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Complementary remarks about the limit point and limit circle theory

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ABSTRACT

A method to obtain bounds for the number of solutions with regular behaviour at infinity of selfadjoint ordinary differential equations is given. In this method, which is presented for a polar problem, the sequence of nested ellipsoids in a previous paper of mine is replaced by the use of certain projections. At the same time I take the opportunity to complete the unsatisfactory bibliography of my earlier paper.

0. Introduction

The first complete generalization of H. Weyl's spectral theory of ordinary differential operators was given by K. Kodaira in a celebrated paper in the American Journal of Mathematics [8]. In the same year 1950 I. M. Glazman's treatment appeared. The problem has also been considered by E. A. Coddington [1], by W. N. Everitt [5–6] and others. T. Kimura and M. Takahasi [7] have treated the eigenvalue problems of the equation $Lu = \lambda u$ in full generality i.e. for any formally selfadjoint ordinary differential operator L.

Bounds for the number of integrable-square solutions of $Lu = \lambda u$, $\operatorname{Im} \lambda \neq 0$, were deduced by W. N. Everitt in three papers [2–4] by different methods. In my previous article [9] the corresponding results were obtained for the polar equation $Lu = \lambda pu$, where p does not have a constant sign. In this case L is assumed to have a positive definite Dirichlet integral and the solutions are requested to give finite values to this Dirichlet integral instead of being square integrable. The present paper gives a different approach to the same result. The method was presented at the Scandinavian Congress of Mathematicians at Oslo in 1968. As L. Hörmander remarked in a private communication at the congress, the Weyl–Everitt result for $Lu = \lambda u$ is a consequence of the existence of closed symmetric extensions of L in $\mathfrak{L}^2$. For general polar problems no extension theory of this kind has been established.

1. The method

Let L be a linear, formally selfadjoint, ordinary differential operator defined and with "sufficiently regular" coefficients on the half-infinite interval $a \leqslant x < \infty$. It is assumed that L has a Dirichlet integral ${}_a^b(.\,,.)$ so that

$$\int_a^b Lu\,\bar{v} = i\,{}_a^b[B(u,v)] + {}_a^b(u,v). \tag{1}$$

This Dirichlet integral shall be increasing (non-decreasing) with b and positive definite in C^m for large values of b. The number m is the half of the necessarily even order $M=2m$ of L. Assume $\operatorname{Im}\lambda>0$. On the solution space $l=\{u\,|\,Lu=\lambda pu\}$ the formula (1) leads to

$$\frac{\lambda-\bar{\lambda}}{i}\,{}_a^b(u,v) = h_b(u,v) - h_a(u,v), \tag{2}$$

where $h(u,v)$ is a quadratic (Hermitean) form in u, v and certain of their derivatives and with coefficients depending on x. As in Pleijel [9], it is proved that h has the maximum signature $[m,m]$ and that it assumes this signature on the solution space.

Let P be a linear subspace of l on which $h_a(u,u)$ is non-negative and which has the maximum dimension m. As a consequence of (2), the form $h_b(u,u)$ is positive definite on P if $a<b<\infty$. On the linear subspace of l

$$N(b)=\{u\,|\,h_b(u,P)=0\},$$

the form $h_b(u,u)$ is negative definite and the dimension of $N(b)$ is m. It follows that $l=P\dotplus N(b)$ as a direct sum. Let $\psi\in l$ and write $\psi=V(b)+U(b)$, where $V(b)\in P$, $U(b)\in N(b)$. Since $h_b(P,N(b))=0$, we find for any $V\in P$ that

$$h_b(\psi-V,\psi-V)=h_b(\psi-V(b),\psi-V(b))+h_b(V(b)-V,\,V(b)-V). \tag{3}$$

Since $V(b)-V\in P$, the last term is non-negative and $h_b(\psi-V(b),\psi-V(b))$ is the minimum of $h_b(\psi-V,\,\psi-V)$ when V varies in P. Let $a<b<b'$. Formula (2) shows that $h_b(u,u)$ increases with b. We conclude that

$$\begin{aligned} h_{b'}(\psi-V(b'),\,\psi-V(b')) &\geqslant h_b(\psi-V(b'),\,\psi-V(b')) \\ &= h_b(\psi-V(b),\,\psi-V(b))+h_b(V(b)-V(b'),\,V(b)-V(b')), \end{aligned} \tag{4}$$

so that $h_b(\psi-V(b),\,\psi-V(b))$ is also increasing. But $h_b(\psi-V(b),\,\psi-V(b))=h_b(U(b),\,U(b))\leqslant 0$. Thus

$$\lim_{b\to\infty} h_b(\psi-V(b),\;\psi-V(b))=G(\psi)\leqslant 0. \tag{5}$$

According to (4)

$$\begin{aligned} &h_{b'}(\psi-V(b'),\,\psi-V(b'))-h_b(\psi-V(b),\,\psi-V(b)) \\ &\qquad\geqslant h_b(V(b)-V(b'),\,V(b)-V(b'))\geqslant h_X(V(b)-V(b'),\,V(b)-V(b'))\geqslant 0, \end{aligned} \tag{6}$$

where X is a momentarily fixed number, $a<X<b<b'$. From (6) it follows that

$$\lim_{b,\,b'\to\infty} h_X(V(b)-V(b'),\;V(b)-V(b'))=0.$$

But h_X defines a positive definite metric on the finite dimensional and therefore closed space P. Hence, with a certain V in P,

$$\lim_{b\to\infty} V(b) = V \in P$$

holds in the sense that $\lim_{b\to\infty} h_X(V(b)-V,\ V(b)-V)=0$ for every $X>a$. From (6) we can then conclude that

$$\lim_{b\to\infty} h_b(V(b)-V,\ \ V(b)-V)=0 \tag{7}$$

by performing the transitions to the limit $b'\to\infty$, $b\to\infty$ in this order.

From (3), (5), (7) it then follows that $\lim_{b\to\infty} h_b(\psi-V,\psi-V)=G(\psi)$. According to (2) this shows that ${}^b_a(\psi-V,\psi-V)$ tends to a finite limit when $b\to\infty$. Thus to any solution ψ of $Lu=\lambda pu$ we can find another solution V in P such that the difference $\psi-V$ gives a finite value to the Dirichlet integral taken over $a\leqslant x<\infty$. From this it follows that there are at least m linearly independent solutions of $Lu=\lambda pu$ which have finite Dirichlet integrals over $a\leqslant x<\infty$.

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