ney)^n e^{1/8y}. \tag{4}$$

Now (4) holds for all n. In particular, we may choose n such that

$$\frac{e^{-2}}{y}-1<n\leqslant\frac{e^{-2}}{y}.$$

Then, the second term on the right of (4) does not exceed

$$2e^{-n+(1/8y)}<2 \text{ if } y<e^{-2}-\tfrac{1}{8}.$$

Moreover,

$$|S_{n-1}(iy)|\leqslant\sum_{k=0}^{n-1}\frac{k^{2k}y^k}{k!}\leqslant\sum_{k=0}^{n-1}(key)^k<1+\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^k<4,\quad\text{by the lemma.}$$

Thus, the left side of (4) is less than 6 for small y, and the theorem is proved.

Theorem 2. *Let p_k be any decreasing sequence of positive numbers tending to zero. There exists a meromorphic function $f(z)$ of bounded type in $|z|<1$ whose poles cluster at every point of $|z|=1$ and whose boundary function coincides* a.e. *with a function $F(\theta)=\sum_{k=1}^{\infty}c_k e^{-ik\theta}$, where c_k are complex numbers such that $|c_k|\leqslant e^{-kp_k}$.*

Proof. Let α be any complex number of modulus one which is not a root of unity, and let $r_n=e^{-1/n^2}$. Consider $f(z)=\sum_{n=1}^{\infty}a_n/(z-z_n)$ where $z_n=r_n\alpha^n$. We shall show that positive numbers a_n can be chosen so that the series converges uniformly on every compact subset of $|z|<1$ not containing any of the points z_n and represents a meromorphic function of bounded type with the asserted boundary behaviour. First of all, suppose $\sum n^2 a_n < \infty$. Then the series converges as stated, and moreover, if $B(z)$ denotes the Blaschke product formed with zeros at $\{z_n\}$ we have for $|z|<1$,

$$|f(z)\,B(z)| \leqslant \sum a_n \left|\frac{B(z)}{z-z_n}\right| \leqslant \sum \frac{a_n}{1-|z_n|} \leqslant \sum n^2 a_n < \infty.$$

Thus fB is a bounded holomorphic function in $|z|<1$, and f is of bounded type. Now, it is readily verified that the boundary values of zf coincide a.e. with the function

$$\sum_{k=1}^{\infty} c_k e^{-ik\theta}, \quad \text{where} \quad c_k = \sum_{n=1}^{\infty} a_n z_n^k.$$

Thus, to complete the proof it suffices to show that if the a_n tend to zero fast enough, we have

$$\sum_{n=1}^{\infty} a_n e^{-(k/n^2)} \leqslant e^{-kp_k}. \tag{5}$$

Now define the function $M(k)$ to be the greatest integer not exceeding $p_k^{-\frac{1}{2}}$. Clearly $M(k)$ is non-decreasing and tends to infinity with k. Now, we can choose positive numbers a_n so that

$$\sum_1^{\infty} n^2 a_n < \infty \quad \text{(to satisfy the earlier requirement)}, \tag{6}$$

$$\sum_1^{\infty} a_n \leqslant \tfrac{1}{2}, \tag{7}$$

$$\sum_{n=M(k)+1}^{\infty} a_n \leqslant \tfrac{1}{2} e^{-kp_k}, \quad k=1,2,3,\ldots. \tag{8}$$

Concerning the last inequality, observe that since $M(k)$ tends to infinity we have for each r only a finite number of conditions imposed on the remainder $\sum_{n=r}^{\infty} a_n$, therefore the inequalities (8) can be satisfied by positive numbers a_n upon which we can moreover impose the conditions (6), (7). Now the left-hand side of (5) does not exceed

$$\sum_{n=1}^{M(k)} a_n e^{-(k/n^2)} + \sum_{n=M(k)+1}^{\infty} a_n.$$

The first sum is bounded by $\frac{1}{2}\exp\left(-k/(M(k))^2 \leqslant \frac{1}{2} e^{-kp_k}\right.$ (by (7)), and the second by $\frac{1}{2} e^{-kp_k}$, by (8). Theorem 2 is proved.

Corollary. *If B_n are any positive numbers increasing to infinity, we can find a meromorphic function satisfying the hypotheses of Theorem 2 and such that*

$$\max |F^{(n)}(\theta)| \leqslant (nB_n)^n.$$

We shall not carry out the details of the proof. Since $|F^{(n)}(\theta)| \leqslant \sum_{k=1}^{\infty} |c_k| k^n$, it is a matter of showing that if p_k tend to zero slowly enough, we have

$$\sum_{k=1}^{\infty} e^{-kp_k} k^n \leqslant (nB_n)^n$$

which is straightforward but tedious. As the result is needed only for a counter-example we carry out the estimation only for the concrete choice $p_k = 1/\log(k+1)$. To estimate $\sum_{k=1}^{\infty} \exp(-k/\log(k+1))k^n$, break up the sum into $\sum_{k=1}^{m}$ and $\sum_{k=m+1}^{\infty}$ where $m = m(n)$ is chosen as the least integer such that $k^n \leqslant \exp(\frac{1}{2}k/\log(k+1))$ holds for all $k > m$. It is easy to see that m is around $2n(\log n)^2$. Then the second sum is bounded uniformly with respect to n and the first sum is majorized by $\sum_{k=1}^{m} e^{-ak}k^n < \sum_{k=1}^{\infty} e^{-ak}k^n$ where $a = 1/(1+\log n)$. Now, comparing the last sum with an integral we show easily that it is $O(n/ea)^{n+1}$, therefore, finally, the original sum and hence $\max|F^{(n)}(\theta)|$ is bounded by $(An \log n)^n$ for some constant A. Thus, F belongs to a Denjoy–Carleman quasianalytic class, which disproves the conjecture in [5].

Concluding remarks

1. We have still not been able to settle whether in Theorem A or Theorem 1 the Gevrey class can be replaced by one of the slightly larger Carleson–Korenblyum quasi-analytic classes (see the discussion in [5], p. 333, also Korenblyum [3]). For example, would Theorem A be true with the right side of (1) replaced by $(An \log n)^{2n}$?

2. The rather drastic smoothness imposed on F could be very much relaxed if we had some *a priori* information about the singular factor U. For instance, it is easy to see by the method used to prove Theorem 1 that if $|U(x+iy)| \geqslant By^{\delta}$ for some positive B, δ (here B, δ may even depend on x) for $x_1 < x < x_2$ then if $F(x)$ satisfies a Lipschitz condition of any positive order on $[x_1, x_2]$ we can conclude that f is bounded (hence uniformly continuous) in a neighborhood of each point x, $x_1 < x < x_2$.

3. Under the hypotheses of Theorem A, f can have only finitely many zeros in $|z| < 1$ (this follows with the help of considerations outlined in [1] p. 331). As B. I. Korenblyum of Kiev has pointed out to the author, it can also be shown in this case that f has no (non-constant) singular factor. Korenblyum has suggested to the author the interesting question whether perhaps, under the assumption that f is *meromorphic* and of bounded type and F satisfies (1), we can conclude that f has no (non-constant) *singular* factors (either in numerator or denominator), i.e. F is a quotient of Blaschke products times an outer function. If this were true, one would have the interpretation that the Gevrey smoothness of the boundary function indicates the absence of singular factors, rather than boundedness in the interior.

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