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On extensions of Lipschitz functions

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Let X be a compact metric space with metric d. A real-valued function f on X is Lipschitz if there exists a constant K such that $|f(x)-f(y)| \leqslant Kd(x, y)$ holds for all points $x, y \in X$. The class of functions Lipschitz on X is an algebra Lip (X, d). On Lip (X, d) we can introduce a norm in the following way. Let $f \in \text{Lip}\,(X, d)$. We put $\|f\|_c = \sup_{x \in X} |f(x)|$ and $\|f\|_d = \sup \{|f(x)-f(y)|/d(x, y) \,|\, x, y \in X \text{ and } x \neq y\}$. Finally we put $\|f\| = \|f\|_c + \|f\|_d$. It is easily seen that Lip (X, d) is a Banach algebra with this norm. A general investigation of Lip (X, d) is given in [1]. We shall freely use results and notations from that paper. In this paper our main result is the following: Let F be a closed subset of a compact metric space X with metric d. Let G be a closed set contained in F. Let $f \in \text{Lip}\,(F, d)$ be such that $\lim |f(x)-f(y)|/d(x, y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero. Then there exists $H \in \text{Lip}\,(X, d)$ such that $H = f$ on F and $\lim |H(x)-H(y)|/d(x, y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero. This result answers a question raised in [2] (see IV, Miscellaneous problems no. 8, p. 355). For the result above implies the following: Let X be a compact metric space with metric d. Let Lip (X, d) be the Banach algebra of functions Lipschitz on X. Let $I(F)$ be the ideal of functions vanishing on a closed set F contained in X. Let G be a closed subset of F. Let $J(G)$ be the smallest closed ideal with hull G. Let u be a continuous linear on $I(F)$ which vanishes on $J(G)$. Then u can be extended to a continuous linear form on $I(G)$ which vanishes on $J(G)$.

Remark. In [1] it is shown that $J(G)$ consists of all functions $f \in \text{Lip}\,(X, d)$ such that $f \in I(G)$ and $\lim |f(x)-f(y)|/d(x,y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero.

In the proof of Theorem 1 we shall need the following result from [3].

Proposition 1. *Let X be a metric space with metric d. Let F be a closed subset of X. Let f be a bounded function Lipschitz on F, i.e.* $\|f\|_c = \sup_{x \in X}|$ *and* $\|f\|_d = \sup \{|f(x) - f(y)|/d(x, y) \,|\, x, y \in F$ and $x \neq y\}$ are *finite. Then there exists a function H on X such that $H = f$ on F and $H \in \text{Lip}\,(X, d)$ with $\|H\|_d = \|f\|_d$ and $\|H\|_c = \|f\|_c$.*

Proof. Let us put $H_1(x) = \sup_{y \in F} \{f(y) - \|f\|_d d(x, y)\}$. It is not hard to see that $H_1 = f$ on F and H_1 is Lipschitz on X with $\|H_1\|_d = \|f\|_d$. Now we only have to put $H(x) = H_1(x)$ if $|H_1(x)| \leqslant \|f\|_c$ and $H(x) = \|f\|_c$ if $H_1(x) > \|f\|_c$ and $H(x) = -\|f\|_c$ if $H_1(x) < -\|f\|_c$.

In the proof of Theorem 1 the following lemma will be useful.

Lemma 1. *Let F and G be two closed subsets of a metric space X with metric d, such that if $x \in F$ and $y \in G$ then there exists $z \in F \cap G$ with $4d(x, y) \geqslant d(x, z)$ and $4d(x, y) \geqslant d(y, z)$. Let $f \in \text{Lip}\,(F, d)$ and $g \in \text{Lip}\,(G, d)$ be such that $f = g$ on $F \cap G$. If we put $h = f$ on F and $h = g$ on G then $h \in \text{Lip}\,(F \cap G, d)$ and $\|h\|_d \leqslant 4(d\|f\|_d + \|g\|_d)$.*

Proof. Obviously h is a well defined function on $F \cap G$ since $f=g$ on $F \cap G$. Let $x \in F$ and $y \in G$. By assumption we can find $z \in F \cap G$ such that $4\,d(x,y) \geqslant d(x,z)$ and $4\,d(x,y) \geqslant d(y,z)$. It follows that $|h(x)-h(y)|/d(x,y) \leqslant |f(x)-f(z)|/d(x,y) + |g(z)-g(y)|/d(x,y) \leqslant 4(\|f\|_d + \|g\|_d)$.

Theorem 1. *Let A be a normed linear space. Let d be the metric on A defined by the norm A. Let F be a compact subset of A containing the origin 0. Let $f \in \operatorname{Lip}(F,d)$ be such that* $\lim |f(x)-f(y)|/d(x,y)=0$ *as $d(x,0)$ and $d(y,0)$ tend to zero. Then there exists $H \in \operatorname{Lip}(A,d)$ such that $H=f$ on F and* $\lim |H(x)-H(y)|/d(x,y)=0$ *as $d(x,0)$ and $d(y,0)$ tend to zero.*

Proof. The case when 0 is an isolated point of F is trivial so now we assume that 0 is not an isolated point of F. If $y \in A$ we put $\|y\|=d(y,0)$. Choose $(x_n)_2^\infty$ from F such that $2\|x_{n+1}\| \leqslant \|x_n\|$ and $F \subset \{y \in A \mid \|y\| \leqslant \|x_2\|\}$. We also assume that $f(0)=0$ here. Let us put $B_n=\{y \in A \mid \|y\| \leqslant \|x_{2n}\|\}$, $S_n=\{y \in A \mid \|y\|=\|x_{2n}\|\}$, $W_1=\{y \in A \mid \|x_3\| \leqslant \|y\| \leqslant \|x_2\|\}$ and for $n \geqslant 2$ we put $W_n=\{y \in A \mid \|x_{2n+1}\| \leqslant \|y\| \leqslant \|x_{2n-1}\|\}$. Using proposition 1 we can extend the restriction of f on $W_n \cap F$ to the set $(W_n \cap F) \cup S_n$. In this way we obtain $g_n \in \operatorname{Lip}((W_n \cap F) \cup S_n, d)$ such that $\lim \|g_n\|_d = \lim \|g_n\|_c = 0$. Let us put $Q_n=\{y \in F \mid \|x_{2n+2}\| \leqslant \|y\| \leqslant \|x_{2n}\|\}$ and $T_n = S_n \cap S_{n+1} \cap Q_n$. On T_n we define h_n as follows: $h_n=g_n$ on S_n, $h_n=g_{n+1}$ on S_{n+1} and $h_n=f$ on Q_n. Because F contains a point x_{2n+1} such that $2\|x_{2n+2}\| \leqslant \|x_{2n+1}\| \leqslant 1/2\|x_{2n}\|$ we see that Lemma 1 can be applied to prove that $\|h_n\|_d \leqslant 4(\|g_n\|_d + \|g_{n+1}\|_d)$. Let us put $R_n=\{y \in A \mid \|x_{2n+2}\| \leqslant \|y\| \leqslant \|x_{2n}\|\}$. Using Proposition 1 we find $H_n \in \operatorname{Lip}(R_n,d)$ such that $H_n=h_n$ on T_n with $\|H_n\|_d=\|h_n\|_d$ and $\|H_n\|_c=\|h_n\|_c$. It follows that $\lim\|H_n\|_d=\lim\|H_n\|_c=0$. Now we define H on B_1 as follows: $H=H_n$ on R_n and $H(0)=0$. Now we prove that $H \in \operatorname{Lip}(B_1,d)$ and that $\lim |H(x)-H(y)|/d(x,y)=0$ as $d(x,0)$ and $d(y,0)$ tend to zero. For suppose that $y \in R_{n+v}$ and $x \in R_n$ with $v>0$. Now the line segment $\{z(t)=tx+(1-t)y \mid 0 \leqslant t \leqslant 1\}$ contains points $z(t_j) \in S_{n+j}$ for $j=1,\dots,v$. Now we get $|H(x)-H(y)| \leqslant |H_n(x)-H_n(z(t_1))| + \dots + |H_{n+v}(z(t_v))-H_{n+v}(y)| \leqslant K_n d(x,y)$ where we have put $K_n=\sup\{\|H_v\|_d \mid v \geqslant n\}$. Finally we can extend H from B_1 to A using Proposition 1 and the theorem is proved.

Theorem 2. *Let X be a compact metric space with metric d. Let F be a closed subset of X. Let G be a closed set contained in F. Let $f \in \operatorname{Lip}(F,d)$ be such that* $\lim |f(x)-f(y)|/d(x,y)=0$ *as $d(x,G)$ and $d(y,G)$ tend to zero. Then there exists $H \in \operatorname{Lip}(X,d)$ such that $H=f$ on F and* $\lim |H(x)-H(y)|/d(x,y)=0$ *as $d(x,G)$ and $d(y,G)$ tend to zero.*

Proof. Let $I(G)$ be the ideal of functions in $\operatorname{Lip}(X,d)$ vanishing on the closed set G. $I(G)$ is now considered as a normed linear space with the norm $\|f\|=\sup\{|f(x)-f(y)|/d(x,y) \mid x,y \in X \text{ and } x \neq y\}$. Every point $x \in X$ defines a linear form $\hat{x}$ on $I(G)$, i.e. we put $\hat{x}(f)=f(x)$ for $f \in I(G)$. Obviously we have $|\hat{x}(f)| \leqslant \|f\| d(x,G)$ for every $f \in I(G)$. Let $I'(G)$ be the normed dual space of $I(G)$. Let $\hat{d}$ be the metric on $I'(G)$ defined by the norm on $I'(G)$. Hence we have the mapping $T: X \to I'(G)$ introduced above. Since we can find $f \in I(G)$ such that $f(x)=d(x,G)$ and $\|f\|_d=1$ when $x \in X-G$, it follows that $\hat{d}(\hat{x},0)=d(x,G)$ holds when $x \in X-G$, while $\hat{x}=0$ if $x \in G$. It is also easily seen that $\hat{d}(\hat{x},\hat{y}) \leqslant d(x,y)$ holds. Let us also consider two points $x,y \in X-G$. We may assume that $d(x,G) \leqslant d(y,G)$. If $d(x,y) < d(y,G)$ we easily find $\hat{d}(\hat{x},\hat{y})=d(x,y)$. Let us now consider a function $f \in \operatorname{Lip}(F,d)$ such that $\lim|f(x)-f(y)|/d(x,y)=0$ as $d(x,G)$ and $d(y,G)$ tend to zero while f vanishes on G. Now we define a function $\hat{f}$

on $T(F)=\{\hat{x}\in I'(G)\,|\,x\in F\}$ as follows: $\hat{f}(\hat{x})=f(x)$ for $x\in F$. We see that $T(F)$ is a compact subset of $I'(G)$ containing the origin 0.

Let $x, y\in F$ with $d(x, G)\leqslant d(y, G)$. If $d(x,y)<d(y,G)$ we have $\hat{d}(\hat{x},\hat{y})=d(x,y)$. In this case we get $|\hat{f}(\hat{x})-\hat{f}(\hat{y})|/\hat{d}(\hat{x},\hat{y})=|f(x)-f(y)|/d(x,y)$. If $d(x,y)\geqslant d(y,G)$ we choose $z, u\in G$ such that $d(x,z)=d(x,G)$ and $d(y,u)=d(y,G)$. Then we get $|\hat{f}(\hat{x})-\hat{f}(\hat{y})/\hat{d}(\hat{x},\hat{y})$ $\leqslant|f(x)-f(z)|/\hat{d}(\hat{x},\hat{y})+|f(y)-f(u)|/\hat{d}(\hat{x},\hat{y})\leqslant|f(x)-f(z)|/d(x,z)+|f(y)-f(y)|/d(y,u)$. It follows that $\hat{f}\in\mathrm{Lip}\,(T(F),\hat{d})$ and $\lim|\hat{f}(\hat{x})-\hat{f}(\hat{y})|/\hat{d}(\hat{x},\hat{y})=0$ as $\hat{y}(\hat{x},0)$ and $\hat{d}(\hat{y},0)$ tend to zero. We also have $\|\hat{f}\|\hat{d}\leqslant 2\|f\|_d$. Now Theorem 1 shows that we can find $Q\in\mathrm{Lip}\,(I'(G),\hat{d})$ such that $Q=\hat{f}$ on $T(F)$ and $\lim|Q(a)-Q(b)|/\hat{d}(a,b)=0$ as $\hat{d}(a,0)$ and $\hat{d}(b,0)$ tend to zero. We also have $\|Q\|_{\hat{d}}\leqslant 0\|\hat{f}\|_{\hat{d}}\leqslant 16\|f\|_d$. Now we define a function H on X as follows: If $x\in X$ we put $H(x)=Q(\hat{x})$. Since $\hat{d}(\hat{x},\hat{y})\leqslant d(x,y)$ it follows that $H\in\mathrm{Lip}\,(X,d)$ with $\|H\|_d\leqslant\|Q\|_{\hat{d}}\leqslant 16\|f\|_d$.

Corollary. *For every $e>0$ we can find an extension H of f in Theorem 2 such that $\|H\|_d<(1+e)\|f\|_d$ and $\|H\|_c=\|f\|_c$.*

Proof. The constructions used in the proof of Theorem 2 show that we can find an extension H of f such that $\|H\|_d\leqslant 16\|f\|_d$. Using this estimate we can reduce the problem to the case when $f\in\mathrm{Lip}\,(F,d)$ is such that f vanishes in a neighborhood of G and this case is easy.

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