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# Poisson processes as renewal processes invariant under translations

By Murali Rao and Hans Wedel

## Introduction

Let $\{X_n : n = \pm 1, \pm 2, \ldots\}$ be a sequence of random variables such that a.s.

$$,,, X_{-2} < X_{-1} < 0 < X_1 < X_2 \ldots$$

Put $Y_0 = X_{-1}$, $Y_1 = X_1$ $Y_n = X_n - X_{n-1}$, $n \neq 0, 1$.
Assume that:

(i) $\{(Y_0, Y_1), Y_n, n \neq 0. 1\}$ is a set of independent random variables.

(ii) $\{Y_n : n \neq 0, 1\}$, are independent, identically distributed positive random variables with $P[Y_n \leqslant y] = F(y)$,

$$F(0) = 0 \quad \text{and} \quad E[Y_n] = \frac{1}{m} < \infty.$$

Let $\{\xi_n, n = \pm 1, \pm 2, \ldots\}$ be a set of independent random variables which is independent of $\{X_n\}$ and $\xi_n$, $n = \pm 1, \pm 2 \ldots$, be identically distributed with the same non-degenerated distribution $G$. Put $Z_n = X_n + \xi_n$ and let $N(I)$ = number of $X_n \in I$ and $\tilde{N}(I)$ = number of $Z_n \in I$.

Doob has shown [1, pp. 404–407] that if all $Y_n$ have an exponential distribution then $N(I)$ and $\tilde{N}(I)$ have the same distribution. Thedéen proved the converse of this statement, namely that every $Y_n$, $n \neq 0, 1$, has an exponential distribution if $N(I)$ and $\tilde{N}(I)$ have the same distribution and if

(iii′) $$P[Y_0 > y_0, Y_1 > y] = m \int_{y_0 + y_1} (1 - F(s))\, ds.$$

We shall here prove that the weaker conditions $E[N(I))] = m\,|I|$ and $E[N(I)N(J)] = E[\tilde{N}(I)\tilde{N}(J)]$ are sufficient to imply exponential distributions of $Y_n$. Our proof is at the same time a simplification of Thedéen's proof.

Let $X_n$ and $Y_n$ be as in the introduction and instead of (iii′) put

(iii) $E[N(I)] = m\,|I|$ where $|I|$ denotes the Lebesgue measure of $I$.

Then (iii) is equivalent to $P[Y_i > u] = \int_u^\infty (1-F(t))\,dt$ for $i=0,1$, see [2], pp. 354.

Let now $\{\xi_n, n = \pm 1, \pm 2, \dots\}$ be a sequence of random variables which is independent of the sequence $\{X_n\}$. We shall assume that for all $n, m, n \neq m$, $(\xi_n, \xi_m)$ have the same joint distribution $G$ and that the support group of G. i.e. the group generated by the support of $G$, has an element of the form $(0, d)$ with $d > 0$; if $\xi_n$ and $\xi_m$ are independent and have a nondegenerate distribution then this is certainly true.

Put $Z_n = X_n + \xi_n$, $n = \pm 1, \pm 2, \dots$, and $\tilde{N}(I) =$ number of $Z_n \in I$.

**Theorem.** *Let* $X_n, \xi_n, Z_n$ *be as above. If* $E[\tilde{N}(I)\,\tilde{N}(J)] = E[N(I)N(J)]$ *for all* $I, J$ *then* $\{X_n\}$ *is Poisson* i.e. $F(y) = 1 - me^{-my}$.

*Proof.* Put $\Phi(I,J) = E[N(I)\,N(J)] - E[N(I \cap J)] = \sum_{n \neq m} P[X_n \in I, X_m \in J]$. Using independence of $\{\xi_n\}$ and $\{X_n\}$ we get $E[\tilde{N}(I)\,\tilde{N}(J)] - E[\tilde{N}(I \cap J)] = \iint \Phi(I-u, J-v)\,dG(u,v)$. The condition $E[N(I)] = m\,|I|$ implies $E[\tilde{N}(I)] = m\,|I|$. Thus $E[\tilde{N}(I)\,\tilde{N}(J)] - E[\tilde{N}(I \cap J)] = E[N(I)\,N(J)] - E[N(I \cap J)]$: which gives $\Phi = \Phi * G$.

A simple consequence of the renewal theorem is that for any pair of finite intervals $I, J$, we see that $E[N(I+h)\,N(J+k)]$ is a bounded function of $(h,k)$. The Choquet–Deny theorem [3, p. 152] applies and we deduce that every point of the support of $G$ is a period for $\Phi$. The set of periods for $\Phi$ is a group and this group contains the element $(0, d)$ and hence $(0, kd)$ where $k$ is any positive integer (indeed any integer). Thus for all $I, J$ and all positive integers $k$, $\Phi(I,J) = \Phi(I, J + kd)$. Take $I = (0, x]$ with $x < kd$. Then $I \cap (I + kd) = \phi$.

Also

$$\Phi(I, I+kd) = \sum_{n \neq m} P[X_n \in I, X_m \in I + kd] = \sum_{m,n \geq 1} P[X_n \in I, X_m \in I + kd]$$

$$= \sum_{n=1}^{\infty} \sum_{m>n} P[X_n \in I, X_m \in I + kd] = \sum_{n=1}^{\infty} \int_0^x H(I + kd - u)\,d(F_0 * F^{(n-1)*})(u)$$

$$= \int_0^x H(I + kd - u)\,du,$$

where $H(x) = \sum_{k=1}^\infty F^{k*}(x)$ and (iii) implies $mx = \sum_{k=0}^\infty F_0 * F^{k*}(x)$, $x > 0$.

Similar calculations give $\Phi(I,I) = 2\int_0^x H(x-u)\,du = 2\int_0^x H(u)\,du$. Thus

$$2\int_0^x H(u)\,du = \int_0^x H(I + kd - u)\,du = \int_0^x [H(x + kd - u) - H(kd - u)]\,du$$

$$= \int_0^x [H(kd + u) - H(kd - u)]\,du.$$

This equality for all $x < kd$ implies $2H(u) = H(kd + u) - H(kd - u)$, $u < kd$.

Suppose $d_0$ is a positive number such that $F(d_0) > 0$ and $F(d_0 -) = 0$. Then $F^{n*}$ has an atom at $nd_0$ and thus $H$ has a mass point at every positive integral multiple of $d_0$, but $H(u) = 0\; u < d_0$. Choose $k$ so that $kd > d_0$. For $u < d_0\; H(u) = 0$ and the functional equation for $H$ shows that $H(kd - u) = H(kd + u)\; u < d_0$. Since every interval of length larger than $d_0$, contains a multiple of $d_0$ and $H$ has a positive mass at such a point, we see that $H(kd-u) < H(kd+u)$ for some $0 < u < d_0$. This is a contradiction and thus $F$ certainly cannot be arithmetic. As $k \to \infty$ Blackwell's theorem [2, p. 347] shows that $H(kd+u) - H(kd-u) \to 2\,um$, and thus $2\,H(u) = 2\,um$. This is equivalent to $F$ being exponential. Q.E.D.

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*University of Göteborg, Department of Mathematics, Göteborg, Sweden*

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