1.68 03 1 Communicated 14 February 1968 by LENNART CARLESON

On a strong form of spectral synthesis

By JOHN E. GILBERT

When G is a locally compact abelian group with character group Γ, the Fourier Transform of a function f in the group algebra $L^1(G)$ is defined by

$$\hat{f}(\lambda) = \int_G f(x)\,\overline{(x,\lambda)}\,dx \quad (\lambda \in \Gamma),$$

and, for a given closed set Ω in Γ, $I(\Omega)$ denotes the closed ideal in $L^1(G)$ of all functions for which $\hat{f}=0$ on Ω. Following Wik ([8], p. 56) we say that

1. *Ω is a Ditkin set if to each $f \in I(\Omega)$ there corresponds a sequence $\{\mu_n\} \subset I(\Omega)$ where $\hat{\mu}_n = 0$ in a neighbourhood of Ω and $\|f - \mu_n * f\| \to 0$ as $n \to \infty$.*
2. *Ω is a Strong Ditkin set if the sequence $\{\mu_n\}$ in condition 1 can be chosen independently of $f \in I(\Omega)$.*

If $\Sigma_d(\Gamma)$ denotes the discrete coset-ring of Γ, i.e., the Boolean algebra generated by cosets of all subgroups of Γ whether closed or not, we prove in this note:

Theorem 1. *Let Γ be a separable, metrizable group. Then each closed subset Ω of Γ in $\Sigma_d(\Gamma)$ is a Strong Ditkin set.*

Using Theorem 1 we obtain immediately the converse of a result of Rosenthal ([6], Theorem 3.1, p. 187) completing the characterization of Strong Ditkin sets in R^n, T^n, ... without interior points.

Theorem 2. *Let $\Gamma = R^n$, T^n or any compact, metrizable group such that the union of all of its finite subgroups is everywhere dense. Then a closed set $\Omega \subset \Gamma$ having no interior points is a Strong Ditkin set if and only if $\Omega \in \Sigma_d(\Gamma)$.*

The principal step in the proof of Theorem 1 is the description of closed sets in $\Sigma_d(\Gamma)$: every closed subset Ω of Γ (not necessarily separable, metrizable) in $\Sigma_d(\Gamma)$ is the finite union of sets of the form $\lambda(\Pi \setminus \Delta)$ where $\lambda \in \Gamma$[1], Π is a closed subgroup of Γ and Δ belongs to the coset-ring $\Sigma(\Pi)$ of Π (Gilbert [1], Theorem 3.1); conversely, every such union is a closed subset of Γ in $\Sigma_d(\Gamma)$. Thus, in R^n for example, a closed set Ω without interior points is a Strong Ditkin set precisely when

$$\Omega = F \cup \left(\bigcup_{i=1}^{m} \Omega_i \right) \tag{1}$$

[1] The group operation in all groups is written multiplicatively.

with F a finite (possibly empty) set and Ω_i an affine transformation of a set Φ_i differing from $R^p \times a_1 Z \times \ldots \times a_q Z$ by at most finitely many cosets of R^p (here $0 \leqslant p = p(i) < n$, $p+q \leqslant n$, $a_j \in R$ and R^0 is interpreted as the trivial subgroup $\{0\}$). The examples given by Wik and Rosenthal follow easily from (1) as do the examples given by Rosenthal for sets which fail to be Strong Ditkin sets ([6], pp. 187, 8).

First we introduce some notation: for a closed subgroup H of G the norm on the group algebra $L^1(H)$ is written $\|(\cdot)\|_H$ while that on $L^1(G)$ or on the measure algebra $M(G)$ is written $\|(\cdot)\|$. The Haar measure on G/H is adjusted so that

$$\int_G k(x)\,dx = \int_{G/H}\left(\int_H k(x\xi)\,d\xi\right)dx' \quad (x' = xH \in G/H)$$

for every k in the space $\mathcal{K}(G)$ of continuous functions with compact support in G. The mapping $f \to f'$ defined by

$$f'(x') = \int_H f(x\xi)\,d\xi \quad (x' \in G/H),$$

is norm decreasing homomorphism from $L^1(G)$ onto $L^1(G/H)$ (cf. Reiter [2], p. 415) which can be extended to the respective measure algebras $M(G)$, $M(G/H)$ (Rudin [7], Theorem 2.7.2; Reiter [5]). The Fourier Transform of f' is the restriction of $\hat{f}$ to the annihilator group of H in Γ.

Proof of Theorem 1. In view of the known structure of closed sets in $\Sigma_d(\Gamma)$ and the fact that finite unions of Strong Ditkin sets are again Strong Ditkin sets (Wik [8], Theorem 3) we need only show that a set of the form $\Pi \backslash \Delta$, $\Delta \in \Sigma(\Pi)$, is a Strong Ditkin set in Γ. For then, certainly, any translate $\lambda(\Pi\backslash\Delta)$ and hence any closed set $\Omega \in \Sigma_d(\Gamma)$ is a Strong Ditkin set. Since already we know that any $\Omega \in \Sigma_d(\Gamma)$ is a Ditkin set (Gilbert [1], Theorem 3.9), given any $\varepsilon > 0$ and $g \in I(\Omega)$, there exists $g_\varepsilon \in I(\Omega)$ such that

$$(a) \quad \hat{g}_\varepsilon = 0 \quad \text{in a neighbourhood of } \Omega, \qquad (b) \quad \|g - g_\varepsilon\| < \varepsilon. \tag{2}$$

Now let $\Omega = \Pi\backslash\Delta$, $\Delta \in \Sigma(\Pi)$. When H is the annihilator group of Π in G, let μ_Ω be the idempotent measure in the measure algebra $M(G/H)$ whose Fourier-Stieltjes Transform is the characteristic function of Δ (as a subset of Π). Further, let μ be any measure in $M(G)$ for which the restriction to Π of the Fourier-Stieltjes Transform of μ coincides with that of μ_Ω. Now assume for the moment that the following result has been proved.

Lemma 3. *For each $A > 1$, there is a sequence $\{\tau_n\} \subset M(G)$ such that*

(i) $\|\tau_n\| \leqslant A$,
(ii) *the Fourier-Stieltjes Transform of τ_n is* 1 *in some neighbourhood of* Π,
(iii) *for any $f \in L^1(G)$,*

$$\lim_{n\to\infty} \int_G |\tau_n * f|\,dx \leqslant A \int_{G/H} |f'(x')|\,dx'. \tag{3}$$

Under the conditions of Γ there is a sequence $\{\delta_n\}\subset L^1(G)$ forming an approximate identity i.e., $\|\delta_n\|=1$ and

$$\|f-\delta_n * f\| \to 0 \quad \text{as} \quad n\to\infty \quad (f\in L^1(G)). \tag{4}$$

Since, clearly, $\mu * \delta_n \in I(\Omega)$, choose a sequence $\{\nu_n\}\subset I(\Omega)$ for which

$$(a)\quad \hat{\nu}_n = 0 \quad \text{in a neighbourhood of } \Omega, \qquad (b)\quad \|\nu_n - \mu * \delta_n\| < 1/n. \tag{5}$$

Setting
$$\mu_n = \delta_n - (\tau_n * \delta_n - \tau_n * \nu_n) \quad (n = 1, 2, \ldots),$$

we can soon check that $\hat{\mu}_n = 0$ in some neighbourhood of Ω. On the other hand, for any $f\in L^1(G)$,

$$\begin{aligned}\|f-\mu_n * f\| &= \|f-\delta_n * f + \tau_n * (\delta_n * f - \nu_n * f)\|\\ &\leqslant \|f-\delta_n * f\| + \|\tau_n * (f-\mu * f)\| + A\|f\|\cdot\|\nu_n - \mu * \delta_n\|,\end{aligned}$$

i.e., by (3), (4), (5)

$$\lim_{n\to\infty} \|f-\mu_n * f\| \leqslant A\int_{G/H} |f' - \mu' * f'|\,dx'. \tag{6}$$

But $\mu' = \mu_\Omega$ and, if $f\in I(\Omega)$, then $\mu_\Omega * f' = f'$, for clearly when $I_H(\Omega)$ is the closed ideal in $L^1(G/H)$ of functions whose Fourier Transforms vanish on Ω (as a subset of Π),

$$f'\in I_H(\Omega), \quad I_H(\Omega) = \mu_\Omega * L^1(G/H)$$

(cf. [4], p. 561). Hence, by (6), if $f\in I(\Omega)$, $\lim_{n\to\infty}\|f-\mu_n * f\| = 0$ which proves that Ω is a Strong Ditkin set.

Remarks. Lemma 3 is only a mild reworking of a generalization by Reiter (unpublished, but see Reiter [3], Lemma 2) of a result of Calderon (cf. Rudin [7], Theorem 2.7.5). From Reiter's version of Lemma 3 it follows, in particular, that if $\Omega\subset\Pi$ is a Ditkin set in Π then Ω is a Ditkin set in Γ. Theorem 1 shows, in effect, that a similar result holds with Ditkin set replaced by Strong Ditkin set at least when $\Omega\in\Sigma(\Pi)$. In general, however, the result fails for, as Rosenthal points out, a finite interval in R^1 is a Strong Ditkin set in R^1 but it cannot be a Strong Ditkin set in R^2 since it then has no interior points and does not belong to the discrete cosetring $\Sigma_d(R^2)$. Notice that, even so, it is a Ditkin set in R^2. It is, perhaps, worth noting that the proof of Theorem 1 for an arbitrary Strong Ditkin set fails because the rate at which

$$\lim_{n\to\infty}\left(\|\tau_n * f\| - A\int_{G/H}|f'(x')|\,dx'\right)$$

becomes 0 or negative depends on f and it should be clear that the sequence $\{\mu_n\}$ cannot always be chosen independently of $f\in I(\Omega)$.

Proof of Lemma 3. When $\{K_n\}$ is an increasing sequence of compact symmetric sets in H for which $\cup_n K_n = H$, let $\{U_n\}$ be a nested sequence of symmetric neighbourhoods of the identity e in $\Gamma/H^\perp$ shrinking to e and such that $|1-(x,y)|<1/n$, $x\in K_n$, $\gamma\in U_n$. Following the construction of approximate units in $L^1(H)$ as in Reiter ([2], p. 405) we obtain, for each $A>1$, a sequence $\{\tau_n\}\subset L^1(H)$ satisfying

(i) $$\int_H |\tau_n(\xi)|\, d\xi \leqslant A,$$

(ii) $$\hat{\tau}_n = 1 \text{ in a neighbourhood of } e \text{ in } \Gamma/\Pi,$$

(iii) $$\sup_{\eta \in K_n} \int_H |\tau_n(\eta^{-1}\xi) - \tau_n(\xi)|\, d\xi < 2A/n.$$

When each τ_n is regarded as a measure on G, the first two of these conditions give the corresponding conditions (i), (ii) of the lemma. For condition (iii) suppose first that (3) has been established for each $k \in \mathcal{K}(G)$. Then, by (i), for any $\varepsilon > 0$ and $k \in \mathcal{K}(G)$ satisfying $\|f - k\| < \varepsilon$,

$$\lim_{n\to\infty} \int_G |\tau_n * f|\, dx \leqslant \lim_{n\to\infty} \left\{ \int_G |\tau_n * k\, dx + \|f - k\| . \|\tau_n\|_H \right\}$$

$$\leqslant A\left(\int_{G/H} |f'(x')|\, dx' + \int_{G/H} |f' - k'|\, dx + \varepsilon \right)$$

$$\leqslant A\left(\int_{G/H} |f'(x')|\, dx' \right) + 2A\varepsilon,$$

i.e., (3) then holds for f since ε was arbitrary. Now only minor modifications are required of the second and third steps in Reiter's proof of his Lemma 2 in [3] to obtain (3) for $k \in \mathcal{K}(G)$. For clearly

$$\|k_x * \tau_n\|_H \leqslant \int_H \left| \int_H k_x(\eta) \{\tau_n(\eta^{-1}\xi) - \tau(\xi)\}\, d\eta \,\right| d\xi + A \left| \int_H k(x\eta)\, d\eta \right|$$

$$\leqslant A \left| \int_H k(x\eta)\, d\eta \right| + (2A/n)\, \|k_x\|_H \tag{7}$$

for all sufficiently large n since the function $k(x\eta)$ vanishes outside some fixed compact set as x ranges over the support of k and η over H (in fact, the compact set $C^{-1} \cdot C \cap H$ where C is the support of K). As both sides of (7) are functions on G/H we may integrate both sides with respect to dx' obtaining

$$\int_{G/H} \|k_x * \tau_n\|_H dx = \int_{G/H} \left\{ \int_H \left| \int_H k_x(\eta)\, \tau_n(\eta^{-1}\xi)\, d\eta \right\} dx\right.$$

$$= \int_{G/H} \left\{ \int_H \left| \int_H k(x\xi\eta)\, \tau_n(\eta^{-1})\, d\eta \right| d\xi \right\} dx$$

$$= \int_G \left| \int_H k(x\eta)\, \tau_n(\eta^{-1})\, d\eta \right| dx = \|\tau_n * k\|$$

$$\leqslant A \int_{G/H} |k'(x')|\, dx' + (2A/n) \int_G |k(x)|\, dx.$$

Taking the limit as $n \to \infty$ we obtain (3). This completes the proof of the lemma.

The University, Newcastle-upon-Tyne, England

REFERENCES

1. Gilbert, J. E., On projections of $L^\infty(G)$ onto translation-invariant subspaces. To appear in Proc. Lond. Math. Soc.
2. Reiter, H., Investigations in harmonic analysis. Trans. Amer. Math. Soc. *73*, 401–427 (1952).
3. —— The convex hull of translates of a function in L^1. J. Lond. Math. Soc. *35*, 5–16 (1960).
4. —— Contributions to harmonic analysis. VI. Ann. Math. *77*, 552–562 (1963).
5. —— Zwei Anwendungen der Bruhatschen Funktion. Math. Annalen *163*, 118–121 (1966).
6. Rosenthal, H. P., On the existence of approximate identities in ideals of group algebras. Arkiv för Mat. *7*, 185–191 (1967).
7. Rudin, W., Fourier Analysis on Groups. Interscience, New York, 1962.
8. Wik, I., A strong form of spectral synthesis. Arkiv för Mat. *6*, 55–64 (1965).

Tryckt den 26 november 1968

Uppsala 1968. Almqvist & Wiksells Boktryckeri AB