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On the joint distribution of crossings of high multiple levels by a stationary Gaussian process

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1. Introduction

Let $\{\xi(t), -\infty<t<\infty\}$ be a real stationary Gaussian process with zero mean function and having continuous sample paths with probability one. Denote the covariance function by r (taking $r(0)=1$ for convenience), and the corresponding spectral distribution function by F. Let μ be the expected number of upcrossings of the level u by $\xi(t)$ in a t-interval of length 1.

Under certain conditions, H. Cramér [2, pp. 258 ff.] has shown that the number of upcrossings by $\xi(t)$ during a t-interval of length T of a single level tending to infinity is asymptotically Poisson distributed with parameter τ, provided T is chosen tending to infinity according to $T=\tau/\mu$. Cramér's conditions for validity have been weakened, in slightly different directions, by Belayev [1] and the author [3].

In this paper we show that a multivariate Poisson distribution is obtained in the analogous situation for upcrossings of multiple levels. The conditions for validity are the weakened ones of [3]. For the following precise statement of the result we need some notation. Let $0<p_l\leqslant p_{l-1}\leqslant \dots \leqslant p_1\leqslant p_0=1$, and consider the levels $u, u-(\ln p_1)/u, \dots, u-(\ln p_l)/u$, and the t-interval $(0, T)$ where $T=\tau/\mu$, $\tau>0$.

Let $N_0, N_1, \dots, N_l$ be the numbers of upcrossings by $\xi(t)$ during time T of these $l+1$ levels in the order listed.

Theorem 1.1. *If the stationary Gaussian process $\xi(t)$ satisfies*

(1) $\lambda_2=-r''(0)$ *exists and* $\int_0^\delta (\lambda_2+r''(t))/t\,dt<\infty$, *for some* $\delta>0$,

$$\text{or equivalently, } \int_0^\infty \log\,(1+\lambda)\,\lambda^2 dF(\lambda)<\infty, \text{ and}$$

(2) $r(t)=0(t^{-\alpha})$ *as* $t\to\infty$ *for some* $\alpha>0$,

then

$$\lim_{u\to\infty} P\{N_i=k_i,\ i=0,\ldots,l\}=\begin{cases} p(k_0;\ \tau)\prod_{i-1}^{l} b(k_i;\ k_{i-1},\ p_i/p_{i-1}) \\ (\textit{if } 0\leqslant k_l\leqslant\ldots\leqslant k_0 \textit{ and } k_i \textit{ are integers}) \\ 0 \textit{ otherwise}\end{cases}$$

In this theorem $p(k;\tau)$ and $b(r;n,p)$ are Poisson and binomial probabilities respectively.

The proof of this theorem is accomplished by dividing $(0, T)$ alternately into two types of subintervals. The t_1-intervals are chosen long enough to have one upcrossing of the different levels with appropriate small probabilities, but not more than one upcrossing. The interspaced t_2-intervals are chosen short enough to have no upcrossings, but long enough that the upcrossings of different t_1-intervals are asymptotically independent. This binomial situation leads to the Poisson distribution in the limit. This is the method of proof used by Cramér.

In the concluding remarks we note that the multiple levels given above are the only meaningful choices.

2. Preliminaires

Choose β such that $0<(k_0+\ldots+k_l+4)\beta<\alpha<1$. Let $M=[T\mu^{-\beta}]$ where [ ] denotes the greatest integer function, $q=T/M$. Let $m_1=[\mu^{-1}]$, $t_1=m_1q$, $m_2=[\mu^{\beta-1}]$, $t_2=m_2q$, and $n=[M/(m_1+m_2)]+1$. Note that as $u\to\infty$, $q\sim\mu^{\beta}\to 0$, $t_1\sim\mu^{\beta-1}\to\infty$, $t_2\sim\mu^{2\beta-1}\to\infty$, and $n\sim\tau\mu^{-\beta}\to\infty$. Let $\xi_q(t)=\xi(t)$ for $t=kq$, $k=0, \ldots, M$ and be the linear interpolation between the $\xi(kq)$. Let N_i^q be the number of upcrossings of the ith level by $\xi_q(t)$ during $(0, T)$.

Lemma 2.1. *If $\xi(t)$ satisfy the condition that $r''(0)$ exists, then* $\lim_{u\to\infty}(P\{N_i=k_i,$ $i=0, \ldots, l\}-P\{N_i^q=k_i, i=0, \ldots, l\})=0$. The proof follows easily from the single level proof [see 2, p. 260].

Lemma 2.2. *Under conditions 1) and 2) of Theorem 1.1, we have as $u\to\infty$*

$$P\{N_i^q(t_1)=0\}=1-p_iq+o(q)$$

$$P\{N_i^q(t_1)=1\}=p_iq+o(q).$$

Proof. For any non-negative integer valued random variable ν, we have

$$E\nu-E\nu(\nu-1)\leqslant P\{\nu=1\}\leqslant 1-P\{\nu=0\}\leqslant E\nu.$$

Take $\nu=N_i^q(t_1)$, and divide the above string of inequalities by p_iq. Note that a modification of lemma 2.1 of [3] gives $EN_i^q(q)=p_iq\mu+o(q\mu)$ and therefore

$$EN_i^q(t_1)=p_iq+o(q),$$

as $u\to\infty$. The final and crucial step is to apply a modification of theorem 2.2 of [3], which states $(1/q)E\nu(\nu-1)\to o$ as $u\to\infty$. Q.E.D.

We shall need the following additional notation.

On the $(0, T)$ interval mark off intervals of lengths t_1 and t_2 alternately, beginning with a t_1-interval. Define for $r=1, 2, \ldots, n$ and $i=0, \ldots, l$

${}_ic_r$ $=\{$exactly one ξ_q upcrossing of the ith level in the rth t_1-interval$\}$

${}_id_r$ $=\{$at least one ξ_q upcrossing of the ith level in the rth t_1-interval$\}$

${}_ie_r$ $=\{\xi(\nu q) > u - \ln p_i/u$ for at least one νq in the closed rth t_1-interval$\}$

${}_iC_{k_i}$ $=\{{}_ic_r$ occurs in exactly k_i of the n t_1-intervals of $(0, T)$ and ${}_ic_r^*$ occurs in the remaining $n-k_i$ t_1-intervals$\}$

$\left.\begin{matrix} {}_iD_{k_i} \\ {}_iE_{k_i} \end{matrix}\right\}$ similarly defined.

3. Proof of Theorem 1.1

Lemma 3.1. *Under conditions 1) and 2) of theorem 1.1, we have*

$$\lim_{u\to\infty} [P\{N_i^q = k_i,\ i=0, \ldots, l\} - P(\bigcap_{i=0}^{l} {}_iE_{k_i})] = 0.$$

Proof. We shall proof

$$\Delta_1 = P\{N^q = k_i,\ i=0, \ldots, l\} - P(\bigcap_{i=0}^{l} {}_iC_{k_i}),$$

$$\Delta_2 = P(\bigcap_{i=0}^{l} {}_iC_{k_i}) - P(\bigcap_{i=0}^{l} {}_iD_{k_i}),$$

and

$$\Delta_3 = P(\bigcap_{i=0}^{l} {}_iD_{k_i}) - P(\bigcap_{i=0}^{l} {}_iE_{k_i})$$

all approach zero as $u\to\infty$. First, we need only consider upcrossings in the t_1-intervals. Since

$$1 - P\{N_i^q(t_2) = 0\} \leqslant EN_i^q(t_2) = p_i t_2 \mu + o(t_2\mu) = 0(t_2\mu),$$

$P\{$at least one ξ_q upcrossing of at least one level in at least one of the n t_2-intervals of $(0, T)\} = 0(n\cdot l\cdot t_2\mu) = 0(\mu^\beta)$, which approaches zero as $u\to\infty$. Second, $P\{$more than one ξ_q upcrossing in at least one of the $n\cdot(l+1)$ t_1-intervals$\} = n\cdot(l+1)\cdot o(q) = o(nq) = o(1)$ as $u\to\infty$. These facts show that $\Delta_1\to 0$ and $\Delta_2\to 0$ as $u\to\infty$.

To evaluate Δ_3, we see that the event $\bigcap_{i=0}^{l} {}_iE_{k_i}$ is the union of $\binom{n}{k_0}\binom{n}{k_1}\cdots\binom{n}{k_l}$ different combinations of the more elementary events ${}_ie_r$. Without loss of generality consider the particular combination

$$G = {}_0e_1 \cdots {}_0e_{k_0}\, {}_0e^*_{k_0+1} \cdots {}_0e^*_n \cdots {}_le_1 \cdots {}_le_{k_l}\, {}_le^*_{k_l+1} \cdots {}_le^*_n.$$

The event $\bigcap_{i=0}^{l} {}_iD_{k_i}$ can be similarly decomposed with H corresponding to the event G with e's replaced by d's.

Now ${}_id_r \subset {}_ie_r$, ${}_ie_r^* \subset {}_id_r^*$, and ${}_id_r^* - {}_ie_r^* = {}_ie_r - {}_id_r$, so it is easy to show that ${}_ie_r A - {}_id_r B \subset ({}_ie_r - {}_id_r) \cup (A-B)$, ${}_ie_r^* A - {}_id_r^* B \subset A - B$, ${}_id_r^* A - {}_ie_r^* B \subset ({}_ie_r - {}_id_r) \cup (A-B)$, and

${}_id_rA - {}_ie_rB \subset A - B$. Therefore $G - H \subset \bigcup_{i=0}^{l} \bigcup_{r=1}^{k_i} ({}_ie_r - {}_id_r)$, and $H - G \subset \bigcup_{i=0}^{l} \bigcup_{r=k_i+1}^{n} ({}_ie_r - {}_id_r)$. Consequently

$$|PH - PG| \leqslant P(H-G) + P(G-H)$$
$$\leqslant \sum_{i=0}^{l} \sum_{r=1}^{n} P({}_ie_r - {}_id_r) \leqslant (l+1)\,nP({}_ig_r),$$

where ${}_ig_r = \{\xi(\nu_r g) \geqslant u - \ln p_i/u\}$, $\nu_r q$ is the left end point of the rth t_1-interval, and ${}_ie_r - {}_id_r \subset {}_ig_r$.

Finally since $P({}_ig_r) = 0(u^{-1} \exp\{-u^2/_2\})$ and $\binom{n}{k_i} = 0(n^{k_i})$ we have

$$\Delta_3 = 0(n^{(\Sigma k_i+1)} u^{-1} \exp\{-u^2/2\})$$
$$= 0\left(\exp\left\{-\frac{1-(\Sigma k_i+1)\beta}{2} u^2\right\}\right)$$

which approaches zero as $u \to \infty$ by the choice of β. Q.E.D.

The events ${}_iE_{k_i}$ have the important simplifying property that ${}_ie_r \subset {}_\varkappa e_r$, if $\varkappa < i$. Therefore we need only consider the case when $0 \leqslant k_l \leqslant \ldots \leqslant k_0$, since $G = \phi$ otherwise.

Further, in light of Lemma 2.1, Lemma 3.1, and the fact that Theorem 1.1 has been established for the single level [3], it suffices to show

Lemma 3.2. *For* $0 \leqslant k_l \leqslant \ldots \leqslant k_0$, k_i *integers, we have*

$$P[{}_iE_{k_i} / \bigcap_{j=0}^{i-1} {}_jE_{k_j}] \to b(k_i;\, k_{i-1},\, p_i/p_{i-1}) \quad \text{as} \quad u \to \infty.$$

Proof. Since ${}_je_r \subset {}_ke_r$ for $k < j$, the conditioning event $\bigcap_{j=0}^{i-1} {}_jE_{k_j}$ simplifies to a union of $\binom{n}{k_0}\binom{k_0}{k_1}\cdots\binom{k_{i-2}}{k_{i-1}}$ disjoint events G_ϕ where $G_\phi = \bigcap_{r=1}^{n} g_r$ and g_r equals ${}_0e_r^*$ for $n - k_0$ of the n r-subscripts, and equals ${}_je_r\, {}_{j+1}e_r^*$ for $k_j - k_{j+1}$ of the remaining k_j r-subscripts $(0 \leqslant j \leqslant i-2)$, and equals ${}_{i-1}e_r$ for the remaining k_{i-1} r-subscripts. Now suppose $P[{}_iE_{k_i}/G_\phi] \to b$ as $u \to \infty$ uniformly for all permutations (or better partitions) ϕ. Since $\bigcap_{j=0}^{i-1} {}_jE_{k_j} = \bigcup_\phi G_\phi$ and

$$P[{}_iE_{k_i} / \bigcup_\phi G_\phi] - b = \frac{\sum_\phi \{P[{}_iE_{k_i}/G_\phi] - b\}\, PG_\phi}{\sum_\phi PG_\phi},$$

it follows that $|P[{}_iE_{k_i}/G_\phi] - b| < \varepsilon$ for all ϕ implies $|P[{}_iE_{k_i}/\bigcup_\phi G_\phi] - b| < \varepsilon$.

Therefore we need only show $P[{}_iE_{k_i}/G_\phi] \to b(k_i;\, k_{i-1},\, p_i/p_{i-1})$ uniformly in ϕ. For convenience in notation in the remainder of this proof, we show

$$P[{}_iE_{k_i}/G] \to b(k_i;\, k_{i-1},\, p_i/p_{i-1})$$

for the particular $G=\bigcap_{r=1}^{n} g_r$ where g_r is ${}_{i-1}e_r$ for $r\leqslant k_{i-1}$, ${}_je_r\,{}_{j+1}e_r^*$ for $k_{j+1}<r\leqslant k_j$ $(0\leqslant j\leqslant i-2)$, and ${}_0e_r^*$ for $r>k_0$; and then note these calculations apply in a uniform manner to all G_ϕ. One other reduction can be made along these lines. Since ${}_iE_{k_i}\cap G$ is a finite union of $\binom{k_{i-1}}{k_i}$ disjoint events and $b(k_i;\,k_{i-1},\,p_i/p_{i-1})$ is the sum of $\binom{k_{i-1}}{k_i}$ equal probabilities, we show without loss of generality that

$$(\dagger)\qquad P[{}_ie_1\dots {}_ie_{k_i}\,{}_{i-1}e_{k_i+1}\,{}_ie^*_{k_i+1}\dots {}_{i-1}e_{k_{i-1}}\,{}_ie^*_{k_{i-1}}\,g_{k_{i-1}+1}\dots g_n]/PG$$
$$\rightarrow (p_i/p_{i-1})^{k_i}\,(1-p_i/p_{i-1})^{k_{i-1}-k_i}.$$

Since the events g_r that determine both the numerator and denominator of (†) depend only on two levels at a time, it turns out that all calculations typical of (†) are shown even if we only demonstrate (†) for $i=1$. Therefore consider

$$\Delta=PK-\pi_1^{k_1}\,(\pi_0-\pi_1)^{k_0-k_1}\,(1-\pi_0)^{n-k_0},$$

where the event

$$K={}_1e_1\dots {}_1e_{k_1}\,{}_0e_{k_1+1}\,{}_1e^*_{k_1+1}\dots {}_0e_{k_0}\,{}_1e^*_{k_0}\,{}_0e^*_{k+1}\dots {}_0e^*_n$$

and $\pi_i=P\{{}_ie_r\}$. Stationarity of $\xi(t)$ assures π_i is independent of r. Suppose there are L points of the form νq belonging to the n closed t_1-intervals of $(0, T)$, then $(n-1)(m_1+1)<L\leqslant n(m_1+1)$. The corresponding L random variables $\xi(\nu q)$ have a Gaussian density $f_1(y_1, \dots, y_L)$ and covariance matrix Λ_1. So

$$PK=\int_K f_1(y_1,\dots,y_L)\,dy_1\dots dy_L\equiv F(1).$$

Now if the random variables $\xi(\nu q)$ corresponding to points νq belonging to different t_1-intervals were independent, the corresponding covariances would be zero. Let Λ_0 be the resulting covariance matrix obtained from Λ_1 by zeroing out these covariances and $f_0(y_1, \dots, y_L)$ the corresponding Gaussian density. Now, by independence, f_0 factors and

$$\begin{aligned}F(0)&\equiv\int_K f_0(y_1,\dots,y_L)\,dy_1\dots dy_L\\&=P({}_1e_1)\dots P({}_1e_{k_1})\,P({}_0e_{k_1+1}\,{}_1e^*_{k_1+1})\dots P({}_0e_{k_0}\,{}_1e^*_{k_0})\,P({}_0e^*_{k_0+1})\dots P({}_0e^*_n)\\&=\pi_1^{k_1}\,(\pi_0-\pi_1)^{k_0-k_1}\,(1-\pi_0)^{n-k_0}.\end{aligned}$$

Actually $P({}_0e^*_n)=1-P({}_0e_n)\neq 1-\pi_0$ since the nth t_1-interval may be incomplete; but both approach 1 as $u\rightarrow\infty$, therefore we may take $\Delta=F(1)-F(0)$.

Now define $F(h)=\int_K f_h(y_1, \dots, y_L)\,dy_1 \dots dy_L$, where f_h is the Gaussian density corresponding to the symmetric positive definite matrix $\Lambda_h=h\Lambda_1+(1-h)\Lambda_0$, $0\leqslant h\leqslant 1$. So we have

$$(*)\qquad F'(h) = \sum \varrho_{ij} \int_K \frac{\partial f_h}{\partial \lambda_{ij}}\, dy_1 \dots dy_L = \sum \varrho_{ij} \int_K \frac{\partial^2 f_h}{\partial y_i\, \partial y_j}\, dy_1 \dots dy_L,$$

where $\Lambda_1 = (\varrho_{ij})$, $\Lambda_h = (\lambda_{ij})$, and Σ extends over all $i<j$ for which the corresponding νq points belong to different t_1-intervals. The identity $\partial f_h/\partial \lambda_{ij} = \partial^2 f_h/\partial y_i \partial y_j$ can be checked by differentiating the Fourier transform inversion formula for f_h.

In order to estimate the summands of $F'(h)$, we wish to carry out the integration over K with respect to y_i and then y_j. There are three cases for the first integration

$$\text{(i)}\qquad \int_{{}_0e_r^*\hat{K}} \frac{\partial^2 f_h}{\partial y_i\, dy_j}\, dy_i\, dy_j\, dy' = \int_{{}_0e_r^*\hat{K}} \frac{\partial f_h(y_i = u)}{\partial y_j}\, dy_j\, dy'.$$

$$\text{(ii)}\qquad \int_{{}_1e_r\hat{K}} \frac{\partial^2 f_h}{\partial y_i\, \partial y_j}\, dy_i\, dy_j\, dy' = -\int_{{}_1e_r^*\hat{K}} \frac{\partial f_h(y_i = u_1)}{\partial y_j}\, dy_j\, dy',$$

since the l.h.s. (left-hand side) minus the r.h.s. is an integral equal to zero.

$$\text{(iii)}\qquad \int_{{}_0e_r\,{}_1e_r^*\hat{K}} \frac{\partial^2 f_h}{\partial y_i\, \partial y_j}\, dy_i\, dy_j\, dy' = \int_{{}_1e_r^*\hat{K}} \frac{\partial f_h(y_i = u_1)}{\partial y_j}\, dy_j\, dy' - \int_{{}_0e_r^*\hat{K}} \frac{\partial f_h(y_i = u)}{\partial y_j}\, dy_j\, dy'$$

since the l.h.s. plus the second term of the r.h.s. with the use of case (i) and ${}_0e_r^* = {}_0e_r^*\,{}_1e_r^*$ is equal to the first term of the r.h.s. Here $dy' = dy_1 \dots \widehat{dy_i} \dots \widehat{dy_j} \dots dy_L$, $u_1 = u - \ln p_1/u$, $\hat{K}$ is the event K with the e factors referring to the rth t_1-interval being deleted, and y_i corresponds to a $\xi(\nu q)$ belonging to the rth t_1-interval, and y_j corresponds to a different t_1-interval.

For the double integration with respect to y_i and y_j, there are six different cases. With y_i corresponding to the rth t_1-interval and y_j to the sth $(s \neq r)$, the cases are, according to the event being integrated:

$$A = {}_0e_r^*\,{}_0e_s^*,\ B = {}_1e_r\,{}_1e_s;\ C = {}_1e_r\,{}_0e_s^*, D = {}_0e_r\,{}_1e_r^*\,{}_0e_s^*,\ E = {}_1e_r\,{}_0e_s\,{}_1e_s^*,$$

and $F = {}_0e_r\,{}_1e_r^*\,{}_0e_r\,{}_1e_s^*$. Cases A, B, and C are treated by Cramér [2, p. 268] for a single level. Applying cases (i), (ii) and (iii) and similar techniques for the second integration with respect to y_j, we obtain

$$\textit{Case } D\qquad \int_{D\hat{K}} \frac{\partial^2 f_h}{\partial y_i\, \partial y_j}\, dy_i\, dy_j\, dy' = \int_{{}_1e_r^*\,{}_0e_s^*\hat{K}} f_h(y_1 = u_1, y_j = u)\, dy' - \int_{{}_0e_r^*\,{}_0e_s^*\hat{K}} f_h(y_i = u, y_j = u)\, dy',$$

Case E
$$\int_{E\hat{K}} \frac{\partial^2 f_h}{\partial y_i\,\partial y_j}\,dy_i\,dy_j\,dy' = \int_{{}_1e_r^*\,{}_0e_s^*\hat{K}} f_h(y_i=u,\,y_j=u)\,dy'$$
$$-\int_{{}_1e_s^*\,{}_1e_s^*\hat{K}} f_h(y_i=u_1,\,y_j=u_1)\,dy',$$

Case F
$$\int_{F\hat{K}} \frac{\partial^2 f_h}{\partial y_i\,dy_j}\,dy_i\,dy_j\,dy' = \int_{{}_1e_r^*\,{}_1e_s^*\hat{K}} f_h(y_i=u_1,\,y_j=u_1)\,dy'$$
$$-\int_{{}_1e_r^*\,{}_0e_s^*\hat{K}} f_h(y_i=u_1,\,y_j=u)\,dy' - \int_{{}_0e_r^*\,{}_1e_s^*\hat{K}} f_h(y_i=u,\,y_j=u_1)\,dy'$$
$$+\int_{{}_0e_r^*\,{}_0e_s^*\hat{K}} f_h(y_i=u,\,y_j=u)\,dy'.$$

Here $\hat{K}$ is the event K with factors in both r and s deleted. In all six cases

$$\left|\int_K \frac{\partial^2 f_h}{\partial y_i\,\partial y_j}\,dy_i\,dy_j\,dy'\right| \leqslant 4\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_h(y_i=u,\,y_j=u)\,dy'$$
$$\leqslant \frac{4}{2\pi\sqrt{1-h^2\varrho_{ij}^2}}\exp\left\{-\frac{u^2}{1+h\,|\varrho_{ij}|}\right\}$$

where ϱ_{ij} was replaced by $|\varrho_{ij}|$.

Now by condition 2) of Theorem 1.1, we have

$$|\varrho_{ij}| = |r(\nu_i q - \nu_j q)| < C t_2^{-\alpha} \text{ for sufficiently large } u,$$

since $\nu_i q$ and $\nu_j q$ are separated by at least a t_2-interval.

Also
$$\frac{4}{2\pi\sqrt{1-h^2\varrho_{ij}^2}}\exp\left\{-\frac{u^2}{1+h\,|\varrho_{ij}|}\right\} \sim \frac{2}{\pi}\,e^{-u^2} \quad \text{as} \quad u\to\infty.$$

Since there are less than $L^2 \leqslant n^2(m_1+1)^2$ covariances ϱ_{ij}, we obtain from the equation (*),

$$|F'(h)| \leqslant C''\,n^2\,m_1^2\,t_2^{-\alpha}\,e^{-u^2} < C'\,\mu^{\alpha-4\beta},$$

for u sufficiently large.

In order to see that the constant C' does not depend on which G_ϕ was used, consider equation (*). The summation was over all t_1-intervals and ϱ_{ij} was estimated independent of which pair of t_1-intervals were referred to by i and j, so neither depends on the permutation ϕ of the t_1-intervals. For the integration over K, which depends on which G_ϕ was used, cases A through D are estimated in terms of ϱ_{ij} but otherwise the estimates are independent of K.

Therefore $|\Delta| = |F(1) - F(0)| = |\int_0^1 F'(h)dh| < C'\mu^{\alpha-4\beta}$, which approaches zero as $u \to \infty$ for our choice of β, and C' does not depend on ϕ. Finally, the l.h.s. of statement (†) with $i = 1$ is equal to $PK/P[{}_0e_1 \dots {}_0e_{k_0}\, {}_0e^*_{k_0+1} \dots {}_0e^*_n]$, and can be replaced by

$$\frac{\pi_1^{k_1}(\pi_0 - \pi_1)^{k_0-k_1}(1-\pi_0)^{n-k_0} + 0(\mu^{\alpha-4\beta})}{\pi_0^{k_0}(1-\pi_0)^{n-k_0} + 0(\mu^{\alpha-4\beta})}.$$

Since $\pi_i = P\{{}_ie_r\}$ differs from $P\{{}_id_r\}$ by less than $P\{{}_ig_r\} = 0(u^{-1}\exp\{-u^2/2\})$ as in the final lines of the proof of Lemma 3.1, we have $\pi_i \sim p_i q$ as $u \to \infty$ by Lemma 2.2. Since $n \sim \tau\mu^{-\beta}$, $q \sim \mu^{\beta}$, $(1-\pi_0)^n \sim e^{-\tau}$, and $\pi_0^{k_0} \sim \mu^{\beta k_0}$, we divide the above expression by $\pi_0^{k_0}(1-\pi_0)^{n-k_0}$ to obtain

$$\frac{\left(\dfrac{\pi_1}{\pi_0}\right)^{k_1}\left(1 - \dfrac{\pi_1}{\pi_0}\right)^{k_0-k_1} + 0(u^{\alpha-(k_0+4)\beta})}{1 + 0(\mu^{\alpha-(k_0+4)\beta})}.$$

Now $\pi_i/\pi_{i-1} \to p_i/p_{i-1}$ and by our choice of β the error terms approach zero as $u \to \infty$ to establish statement (†) with $i = 1$ uniformly in ϕ.

The general case $i > 1$ differs slightly from $1 = i$ in cases A through D by added notation and in the fact that the density f_h is evaluated at differing levels u_j. In estimating formula (*) we replace all levels by the least one u, and consequently the l.h.s. of (†) can be replaced by

$$\frac{\pi_i^{k_i}(\pi_{i-1} - \pi_i)^{k_{i-1}-k_i} \dots (\pi_0 - \pi_1)^{k_0-k_1}(1-\pi_0)^{n-k_0} + 0(\mu^{\alpha-4\beta})}{\pi_{i-1}^{k_{i-1}}(\pi_{i-2} - \pi_{i-1})^{k_{i-2}-k_{i-1}} \dots (\pi_0 - \pi_1)^{k_0-k_1}(1-\pi_0)^{n-k_0} + 0(\mu^{\alpha-4\beta})}$$

where the error terms again imply a constant C' which doesn't depend on ϕ. Since $(\pi_j - \pi_{j+1})^{k_j-k_{j-1}} \sim C'' q^{k_j-k_{j-1}}$, the above expression becomes

$$\frac{\left(\dfrac{\pi_i}{\pi_{i-1}}\right)^{k_i}\left(1 - \dfrac{\pi_i}{\pi_{i-1}}\right)^{k_{i-1}-k_i} + 0(\mu^{\alpha-(k_0+4)\beta})}{1 + 0(\pi^{\alpha-(k_0+4)\beta})}.$$

Taking the limit, one establishes statement (†) uniformly in ϕ, Lemma 3.2, and consequently Theorem 1.1. Q.E.D.

4. Comments

Note that the choice of levels in the form $u + k/u$ is comprehensive. If the levels were written as $u + f_i(u)/u$, $f_i(u) > 0$, then we would consider the limiting behavior of $f_i(u)$ as $u \to \infty$. For $f_i(u)$ having a limit and treating degenerate cases separately, we may replace $f_i(u)$ by its limit in Theorem 1.1. If $f_i(u)$ oscillates, then there is no limiting distribution of the number of upcrossings.

In the particular case of two levels u and $u + \varepsilon$, $\varepsilon > 0$, there are asymptotically no upcrossings of $u + \varepsilon$ during $(0, T)$. This is easily proved by showing the expected number of upcrossings of the level $u + \varepsilon$ during time T approaches zero as $u \to \infty$.

It is easy to check that the number of downcrossings is equal to the number of upcrossings, and consequently, the limiting multivariate distribution of the numbers of downcrossings of multiple levels is also given by Theorem 1.1.

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