Lower bounds for pseudo-differential operators

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0. Introduction

Let A be a classical scalar pseudo-differential operator of order m (cf. Kohn — Nirenberg [7]) in an open subset Ω of R^n. We are interested in estimates of A from below of the form

$$\operatorname{Re}(Au, u) \geq C|u|_{(s)}^2, \quad u \in C_0^\infty(K) \tag{0.1}$$

where K is a compact subset of Ω and $|u|_{(s)}$ is the norm of u in the space $H_{(s)}$ of functions with derivatives of order s in L^2 and $(v, u) = \int v\bar{u}\,dx$. If $s \geq m/2$ the estimate is always true for some C since (Au, u) is continuous in $H_{(m/2)}$. On the other hand, if $s < m/2$ it is easy to see that (0.1) implies

$$\operatorname{Re} a_m(x, \xi) \geq 0 \tag{0.2}$$

where a_m is the principal symbol of A. In the opposite direction Gårding [3] proved that if (0.2) is valid, then we can for every $\varepsilon > 0$ and every s find a constant $C = C(K, \varepsilon, s)$ such that

$$\operatorname{Re}(Au, u) + \varepsilon|u|_{(\mu)}^2 \geq C|u|_{(s)}^2, \quad u \in C_0^\infty(K) \tag{0.3}$$

if $\mu = m/2$. A simple modification of the proof gives the same result for any $\mu > (m-1)/2$. In fact if A satisfies (0.2) and $m/2 \geq \mu > (m-1)/2$ then we can write

$$(A + A^*)/2 + \varepsilon(1 + |D|^2)^\mu = P^*P + Q$$

where P and Q are pseudo-differential operators in Ω and the order of Q does not exceed $m - 1$.

However the situation becomes more complex when $\mu = (m-1)/2$. It was proved by Hörmander [5] that (0.2) does imply that (0.3) is valid for some $\varepsilon > 0$, but to have (0.3) for every $\varepsilon > 0$ we must clearly in addition to (0.2) place a restriction on the terms in A of order $m - 1$. In this paper we shall study necessary and sufficient conditions on A for (0.3) to be valid for every $\varepsilon > 0$ when

$\mu = (m - 1)/2$. The proof depends on the localization technique introduced in Hörmander [5, 6] but requires more careful estimates of remainder terms. In addition we have to make a complete study of inequalities of the following kind,

$$\operatorname{Re} \int \overline{v(y)} \sum_{|\alpha+\beta| \leq 2} (1/\alpha!\beta!) a_\beta^\alpha y^\beta D^\alpha v(y) dy \geq 0 , \quad v \in C_0^\infty(R^n) \tag{0.4}$$

where a_β^α are complex numbers.

This will be done in section 2 and the study of (0.3) when $\mu = (m - 1)/2$ will be carried out in section 3. In section 4 we shall use our results to prove a theorem due to Radkevič [10] about hypoellipticity for a certain class of classical pseudo-differential operators with non-negative principal symbol. We also mention that the inequality (0.3) with $\mu = (m - 1)/2$ has been treated in the vector valued case by Lax-Nirenberg [9]. (See also [2] and [11].)

Finally I want to thank my teacher prof. L. Hörmander for his kind interest in this work and many suggestions for improvements.

1. Notation and preliminaries

We shall make use of the familiar notation $D_j = - i\, \partial/\partial x_j$, where $i = \sqrt{-1}$ and if $D = (D_1, \ldots, D_n)$ is the gradient vector and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with the α_j non-negative integers then D^α denotes the differential operator $D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. We set $|\alpha| = \sum \alpha_j$ and $\alpha! = \alpha_1! \ldots \alpha_n!$. If $y = (y_1, \ldots, y_n)$ we define y^α in a similar way. By $\mathcal{S}$ or $\mathcal{S}(R^n)$ we denote the set of all functions $\Phi \in C^\infty(R^n)$ such that

$$\sup_x |x^\beta D^\alpha \Phi(x)| < \infty \tag{1.1}$$

for all multi-indices α and β. $H_{(s)}$ is the completetion of $\mathcal{S}$ in the norm

$$|u|_{(s)}^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \tag{1.2}$$

where $\hat{u}$ denotes the Fourier transform of u

$$\hat{u}(\xi) = \int e^{-i\langle x, \xi \rangle} u(x) \, dx .$$

If X is an open subset of R^n and a belongs to $C^\infty (X \times R^n)$ and satisfies the inequality

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|} , \quad x \in K , \quad \xi \in R^n \tag{1.3}$$

when K is a compact subset of X, then $a(x, D)$ will denote the corresponding pseudo-differential operator (of order m)

$$a(x , D)u(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x , \xi)\hat{u}(\xi)d\xi , \quad u \in C_0^\infty(R^n) .$$

In particular, if ψ and u belong to $\mathcal{S}$ then $\psi(D)u$ will denote the function whose Fourier transform equals $\psi\hat{u}$.

If A is a classical pseudo-differential operator of order m with symbol a, then a_j (with $j = m,\ m - 1 , \ldots$) will denote the part of a that is homogeneous of degree j with respect to ξ. Finally with a as in (1.3) we shall use the notation $a_\beta^\alpha = (iD_x)^\beta(iD_\xi)^\alpha a$.

2. A study of the inequality (0.4)

Let a_β^α denote complex numbers and consider the inequality

$$\mathrm{Re} \int \overline{v(y)} \sum_{|\alpha+\beta| \leq k} (1/\alpha!\beta!) a_\beta^\alpha y^\beta D^\alpha v(y)dy \geq 0 , \quad v \in C_0^\infty(R^n) \tag{2.1}$$

where k is a non-negative integer. Of course the form defined in (2.1) does not determine the coefficients a_β^α uniquely, but we claim that there exist uniquely determined real coefficients b_β^α such that

$$\mathrm{Re} \int \overline{v(y)} \sum_{|\alpha+\beta| \leq k} (1/\alpha!\beta!) b_\beta^\alpha y^\beta D^\alpha v(y)dy , \quad v \in C_0^\infty(R^n) \tag{2.2}$$

defines the same form. For the existence of such b_β^α it is enough to prove that

$$\mathrm{Im} \int \overline{v(y)} y^\beta D^\alpha v(y)dy = \mathrm{Re} \int \overline{v(y)} \{(y^\beta D^\alpha - D^\alpha y^\beta)/2i\}\, v(y)dy$$

can be written in this form, but this follows easily from the fact that

$$y_j D_k - D_k y_j = i\delta_{jk} \tag{2.3}$$

To prove the uniqueness when the coefficients are real we replace $v(y)$ in (2.1) by $\varphi(y - tx)e^{it<y,\xi>}$ where $\int |\varphi(y)|^2 dy = 1$ and get the expression

$$\mathrm{Re} \int \overline{\varphi(y)} \sum_{|\alpha+\beta| \leq k} (1/\alpha!\beta!) a_\beta^\alpha (y + tx)^\beta (D + t\xi)^\alpha \varphi(y)dy$$

which is a polynomial in t with the leading coefficient

$$h(x , \xi) = \sum_{|\alpha+\beta| = k} a_\beta^\alpha x^\beta \xi^\alpha / \alpha!\beta! \tag{2.4}$$

Hence by letting t tend to infinity we can conclude that two sets of real coefficients defining the same form must coincide and that the validity of (2.1) implies that the form h is positive semi-definite. When $k = 2$ and there are no lower terms the converse statement is valid for by formula (2.3)

$$\int \overline{v(y)} y_j D_k v(y) dy = \int \overline{v(y)} D_k y_j v(y) dy + i\delta_{jk} \int |v(y)|^2 dy\,, \quad v \in C_0^\infty(R^n) \tag{2.5}$$

so when taking real parts in (2.1) y and D can be treated as commutating operators. In particular, if $h(x, \xi)$ is a real quadratic form on $R^n \oplus R^n$, then we have a well defined form

$$\operatorname{Re} \int \overline{v(y)}\, h(y, D) v(y) dy\,, \quad v \in C_0^\infty(R^n)$$

and by a diagonalization of h one immediately sees that the expression is non-negative if h is positive semi-definite. Also, when $k = 2$,

$$h_1 \leq h_2 \text{ implies } \operatorname{Re} \int \overline{v(y)} h_1(y, D) v(y) dy \leq \operatorname{Re} \int \overline{v(y)} h_2(y, D) v(y) dy\,. \tag{2.6}$$

When $k = 2$ we get the correspondence between the coefficients in (2.1) and (2.2) by the equations

$$b_\beta^\alpha = \operatorname{Re} a_\beta^\alpha \text{ if } \alpha + \beta \neq 0\,, \quad b_0^0 = \operatorname{Re} a_0^0 - \sum_{|\beta|=1} \operatorname{Im} a_\beta^\beta / 2\,. \tag{2.7}$$

We shall now confine ourselves to the case when k equals 2 in formula (2.1) and let h and f denote the quadratic respectively the linear part of the corresponding polynomial with real coefficients on $R^n \oplus R^n$. If $\langle u, v \rangle$ denotes the standard Euclidean scalar product on $R^n \oplus R^n$, then there is a unique symmetric transformation H such that

$$h(u) = \langle Hu, u \rangle\,, \quad u \in R^n \oplus R^n\,.$$

By i we shall also denote multiplication by the imaginary unit in $R^n \oplus R^n$ when this space is identified with C^n as a real vector space under the isomorphism

$$R^n \oplus R^n \ni (x, \xi) \to x + i\xi \in C^n\,.$$

We shall examine the invariance of (2.1) when the symbol (y, D) is transformed as a vector in $R^n \oplus R^n$ and find that our inequality is invariant under the symplectic group of $R^n \oplus R^n$.

Definition 2.1. We define the symplectic bilinear form σ on $R^n \oplus R^n$ by the equation

$$\sigma(u, v) = \operatorname{Im}(u, v) = \langle -iu, v \rangle = \langle u, iv \rangle\,, \quad u, v \in R^n \oplus R^n \tag{2.8}$$

where (u, v) denotes the Hermitian scalar product on C^n. The corresponding linear transformations on $R^n \oplus R^n$ under which $\sigma(u, v)$ remains unaltered are called canonical transformations. They form the symplectic group $Sp(R^n \oplus R^n)$.

If H is strictly positive, then $\langle Hu, v \rangle$ defines a scalar product b on $R^n \oplus R^n$ and since

$$\langle HiHu, v \rangle = \langle Hu, (-iH)v \rangle\,, \quad u, v \in R^n \oplus R^n$$

iH is skew-symmetric with respect to b and it follows that the spectrum of iH is situated on the imaginary axis, symmetrically around the origin. The last statement is clearly valid even if H is only positive semi-definite since the eigenvalues of a linear transformation depend continuously on the transformation and $H = \lim_{\varepsilon \to 0} H + \varepsilon I$, where I is the identity transformation.

Definition 2.2. By $\widetilde{\mathrm{Tr}}\, H$ ($\widetilde{\mathrm{Tr}}\, h$) we shall mean the sum of the positive elements in $i \cdot$ Spectrum (iH) where each eigenvalue is counted with its multiplicity.

It follows from the argument above that $\widetilde{\mathrm{Tr}}\, H$ depends continuously on H.

Remark 2.3. In order to illustrate the natural role of $\widetilde{\mathrm{Tr}}\, H$ we shall prove that it is invariant under symplectic transformations. If $\chi \in Sp(R^n \oplus R^n)$ then the polynomial $h(\chi u)$ corresponds to the symmetric transformation $\chi^t H \chi$, where χ^t is the adjoint of χ with respect to the standard Euclidean structure of $R^n \oplus R^n$, so we have to show that

$$\widetilde{\mathrm{Tr}}\, \chi^t H \chi = \widetilde{\mathrm{Tr}}\, H \,. \tag{2.9}$$

Now by the definition of $Sp(R^n \oplus R^n)$

$$i\, \chi^t = \chi^{-1} i$$

hence

$$i\, \chi^t H \chi = \chi^{-1} i\, H \chi$$

which implies that $i\chi^t H\chi$ and iH have the same characteristic polynomial.

THEOREM 2.4. *The inequality* (2.1) *with* $k = 2$ *is valid if and only if*

(i) $h(x, \xi) = \sum_{|\alpha+\beta|=2} \mathrm{Re}\, a_\beta^\alpha x^\beta \xi^\alpha / \alpha! \beta! \geq 0$

(ii) $f(x, \xi) = \sum_{|\alpha+\beta|=1} \mathrm{Re}\, a_\beta^\alpha x^\beta \xi^\alpha$

vanishes in the null space of h, *and*

(iii) $\mathrm{Re}\, a_0^0 - \sum_{|\beta|=1} \mathrm{Im}\, a_\beta^\beta / 2 - \langle H^{-1} f, f \rangle / 4 + \widetilde{\mathrm{Tr}}\, H \geq 0$.

Remark 2.5. Although H^{-1} does not have to exist the expression $\langle H^{-1}f, f\rangle$ is well defined by (ii) and as is easily seen

$$\langle H^{-1} f, f \rangle = \sup_{h(u) < 1} \langle u, f \rangle^2 \,. \tag{2.11}$$

In view of (2.7) we may assume that the coefficients a_β^α are real in the proof of Theorem 2.4. Of course we shall make use of the trivial fact that we could as well let v run through $\mathcal{S}$ in (2.1).

Definition 2.6. We shall say that two polynomials p_1 and p_2 on $R^n \oplus R^n$ are symplectically equivalent if there is a canonical transformation χ such that

$$p_1(\chi(v)) = p_2(v)\,, \quad v \in R^n \oplus R^n \,.$$

By a symplectic basis for $R^n \oplus R^n$ we shall mean a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ with the property

$$\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0\,, \quad \sigma(e_j, f_k) = -\,\delta_{jk}\,.$$

LEMMA 2.7. *Let p_1 and p_2 be symplectically equivalent polynomials of the second degree with real coefficients. Then the inequality*

$$\operatorname{Re} \int \overline{v(y)} p_1(y , D)\, v(y)\, dy \geq 0 , \quad v \in \mathcal{S} \tag{2.12}$$

is valid for p_1 if and only if it is valid for p_2.

We have already proved the necessity of condition (i) in Theorem 2.4. Using Lemma 2.7 it will be easy to finish the proof of Theorem 2.4 in the case where h is positive definite. We observe that

$$\sigma(u , v) = \langle HSu , v \rangle = b(Su , v)$$

where $S = (iH)^{-1}$ is skew-symmetric with respect to b. By a standard result concerning normal transformations on a Hilbert space we shall be able to reduce our problem to the case $n = 1$. In fact, we have the following

LEMMA 2.8. *Let $\{\lambda_1 , \ldots , \lambda_n\}$ be the positive elements (counted with multiplicity) in $i \cdot$ Spectrum (S). Then $R^n \oplus R^n$ can be written as a direct sum*

$$R^n \oplus R^n = \sum \oplus V_j$$

which is orthogonal with respect to b and where V_j has an orthonormal basis $\{e_j , f_j\}$ in the same sense such that

$$Se_j = - \lambda_j f_j , Sf_j = \lambda_j e_j , j = 1 , \ldots , n . \tag{2.13}$$

Notice that this condition implies that $\{E_1 , \ldots , E_n , F_1 , \ldots , F_n\}$ with $E_j = \lambda_j^{-1/2} e_j , F_j = \lambda_j^{-1/2} f_j$ is a symplectic basis for $R^n \oplus R^n$. If $\{E_1^0 , \ldots , E_n^0\}$ is the standard orthonormal basis for C^n, then $\{E_1^0 , \ldots , E_n^0, F_1^0 , \ldots , F_n^0\}$ with $F_j^0 = iE_j^0$ is a symplectic basis for $R^n \oplus R^n$. Let χ denote the canonical transformation which maps E_j^0 on E_j and F_j^0 on F_j for $j = 1 , \ldots , n$. If

$$H^{-1} f = t_1 E_1 + \ldots + t_n E_n + \tau_1 F_1 + \ldots + \tau_n F_n$$

and

$$v = x_1 E_1^0 + \ldots + x_n E_n^0 + \xi_1 F_1^0 + \ldots + \xi_n F_n^0$$

then an easy computation yields

$$a_0^0 + \langle f , \chi v \rangle + h(\chi v) = a_0^0 + \sum_{j=1}^{n} \lambda_j^{-1} (t_j x_j + \tau_j \xi_j + x_j^2 + \xi_j^2) . \tag{2.14}$$

According to Lemma 2.7 we have to study the inequality

$$\operatorname{Re} \int \overline{v(y)} \{a_0^0 + \sum_{j=1}^{n} \lambda_j^{-1} (t_j y_j + \tau_j D_j + y_j^2 + D_j^2)\}\, v(y)\, dy \geq 0 , \quad v \in \mathcal{S} . \tag{2.15}$$

Replacing v by

$$e^{-i(\Sigma \tau_j y_j)/2} v(y)$$

in (2.15) and repeating the same argument after a Fourier transformation we get the equivalent inequality,

$$(a_0^0 - \langle H^{-1}f, f\rangle/4) \int |v(y)|^2 dy + \int \overline{v(y)} \sum \lambda_j^{-1}(y_j^2 + D_j^2)v(y)dy \geq 0 , \quad v \in \mathcal{S} . \quad (2.16)$$

Here we have used the equation

$$\sum \lambda_j^{-1}(t_j^2 + \tau_j^2) = b(H^{-1}f, H^{-1}f) = \langle H^{-1}f, f\rangle .$$

Let $u \in C_0^\infty(R)$. By expanding the right-hand side of the inequality

$$0 \leq \int |(D - \lambda iy)u|^2 \, dy$$

we get for real λ

$$\int \bar{u}(D^2 + \lambda^2 y^2)u \, dy \geq \lambda \int |u|^2 \, dy .$$

(This is essentially the proof of the uncertainty relation.) Hence we obtain by choosing $\lambda = 1$

$$\int \overline{v(y)} \sum \lambda_j^{-1}(y_j^2 + D_j^2) \, v(y) \, dy \geq (\sum \lambda_j^{-1}) \int |v(y)|^2 \, dy \quad (2.17)$$

and choosing $v(y) = e^{-|y|^2/2}$ we see that the inequality cannot be improved.

Since $\widetilde{\mathrm{Tr}}\, H = \sum \lambda_j^{-1}$ this will complete the proof of Theorem 2.4 in case H^{-1} exists. The general case follows from a continuity argument. We replace the coefficients a_β^α by $a_\beta^\alpha(\varepsilon)$ in such a way that H will be replaced by $H_\varepsilon = H + \varepsilon I$ while f and a_0^0 are conserved. From (2.6) we get (for $\varepsilon > 0$)

$$\mathrm{Re} \int \overline{v(y)} \sum_{|\alpha+\beta| \leq 2} (1/\alpha!\beta!) a_\beta^\alpha y^\beta D^\alpha v(y) \, dy \leq \mathrm{Re} \int \overline{v(y)} \sum_{|\alpha+\beta| \leq 2} (1/\alpha!\beta!) a_\beta^\alpha(\varepsilon) y^\beta D^\alpha v(y) \, dy$$

and since $g(\varepsilon) = a_0^0 - \langle H_\varepsilon^{-1}f, f\rangle/4 + \widetilde{\mathrm{Tr}}(H_\varepsilon)$ tends to $g(0)$ when ε tends to zero the necessity of (ii) and (iii) follows.

Conversely, if (ii) and (iii) are fulfilled we can choose a function $\varphi(\varepsilon)$ tending to zero when ε tends to zero such that

$$a_0^0 + \varphi(\varepsilon) - \langle H_\varepsilon^{-1}f, f\rangle/4 + \widetilde{\mathrm{Tr}}\,(H_\varepsilon) \geq 0$$

and then the sufficiency of our conditions follows since for fixed v the left hand side of (2.1) depends continuously on the coefficients a_β^α.

Proof of Lemma 2.7. The idea is to subject v in formula (2.12) to isomorphisms of $\mathcal{S}$ which lead to linear transformations of (y, D) generating $Sp = Sp(R^n \oplus R^n)$. We shall introduce three subsets G_1, G_2 and G_3 of Sp:

G_1 is the group of all transformations of the form $T^{-1} \oplus T^t$ where T is an isomorphism in R^n and T^t denotes the adjoint transformation. This is the group of linear transformations in $R^n \oplus R^n$ obtained by regarding the second copy as the dual space of the first.

G_2 is the set of all elementary canonical transformations, i.e. transformations of the form

$$C^n \ni (z_1, \dots, z_n) \to (i^{\psi(1)} z_1, \dots, i^{\psi(n)} z_n) \tag{2.18}$$

where ψ is the characteristic function of a subset of $\{1, \dots, n\}$.

G_3 is the group of transformations

$$R^n \oplus R^n \ni (x, \xi) \to (x, \xi - Sx) \tag{2.19}$$

where S is a symmetric linear transformation on the Euclidean space R^n.

If the validity of (2.12) for p_1 implies the same for $p_2 = p_1 \circ \chi$ when $\chi \in G_j$, then we shall say that (2.12) is G_j-invariant.

To prove that (2.12) is G_1 — invariant we replace $v(y)$ in (2.12) by $v^T(y) = v(Ty)$ where T is an isomorphism in R^n. By the chain rule

$$D^\alpha(v^T) = ((T^t D)^\alpha v)^T$$

where $(T^t D_x) e^{i<x, \xi>} = (T^t \xi) e^{i<x, \xi>}$. A substitution of y by $T^{-1}x$ in the integral yields us the equation

$$\operatorname{Re} \int \overline{v^T(y)} p_1(y, D) v^T(y)\, dy = |\det T|^{-1} \operatorname{Re} \int \overline{v(x)} p_1(T^{-1}x, T^t D_x) v(x)\, dx$$

from which the G_1-invariance follows.

When proving the G_2-invariance we shall construct the partial Fourier transform that corresponds to ψ in (2.18),

$$\mathcal{F}_\psi = \prod_{\psi(j)=1} \mathcal{F}_j$$

where

$$\mathcal{F}_j\, u(\xi) = \int e^{-i<x_j, \xi_j>} u(\xi_1, \dots, \xi_{j-1}, x_j, \xi_{j+1}, \dots, \xi_n) dx_j, \quad u \in \mathcal{S}.$$

Replacing v in (2.12) by $\mathcal{F}_\psi(v)$ and using the formulas

$$y_j \mathcal{F}_\psi = \mathcal{F}_\psi D_j, \quad D_j \mathcal{F}_\psi = -\mathcal{F}_\psi y_j \quad \text{if} \quad \psi(j) = 1 \tag{2.20}$$

$$y_j \mathcal{F}_\psi = \mathcal{F}_\psi y_j, \quad D_j \mathcal{F}_\psi = \mathcal{F}_\psi D_j \quad \text{if} \quad \psi(j) = 0$$

$$\int \overline{\mathcal{F}_\psi v(y)}\, \mathcal{F}_\psi u(y)\, dy = (2\pi)^{\Sigma \psi(j)} \int \overline{v(y)} u(y)\, dy,$$

we see that (2.12) is also G_2-invariant.

In the proof of the G_3-invariance we replace v in (2.12) by $e^{-i<Sy, y>/2} v(y)$ with S as in (2.19). Now

$$D_j(e^{-i<Sy, y>/2} v(y)) = e^{-i<Sy, y>/2}(D_j - S_j(y)) v(y) \tag{2.21}$$

where $S_j(y)$ is the jth coordinate of $S(y)$ and since y and D commute in (2.1) when $k=2$ and the coefficients are real we get

$$\operatorname{Re}\int \overline{e^{-i<Sy,y>/2}v(y)}D^{\alpha}(e^{-i<Sy,y>/2}v(y))\,dy = \tag{2.22}$$

$$= \operatorname{Re}\int \overline{v(y)}(D-S(y))^{\alpha}v(y)dy\,,\quad |\alpha|\leq 2\,.$$

The G_3-invariance now follows immediately from (2.21) and (2.22). In order to complete the proof of Lemma 2.7 we have to show

LEMMA 2.9. *The sets* G_1, G_2 and G_3 *generate together the symplectic group* $Sp(R^n\oplus R^n)$.

We shall use the symbol $\Lambda(n)$ to denote the set of all subspaces λ of $R^n\oplus R^n$ which are isotropic in the symplectic geometry and of maximal dimension, i.e. $\lambda\in\Lambda(n)$ if and only if $\dim(\lambda)=n$ and

$$\sigma(u\,,v)=0 \quad\text{when}\quad u\,,v\in\lambda\,. \tag{2.23}$$

$R^n=[E_1^0\,,\dots,E_n^0]$ and $iR^n=[F_1^0\,,\dots,F_n^0]$ belong to $\Lambda(n)$. It is easy to see that if $\lambda_0\in\Lambda(n)$ is transversal to iR^n, that is, the projection $\lambda_0\to R^n$ along iR^n is surjective, then there is a symmetric linear transformation S on R^n such that

$$\lambda_0=\{(x\,,Sx)\,;x\in R^n\}\,.$$

Let $\lambda_0\in\Lambda(n)$. By looking at the image of the projection

$$\lambda_0\ni(x\,,\xi)\to x\in R^n$$

we realize that there exists an elementary canonical transformation χ such that $\chi(\lambda_0)$ is transversal to iR^n. (Cf. [1, § 96]).

Proof of Lemma 2.9. Let $\chi\in Sp$. By the remark above there is a χ_1 in G_2 and a symmetric linear transformation S such that

$$\chi_1\circ\chi(iR^n)=\{(x\,,Sx)\,;x\in R^n\}\,.$$

Choosing χ_2 as in (2.19) we get

$$i\chi_2\circ\chi_1\circ\chi(iR^n)=iR^n\,.$$

Since R^n is transversal to iR^n it follows that $\chi'R^n$ is transversal to $\chi'iR^n$ for any linear bijective χ'. Hence there is a symmetric linear transformation S_1 such that

$$i\chi_2\circ\chi_1\circ\chi(R^n)=\{(x\,,S_1x)\,;x\in R^n\}\,.$$

If χ_3 denotes the corresponding map defined in (2.19) and $\chi_4=\chi_3\circ i\circ\chi_2\circ\chi_1\circ\chi$ we have

$$\chi_4(R^n)=R^n\,,\ \chi_4(iR^n)=iR^n\,.$$

Thus $\chi_4 = A \oplus B$ where A and B are linear transformations on R^n. Since $\chi_4 \in Sp$ we must have $\chi_4 \in G_1$ and we conclude that χ belongs to the group generated by G_1, G_2 and G_3.

Remark 2.10. It is easy to describe those positive semi-definite linear transformations H of $R^n \oplus R^n$ for which $\widetilde{\mathrm{Tr}}$ vanish. The following statements are equivalent

1° $\widetilde{\mathrm{Tr}}\, H = 0$

2° $H(R^n \oplus R^n) \subset \lambda$ for some $\lambda \in \Lambda(n)$

3° $H^{-1}(0) \supset \lambda$ for some $\lambda \in \Lambda(n)$.

The equivalence between 2° and 3° follows from the fact that $\lambda^\perp = i\lambda$, where $\lambda^\perp$ denotes the orthogonal complement of λ with respect to the standard Euclidean structure. Let G denote the positive square root of H. Since GiG is skew-symmetric the equivalence between 1° and 2° will follow from the following chain of implications:

1° $\Leftrightarrow iH$ is nilpotent $\Rightarrow GiG$ is nilpotent $\Rightarrow GiG = 0 \Rightarrow HiH = 0 \Leftrightarrow$ 2°.

We shall end this section with an application of our results to the inequality

$$\int |\sum_1^n A_\nu y_\nu v(y) + \sum_1^n B_\nu D_\nu v(y)|^2\, dy \geq c \int |v(y)|^2\, dy\,, \quad v \in C_0^\infty(R^n)\,. \tag{2.24}$$

Here A_ν and B_ν are vectors in a (complex) Hilbert space $\mathcal{H}$ and since there is no ambiguity we use the symbols $(A\,, B)$ for the scalar product and $|A|$ for the norm in $\mathcal{H}$.

By using (2.5) we get

$$\begin{aligned}\int |\sum_1^n A_\nu y_\nu v(y) + \sum_1^n B_\nu D_\nu v(y)|^2\, dy = \qquad\qquad\qquad\qquad (2.25)\\ = \mathrm{Re}\,(-i\sum_1^n (A_\nu\,, B_\nu))\int |v(y)|^2\, dy + \mathrm{Re}\int \overline{v(y)}\,\{\sum_{\nu,\mu} \mathrm{Re}\,(A_\nu\,, A_\mu) y_\nu y_\mu v(y) + \\ + 2\sum_{\nu,\mu} \mathrm{Re}\,(A_\mu\,, B_\nu) y_\mu D_\nu v(y) + \sum_{\nu,\mu} \mathrm{Re}\,(B_\mu\,, B_\nu) D_\mu D_\nu v(y)\}\, dy\,.\end{aligned}$$

We now introduce the Gramian of $h_1, \ldots, h_m \in \mathcal{H}$ as the matrix

$$G(h_1, \ldots, h_m) = ((h_j\,, h_k))_{j,k=1}^m\,.$$

Notice that $\mathrm{Re}\, G(h_1, \ldots, h_m)$ is a real positive semi-definite matrix since $\mathrm{Re}\,(g\,, h)$ is an Euclidean scalar product in the real subspace of $\mathcal{H}$ spanned by $\{h_1, \ldots, h_m\}$ over R. Let $H = \mathrm{Re}\, G(A_1, \ldots, A_n, B_1, \ldots, B_n)$, then by (2.25) the inequality (2.24) is equivalent with

$$\begin{aligned}\mathrm{Re}\,(-i\sum_1^n (A_\nu\,, B_\nu))\int |v(y)|^2\, dy + \mathrm{Re}\int \overline{v(y)}\langle H\binom{y}{D}, \binom{y}{D}\rangle v(y)\, dy \geq \qquad (2.26)\\ \geq c\int |v(y)|^2\, dy\,, \quad v \in C_0^\infty(R^n)\,.\end{aligned}$$

Theorem 2.4 then gives us

PROPOSITION 2.11. *The inequality (2.24) is valid if and only if*

$$c \leq \operatorname{Re}\,(-\,i \sum_1^n (A_\nu\,, B_\nu)) + \widetilde{\operatorname{Tr}}\,\operatorname{Re}\,G(A_1\,, \ldots\,, A_n\,, B_1\,, \ldots\,, B_n)\,. \qquad (2.27)$$

In particular, the best constant in (2.24) is a continuous function of the vectors A_ν and B_ν.

Remark 2.12. Hörmander [5] has given necessary and sufficient conditions for (2.24) to hold with a positive constant c. We have however not managed to derive an easy criterion for this directly from (2.27).

Remark 2.13. The validity of the inequality (2.1) when $k = 2$ and conditions (i) and (iii) of Theorem 2.4 are fulfilled (with h positive definite) can also be proved in the following more general way. For the sake of simplicity we assume that the coefficients are real. Let X_i $(i = 1\,, \ldots\,, 2n)$ be symmetric linear operators defined in a dense set F of a complex Hilbert space $\mathcal{H}$ and suppose that $X_i(F)$ is contained in F for all X_i and that the following relations hold (cf. (2.3)),

$$X_j X_k - X_k X_j = [X_j\,, X_k] = -\sqrt{-1}J_{jk}I\,. \qquad (2.28)$$

Here $J = (J_{jk})$ denotes the matrix for i, the multiplication by the imaginary unit in $R^n \oplus R^n$, and I is the identity transformation in $\mathcal{H}$. We shall prove the inequality

$$\operatorname{Re}\,(p(X)v\,, v) \geq 0\,,\quad v \in F\,, \qquad (2.29)$$

where $X = (X_1\,, \ldots\,, X_{2n})$ and $p(u) = \langle Hu\,, u\rangle + \langle f\,, u\rangle + a_0^0$ is defined as in Theorem 2.4. It follows from (2.28) that the left hand side of (2.29) is well defined.

By the computations after Lemma 2.8 we can find a symplectic matrix χ such that

$$\langle H\chi^{-1}u\,, \chi^{-1}u\rangle = \sum_{i=1}^n \lambda_i(u_i^2 + u_{n+i}^2)\,;\ \sum_{i=1}^n \lambda_i = \widetilde{\operatorname{Tr}}\,H\,.$$

We now introduce new operators $\bar{X}_i$ by the equations

$$\bar{X}_j = \sum_{v=1}^{2n} \chi_{jv}X_v + g_j I\,,$$

where $g = 1/2\chi H^{-1}f$. It is then easy to verify that the operators $\bar{X}_i$ will satisfy (2.28). Since we have

$$p(X) = \sum_{i=1}^n \lambda_i(\bar{X}_i^2 + \bar{X}_{n+i}^2) - \tfrac{1}{4}\langle H^{-1}f, f\rangle I + a_0^0 I$$

the inequality (2.29) will follow from the uncertainty relation.

3. Localization of lower bounds for pseudo-differential operators

We shall now apply the results of section 2 to a localization of the inequality (0.3). Let φ be a twice continuously differentiable function defined in an open set in R^M. We then define the Hessian of φ as the matrix

$$H_\varphi(x) = ((iD_j)(iD_k)\varphi(x))_{j,k=1}^M . \tag{3.1}$$

THEOREM 3.1. *Assume that $a_m \geq 0$. Then the inequality (0.3) with $\mu = (m-1)/2$ holds for every $\varepsilon > 0$ and every s with suitable C when K is any compact set in Ω if and only if*

$$\operatorname{Re} a_{m-1} + 1/2 \,\widetilde{\operatorname{Tr}}\, H_{a_m} \geq 0$$

at the zeros of a_m in $\Omega \times S^{n-1}$.

Remark 3.2. The condition $a_m \geq 0$ is a consequence of (0.3) if we assume that A is self-adjoint, which is no restriction. If we do not introduce any normalization our conditions are:

$$\operatorname{Re} a_m \geq 0 \tag{3.2}$$

and

$$\operatorname{Re} a_{m-1} + (1/2) \operatorname{Im} \sum_{j=1}^n a_{m(j)}^{(j)} + (1/2)\, \widetilde{\operatorname{Tr}}\, H_{\operatorname{Re} a_m} \geq 0 \tag{3.3}$$

at the zeros of $\operatorname{Re} a_m$ in $\Omega \times S^{n-1}$.

Remark 3.3. From (2.9) and the identity

$$H_{a_m}(x, r\xi) = r^{m-1}\chi^t H_{a_m}(x, \xi)\chi , \quad r > 0 ,$$

where $\chi = r^{1/2}E \oplus r^{-1/2} E$ and E is the unit matrix of order n, we conclude that $\widetilde{\operatorname{Tr}}\, H_{a_m}$ in Theorem 3.1 is homogeneous of degree $(m-1)$ in ξ.

In the proof of Theorem 3.1 we shall reduce ourselves to the case $m = 1$. Let R^ϱ denote properly supported pseudo-differential operators in Ω with symbol $(1 + |\xi|^2)^{\varrho/2}$ and set $A_\varrho = R^{-\varrho}AR^{-\varrho}$. Recall that a pseudo-differential operator A in Ω is called properly supported if both projections $\operatorname{supp} K_A \to \Omega$ are proper, where K_A is the distribution kernel of A. See also [6], p. 148. Then A satisfies (0.3) with $\mu = (m-1)/2$ for every $\varepsilon > 0$ and every s when C is suitably chosen if and only if the same holds for A_ϱ with $\mu = (m - 2\varrho - 1)/2$. Since $a_m(x, \xi) = 0$ implies $\operatorname{grad}_{(x,\xi)} a_m(x, \xi) = 0$ we also find that the conditions in Theorem 3.1 are left invariant if we replace A by A_ϱ. Henceforth we shall therefore assume that $m = 1$.

Proof that the condition in Theorem 3.1 is necessary

Assume that $a_1(x^0, \eta^0) = 0$ with (x^0, η^0) in $\Omega \times S^{n-1}$. Choose any $\varepsilon > 0$ and let K be a compact set in Ω containing x^0 as an interior point. We want to show that (0.3) implies

$$\operatorname{Re} a_0(x^0, \eta^0) + 1/2\, \widetilde{\operatorname{Tr}}\, H_{a_1}(x^0, \eta^0) \geq 0 . \tag{3.4}$$

In order to get (3.4) we shall construct a class of functions with support near x^0 and with Fourier transform concentrated close to the half-ray generated by η^0 in R^n. For the sake of simplicity we assume that $x^0 = 0$. Let $v \in C_0^\infty(R^n)$ and set

$$u_\lambda(x) = \lambda^{n/2} v(\lambda x) e^{i<x, \lambda^2\eta^0>} .$$

Then $|u_\lambda|_{(0)} = |v|_{(0)}$ and $u_\lambda \in C_0^\infty(K)$ for large λ. A simple calculation yields

$$(Au_\lambda, u_\lambda) = (2\pi)^{-n} \int\int e^{i<x,\xi>} a(x/\lambda, \lambda^2\eta^0 + \lambda\xi) \hat{v}(\xi) \overline{v(x)} dx d\xi . \tag{3.5}$$

Since the first-order derivatives of a_1 at $(0, \eta^0)$ vanish and since

$$(|\lambda^2\eta^0 + \lambda\xi| + 1)^{-1} = O(1)\lambda^{-1}(1 + |\xi|)$$

a Taylor expansion of a about $(0, \eta^0)$ will give

$$a(x/\lambda, \lambda^2\eta^0 + \lambda\xi) = \sum_{|\alpha+\beta|=2} a_{1\beta}^{\alpha}(0, \eta^0) x^\beta \xi^\alpha / \alpha!\beta! + a_0(0, \eta^0) + O(1)\lambda^{-1}(1 + |\xi|)^4 . \tag{3.6}$$

Then applying the Fourier inversion formula and using (0.3) with $s = -1$ we get with some constant C that does not depend on λ

$$\operatorname{Re} \int \overline{v(x)} \sum_{|\alpha+\beta|=2} (1/\alpha!\beta!) a_{1\beta}^{\alpha}(0, \eta^0) x^\beta D^\alpha v(x) dx +$$
$$+ (\varepsilon + \operatorname{Re} a_0(0, \eta^0)) \int |v(x)|^2 dx + \lambda^{-1}O(1) \geq C|u_\lambda|_{(-1)}^2 .$$

Now $|u_\lambda|_{(-1)}^2 \to 0$ when $\lambda \to \infty$. Hence by first letting λ tend to infinity and then letting ε tend to zero we arrive at the inequality

$$\operatorname{Re} \int \overline{v(x)} \sum_{|\alpha+\beta|=2} (1/\alpha!\beta!) a_{1\beta}^{\alpha}(0, \eta^0) x^\beta D^\alpha v(x)\, dx + \operatorname{Re} a_0(0, \eta^0) \int |v(x)|^2 dx \geq 0 \tag{3.7}$$

when v belongs to $C_0^\infty(R^n)$. The inequality (3.4) now follows by Theorem 2.4.

Proof that the condition in Theorem 3.1. is sufficient

Following Hörmander [5] we shall make a localization of the estimates by means of partitions of unity in the variables x and ξ. There exist sequences $(\varphi_k)_{k=0}^\infty$ and $(\psi_k)_{k=0}^\infty$ of non-negative functions belonging to $C_0^\infty(R^n)$ such that the following conditions are fulfilled (See [5], p. 141—142):

$$\psi_k(\xi) = \varphi_k(\xi \,.\, |\xi|^{-1/2}) , \text{ and } \sum_0^\infty \varphi_k^2(x) = \sum_0^\infty \psi_k^2(\xi) = 1 \tag{3.8}$$

(and at most 2^n supports overlap)

$$|x - y| \leq C \quad \text{if} \quad x, y \in \operatorname{supp}(\varphi_k) \tag{3.9}$$

$$\sum_0^\infty |D^\alpha \varphi_k(x)|^2 \leq C_\alpha \tag{3.10}$$

$$|\xi - \eta| \leq C|\xi|^{1/2} \quad \text{if} \quad \xi, \eta \in \operatorname{supp}(\psi_k) \quad \text{and} \quad k \neq 0 \tag{3.11}$$

and finally

$$\sum (\psi_j(\xi) - \psi_j(\eta))^2 \leq C|\xi - \eta|^2 |\xi|^{-1} \quad \text{if} \quad |\xi - \eta| \leq |\xi|/2 . \tag{3.12}$$

When proving this part of Theorem 3.1 we make the convention that constants depending only on A and the chosen partitions of unity are to be denoted by the same symbol C. Of course, we may restrict ourselves to the case $s = -1/4$ in (0.3), so we have to show that if $\varepsilon > 0$ is given, and K is a compact set in Ω then there exists a constant C such that

$$\operatorname{Re}(Au, u) + \varepsilon|u|^2_{(0)} \geq C|u|^2_{(-1/4)}, \quad u \in C_0^\infty(K) . \tag{3.13}$$

Furthermore, since the contribution to $\operatorname{Re}(Au, u)$ that comes from $\operatorname{Im} a_0$ and terms of negative degree is continuous in $H_{(-1/2)}$, we may also assume that $A = a(x, D)$ with $a = a_1 + a_0$ and a_0 real valued.

Finally we notice that the validity of (3.13) is not affected if the symbol of A is changed outside a neighbourhood of K. Hence replacing a by $(1 - \psi)(1 + |\xi^2|)^{1/2} + \psi a(x, \xi)$, where ψ belongs to $C_0^\infty(\Omega)$ and $0 \leq \psi(x) \leq 1$ with equality to the right in a neighbourhood of K, we may assume that A is defined in $\Omega = R^n$ and that the symbol of A equals $r(\xi) = (1 + |\xi|^2)^{1/2}$ outside a compact set in R^n. We shall prove (3.13) with K replaced by R^n.

To begin with we shall split up u by its spectrum and make the corresponding approximations of the operator. Let $0 \neq \xi^j$ belong to the support of ψ_j and let δ be a number with $0 < \delta \leq 1$. We introduce the differential operators

$$A_j^\delta v(x) = \sum_{|\alpha| \leq 2} (1/\alpha!) a^\alpha(x, \xi^j/\delta)(D - \xi^j/\delta)^\alpha v(x), \quad v \in C^\infty(R^n) .$$

We shall prove the inequality

$$|(Au, u) - \sum_j (A_j^\delta \psi_j(\delta D)u, \psi_j(\delta D)u)| \leq C\delta|u|^2_{(0)} + C_\delta|u|^2_{(-1/4)} \text{ for } u \in C_0^\infty(R^n) . \tag{3.14}$$

Of course (3.14) is valid if it holds for $A = (1 + |D|^2)^{1/2}$ and when A is replaced by $B = b(x, D)$ where B satisfies (1.3) with $m = 1$ and vanishes when x is outside a compact subset of R^n. When $A = (1 + |D|^2)^{1/2}$ we get by using Parseval's formula and (3.8)

$$(Au, u) - \sum_j (A_j^\delta \psi_j(\delta D)u, \psi_j(\delta D)u) = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 q(\xi) d\xi$$

where with $r(\xi) = (1 + |\xi|^2)^{1/2}$

$$q(\xi) = \sum_j \psi_j^2(\delta\xi)(r(\xi) - \sum_{|\alpha| \leq 2} (1/\alpha!) r^{(\alpha)}(\xi^j/\delta)(\xi - \xi^j/\delta)^\alpha) .$$

From (3.11) and Taylor's formula we deduce the existence of a constant C such that

$$|q(\xi)| \leq C\delta^{-3/2}|\xi|^{-1/2} \text{ if } |\xi| \geq C/\delta$$

and using this inequality we easily prove (3.14) in this case.

We shall now prove (3.14) for B. If $\hat{b}(\eta, \xi)$ denotes the Fourier transform of b with respect to the variables x then

$$(Bu, u) - \sum_j (B_j^\delta \psi_j(\delta D)u, \psi_j(\delta D)u) = (2\pi)^{-2n} \iint H_\delta(\xi, \eta)\hat{u}(\xi)\overline{\hat{u}(\eta)}d\xi d\eta$$

where

$$H_\delta(\xi, \eta) = 1/2 \sum_j (\psi_j(\delta\xi) - \psi_j(\delta\eta))^2 \hat{b}(\eta - \xi, \xi) + \tag{3.15}$$

$$+ \sum_j \psi_j(\delta\xi)\psi_j(\delta\eta)\{\hat{b}(\eta - \xi, \xi) - \sum_{|\alpha| \leq 2} (1/\alpha!)\hat{b}^{(\alpha)}(\eta - \xi, \xi^j/\delta)(\xi - \xi^j/\delta)^\alpha\} .$$

Our approximation will be good for small δ since we have

LEMMA 3.4. *For large N there are constants C_N independent of δ and functions $\varphi_{\delta, N} \in C_0^\infty(R^n)$ such that*

$$|H_\delta(\xi, \eta)| \leq C_N\delta(1 + |\xi - \eta|)^{-N} + \varphi_{\delta, N}(\xi)\varphi_{\delta, N}(\eta) . \tag{3.16}$$

Proof. By carrying out partial integrations one obtains

$$|D_\xi^\alpha \hat{b}(\eta, \xi)| \leq C_{N, \alpha}(1 + |\eta|)^{-N}(1 + |\xi|)^{1-|\alpha|} \tag{3.17}$$

and Taylor's formula then gives us an estimate for the second term in the expression for H_δ when $\delta(|\xi| + |\eta|)$ is large by

$$C_N(1 + |\eta - \xi|)^{-2N} \sum_{j \neq 0} \psi_j(\delta\xi)\psi_j(\delta\eta)|\xi - \xi^j/\delta|^3 q_{j, \delta}(\xi) \tag{3.18}$$

where

$$q_{j, \delta}(\xi) = \sup_{0 < t < 1} (1 + |\xi + t(\xi^j/\delta - \xi)|)^{-2} .$$

From (3.11) we obtain

$$|\xi - \eta| \leq C\delta^{-1/2}|\eta|^{1/2} \leq |\eta|/2 \text{ if } j\psi_j(\delta\xi)\psi_j(\delta\eta) \neq 0 \text{ and } |\eta| > 4C^2\delta^{-1} ,$$

so if $j\psi_j(\delta\xi)\psi_j(\delta\eta) \neq 0$ and $|\xi| + |\eta| > 12C^2\delta^{-1}$ then we must have $|\xi| > 4C^2\delta^{-1}$. By replacing ξ by ξ_j/δ and η by ξ in the inequality above we get

$$|\xi - \xi^j/\delta| \leq C\delta^{-1/2}|\xi|^{1/2} \leq |\xi|/2 \text{ if } |\eta| + |\xi| > 12C^2\delta^{-1} \text{ and } j\psi_j(\delta\xi)\psi_j(\delta\eta) \neq 0 .$$

Then the sum in (3.18) can be estimated by

$$\{C_N\delta^{-3/2}(1 + |\eta - \xi|)^{-N}|\xi|^{-1/2}\}(1 + |\eta - \xi|)^{-N}$$

if $\delta(|\eta| + |\xi|)$ is large enough, and since the first factor tends to zero when $(|\xi| + |\eta|) \to \infty$ we realize that the second sum in (3.15) can be estimated by the right hand side of (3.16).

It remains to estimate the first sum in (3.15). Now (3.12) together with (3.17) gives us

$$\sum (\psi_j(\delta\xi) - \psi_j(\delta\eta))^2 |\hat{b}(\eta - \xi, \xi)| \leq C_N \delta(1 + |\eta - \xi|)^{-N}$$

if $|\xi - \eta| < |\xi|/2$ (and $|\xi| + |\eta|$ is large). Finally if $|\xi - \eta| \geq |\xi|/2$ and $N \geq 2$ then

$$|\hat{b}(\eta - \xi, \xi)| \leq C_N(1 + |\xi|)(1 + |\eta - \xi|)^{-2N} \leq \delta(1 + |\eta - \xi|)^{-N}$$

if $|\xi| + |\eta|$ is large enough. This completes the proof of the lemma.

The validity of (3.14) for B now follows from Lemma 3.4 and the inequality

$$\iint (1 + |\xi - \eta|)^{-(n+1)} |\hat{u}(\xi)||\hat{u}(\eta)| d\xi d\eta \leq C \int |u(x)|^2 dx\,, \quad u \in C_0^\infty(R^n)\,.$$

We shall also carry out a partition of u in the variables x. If $\varphi = \varphi(x)$ is a real valued function in $C_0^\infty(R^n)$ then

$$\operatorname{Re}\left(\int \varphi(A_j^\delta v)\varphi\bar{v}dx - \int A_j^\delta(\varphi v)\varphi\bar{v}dx\right) =$$

$$= \operatorname{Re}\int \varphi\bar{v} \sum_{|\alpha|=2} (1/\alpha!) a^\alpha(x, \xi^j/\delta)\{\varphi(D - \xi^j/\delta)^\alpha v - (D - \xi^j/\delta)^\alpha(\varphi v)\}dx\,.$$

We shall replace φ by $\varphi_k(x|\xi^j|^{1/2})$ and v by $\psi_j(\delta D)u(x)$. In doing so we note that when the differentiations on the right hand side are carried out we obtain in addition to the term

$$-\varphi|v|^2 \sum_{|\alpha|=2} a^\alpha(x, \xi^j/\delta) D^\alpha\varphi/\alpha!$$

only terms containing a factor $\varphi D\varphi$. Since $\sum \varphi_k D\varphi_k = 0$ these will drop out when we sum over k which gives

$$\operatorname{Re}\{(A_j^\delta\psi_j(\delta D)u, \psi_j(\delta D)u) - \sum_k (A_j^\delta u_\delta^{jk}, u_\delta^{jk})\} = \tag{3.19}$$

$$= -\sum_k \sum_{|\alpha|=2} \int (1/\alpha!) a^\alpha(x, \xi^j/\delta)|\xi^j|(\varphi_k D^\alpha\varphi_k)(x|\xi^j|^{1/2})|\psi_j(\delta D)u|^2 dx\,.$$

Here we have introduced

$$u_\delta^{jk}(x) = \varphi_k(x|\xi^j|^{1/2})\psi_j(\delta D)u(x)\,.$$

We shall use the symbol $O(1)$ to denote functions that can be estimated uni formly when u and δ vary. Summing over j in (3.19) and using (3.14) we get

$$\operatorname{Re}(Au, u) = \sum_{j,k} \operatorname{Re}(A_j^\delta u_\delta^{jk}, u_\delta^{jk}) - \tag{3.20}$$
$$- \sum_{j,k} \sum_{|\alpha|=2} \int (1/\alpha!) a^\alpha(x, \xi^j/\delta) |\xi^j| (\varphi_k D^\alpha \varphi_k)(x|\xi^j|^{1/2}) |\psi_j(\delta D) u|^2 dx +$$
$$+ \delta O(1) |u|_{(0)}^2 + C_\delta O(1) |u|_{(-1/4)}^2, \quad u \in C_0^\infty(R^n).$$

The fact that a satisfies (1.3) with $m = 1$ uniformly when $x \in R^n$ and that

$$\sum_j \int |\psi_j(\delta D) u|^2 dx = \int |u(x)|^2 dx$$

implies that the second sum in (3.20) can be written as $O(1)\delta |u|_{(0)}^2$. Hence

$$\operatorname{Re}(Au, u) = \sum_{j,k} \operatorname{Re}(A_j^\delta u_\delta^{jk}, u_\delta^{jk}) + \delta O(1) |u|_{(0)}^2 + C_\delta O(1) |u|_{(-1/4)}^2. \tag{3.21}$$

We shall now make a change of variables and introduce the functions v_δ^{jk} by the equations

$$u_\delta^{jk}(x) = e^{i\langle x, \xi^j/\delta\rangle} v_\delta^{jk}((x - x^{jk})|\xi^j|^{1/2})$$

where

$$x^{jk} = |\xi^j|^{-1/2} x^k$$

and

$$x^k = \xi^k |\xi^k|^{-1/2} \in \operatorname{supp}(\varphi_k).$$

The last relation follows from the definition of ξ^k and since

$$|v_\delta^{jk}(y)| = \varphi_k(x^k + y) |\psi_j(\delta D) u((x^k + y)|\xi^j|^{-1/2})|$$

we also have by (3.9)

$$\operatorname{supp}(v_\delta^{jk}) \subset \{y \in R^n ; |y| \le C\}.$$

This fact together with a change of variables in the integrals $(A_j^\delta u_\delta^{jk}, u_\delta^{jk})$ and a Taylor expansion of a at $(x^{jk}, \xi^j/\delta)$ with respect to the variable x gives us

$$|\xi^j|^{n/2} (A_j^\delta u_\delta^{jk}, u_\delta^{jk}) = \tag{3.22}$$
$$= \sum_{|\alpha| \le 2} \int (1/\alpha!) a^\alpha(x^{jk} + y|\xi^j|^{-1/2}, \xi^j/\delta) |\xi^j|^{|\alpha|/2} (D^\alpha v_\delta^{jk}(y)) \overline{v_\delta^{jk}(y)} dy =$$
$$= \sum_{|\alpha+\beta| \le 2} (1/\alpha!\beta!) a_\beta^\alpha(x^{jk}, \xi^j/\delta) |\xi^j|^{(|\alpha|-|\beta|)/2} \int \overline{v_\delta^{jk}(y)} y^\beta D^\alpha v_\delta^{jk}(y) dy +$$
$$+ O(1)\delta^{-1} |\xi^j|^{-1/2} \sum_{|\alpha| \le 2} \int |D^\alpha v_\delta^{jk}(y)|^2 dy.$$

In order to estimate the remainder in (3.22) we introduce the functions v_j^δ by the equations,

$$\psi^j(\delta D) u(x) = e^{i\langle x, \xi^j/\delta\rangle} v_\delta^j(x|\xi^j|^{1/2}).$$

By (3.11) there is a constant C such that

$$|\delta\xi - \xi^j| \leq C|\xi^j|^{1/2} \quad \text{if} \quad \xi \in \text{Spectrum}\,(\psi_j(\delta D)u)$$

and hence

$$|\xi| \leq C/\delta \quad \text{if} \quad \xi \in \text{Spectrum}\,(v_\delta^j)\,. \tag{3.23}$$

Now v_δ^{jk} differs from $\varphi_k v_\delta^j$ only by a translation and using (3.10) and (3.23) together with Parseval's formula we get since the v_δ^{jk} have their supports in a fixed compact set in R^n

$$\sum_k \int \sum_{|\alpha+\beta|\leq N} |y^\beta D^\alpha v_\delta^{jk}(y)|^2 dy \leq C_N \delta^{-2N} |\xi^j|^{n/2} \int |\psi_j(\delta D)u|^2 dx\,. \tag{3.24}$$

By first summing over all indices j with $|\xi^j| > \delta^{-12}$ we get

$$\sum_j \delta^{-5}|\xi^j|^{-1/2} \int |\psi_j(\delta D)u|^2 dx = O(1)\delta|u|_{(0)}^2 + O(1)C_\delta|u|_{(-1/4)}^2$$

and using this together with (3.21), (3.22) and (3.24) (with $N = 2$) we get

$$\begin{aligned}
&\text{Re}\,(Au\,,u) = \qquad\qquad (3.25)\\
&= \sum_{j,k} |\xi^j|^{-n/2}\,\text{Re} \sum_{|\alpha+\beta|\leq 2} (1/\alpha!\beta!) a_\beta^\alpha(x^{jk}\,,\xi^j/\delta)|\xi^j|^{(|\alpha|-|\beta|)/2} \int \overline{v_\delta^{jk}(y)}\, y^\beta D^\alpha v_\delta^{jk}(y) dy\, +\\
&+ \delta O(1)|u|_{(0)}^2 + C_\delta O(1)|u|_{(-1/4)}^2\,,\quad u \in C_0^\infty(R^n)\,.
\end{aligned}$$

We have now reduced our problem to an estimate of the individual terms in (3.25). In the rest of the proof δ will be kept fixed so that the remainder term $\delta O(1)|u|_{(0)}^2$ in (3.25) is greater than $-(\varepsilon/3)|u|_{(0)}^2$. In order to complete the proof it is enough to find a sequence ϱ_j tending to zero when j tends to infinity such that

$$\begin{aligned}
&\text{Re} \sum_{|\alpha+\beta|\leq 2} (1/\alpha!\beta!) a_\beta^\alpha(x^{jk}\,,\xi^j/\delta)|\xi^j|^{(|\alpha|-|\beta|)/2} \int \overline{v_\delta^{jk}(y)}\, y^\beta D^\alpha v_\delta^{jk}(y) dy\, + \qquad (3.26)\\
&+ (\varepsilon/2) \int |v_\delta^{jk}(y)|^2 dy \geq -\,\varrho_j \int \sum_{|\alpha|\leq 4} |D^\alpha v_\delta^{jk}(y)|^2 dy\,.
\end{aligned}$$

For multiplying both sides of (3.26) by $|\xi^j|^{-n/2}$ after a summation over k and using (3.24) and the equality

$$|\xi^j|^{-n/2} \sum_k \int |v_\delta^{jk}(y)|^2 dy = \int |\psi_j(\delta D)u|^2 dy$$

we realize that (3.13) follows from (3.25) if the resulting terms are summed over j.

Remark 3.5. Since

$$a_{0\beta}^\alpha(x^{jk}\,,\xi^j/\delta)|\xi^j|^{(|\alpha|-|\beta|)/2} \int \overline{v_\delta^{jk}(y)}\, y^\beta D^\alpha v_\delta^{jk}(y) dy$$

can be estimated by a constant times

$$\delta^{|\alpha|}|\xi^j|^{-|\alpha+\beta|/2}\int\sum_{\gamma\leq 2}|D^\gamma v_\delta^{jk}(y)|^2dy$$

when $|\alpha+\beta|\leq 2$, and since $|\xi^j|$ tends to infinity when j tends to infinity we may replace a_β^α in (3.26) by $a_{1\beta}^\alpha$ if we add the term

$$a_0(x^{jk}, \xi^j/\delta)\int |v_\delta^{jk}(y)|^2dy$$

to the left hand side of the formula.

All that we need is furnished by the following lemma.

LEMMA 3.6. *Let K_0 be a compact subset of $R^n\setminus\{0\}$, let γ be a real valued continuous function on $R^n\times K_0$, such that γ is constant when x is outside some compact set in R^n, let C be a fixed positive number and suppose that*

$$(x, \eta)\in R^n\times K_0,\quad a_1(x,\eta)=0$$

implies

$$1/2\,\widetilde{\mathrm{Tr}}\,H_{a_1}(x,\eta)+\gamma(x,\eta)>0\,. \tag{3.27}$$

Then there is a function $\varrho(\lambda)\to 0$ when $\lambda\to\infty$ such that

$$\mathrm{Re}\sum_{|\alpha+\beta|\leq 2}(1/\alpha!\beta!)a_{1\beta}^\alpha(x,\eta)\lambda^{2-|\alpha+\beta|}\int\overline{v(y)}y^\beta D^\alpha v(y)dy+ \tag{3.28}$$

$$+\gamma(x,\eta)\int|v(y)|^2dy+\varrho(\lambda)\int\sum_{|\alpha|\leq 4}|D^\alpha v(y)|^2dy\geq 0$$

for every $(x,\eta)\in R^n\times K_0$ and every v in $C_0^\infty(R^n)$ with support in $\{y\,;\,|y|\leq C\}$.

In order to see that Lemma 3.6 implies the existence of a sequence ϱ_j tending to zero in (3.26) we change (3.26) in accordance with Remark 3.5 and set $K_0=\delta^{-1}S^{n-1}$, $\lambda=\lambda_j=|\xi^j|^{1/2}$, $\varrho_j=\varrho(\lambda_j)$, $(x,\eta)=(x^{jk},\xi^j\delta^{-1}|\xi^j|^{-1})$, $\gamma=(\varepsilon/2)+a_0$ and $v=v_\delta^{jk}$.

Proof of Lemma 3.6. Our proof will be indirect. We shall assume that it is not possible to choose $\varrho(\lambda)$ in such a way that (3.28) holds and $\varrho(\lambda)\to 0$ when $\lambda\to\infty$ and see that this leads to a contradiction. Our assumption means precisely that there exist a $\varrho>0$, sequences $\lambda_j\to\infty$ when $j\to\infty$ and $(x_j,\eta_j)\in R^n\times K_0$, and functions v_j belonging to $C_0^\infty(R^n)$ with supports in $\{y\,;\,|y|\leq C\}$ such that

$$\int|v_j(y)|^2dy=1$$

and

$$\mathrm{Re}\sum_{|\alpha+\beta|\leq 2}(1/\alpha!\beta!)a_{1\beta}^\alpha(x_j, n_j)\lambda_j^{2-|\alpha+\beta|}\int\overline{v_j(y)}y^\beta D^\alpha v_j(y)dy+ \tag{3.29}$$

$$+\gamma(x_j,\eta_j)\int|v_j(y)|^2dy+\varrho\int\sum_{|\alpha|\leq 4}|D^\alpha v_j(y)|^2dy\leq 0\,.$$

We shall use some convexity properties for the derivatives of a positive function. If g is a non-negative function in $C_0^\infty(R^n)$ then there is a constant C_g such that

$$|\text{grad}_y\, g(y)|^2 \leq C_g g(y)\,.$$

It follows that

$$|\text{grad}_{x,\eta}\, a_1(x\,,\eta)|^2 \leq C_{K_0} a_1(x\,,\eta)\,,\ (x\,,\eta) \in R^n \times K_0\,, \tag{3.30}$$

since the inequality is trivially fulfilled when $a_1(x\,,\eta) = |\eta|$.

Hence we get for $|\alpha| = |\beta| = 1$

$$|a_1^\alpha(x_j\,,\eta_j)\lambda_j \int \overline{v_j(y)} D^\alpha v_j(y) dy| \leq \lambda_j |a_1^\alpha(x_j\,,\eta_j)|\,|v_j|_{(0)}|v_j|_{(1)} \leq \tag{3.31a}$$

$$\leq (1/3n) a_1(x_j\,,\eta_j)\lambda_j^2 + C'|v_j|_{(1)}^2$$

$$|a_{1\beta}(x_j\,,\eta_j)\lambda_j \int y^\beta |v_j(y)|^2 dy| \leq C\lambda_j |a_{1\beta}(x_j\,,\eta_j)| \leq (1/3n) a_1(x_j\,,\eta_j)\lambda_j^2 + C' \tag{3.31b}$$

where C' is a constant independent of j.

Using (3.29) and (3.31) together with the facts that γ is a bounded function and that for some constant C''

$$|\sum_{|\alpha+\beta|=2} (1/\alpha!\beta!) a_{1\beta}^\alpha(x_j\,,\eta_j)\lambda_j^{2-|\alpha+\beta|} \int \overline{v_j(y)} y^\beta D^\alpha v_j(y) dy| \leq C''|v_j|_{(2)}^2 \tag{3.32}$$

we obtain the inequality

$$|v_j|_{(4)}^2 \leq C|v_j|_{(2)}^2/\varrho \tag{3.33}$$

where C is a new constant.

From the inequality

$$|v|_{(2)}^2 \leq |v|_{(0)}|v|_{(4)}$$

we conclude that the sequence (v_j) is bounded in the H_4-topology. By passing to a subsequence if necessary we may therefore assume that (v_j) converges to a function v_0 in the H_3-topology.

Using (3.31), (3.32) and the boundedness of the sequence $|v_j|_{(2)}$ we realize that the sequence $\lambda_j^2 a_1(x_j\,,\eta_j)$ must be bounded (and by (3.30) the same statement is valid for the sequence $\lambda_j \,\text{grad}\, a_1(x_j\,,\eta_j)$). Hence $a_1(x_j\,,\eta_j)$ tends to zero when j tends to infinity and since $a_1(x_j\,,\eta_j)$ is greater than some positive constant when x is outside some compact set in R^n we conclude that it is no restriction to assume that there is a $(x_0\,,\eta_0) \in R^n \times K_0$ such that

$$(x_j\,,\eta_j) \to (x_0\,,\eta_0) \text{ when } j \to \infty\,,\ a_1(x_0\,,\eta_0) = 0\,.$$

By choosing ϱ smaller in (3.29) we may replace v_j by v_0 (for large j). Then approximating v_0 by a function in $C_0^\infty(R^n)$ and using the continuity of γ we get the existence of a function v in $C_0^\infty(R^n)$ and a small positive number σ such that

$$\int |v(y)|^2 dy = 1$$

and

$$\operatorname{Re} \sum_{|\alpha+\beta| \le 2} (1/\alpha!\beta!) a_{1\beta}^{\alpha}(x_j, \eta_j) \lambda_j^{2-|\alpha+\beta|} \int \overline{v(y)} y^\beta D^\alpha v(y) dy + \tag{3.34}$$
$$+ (\sigma/2) \sum_{j=1}^{n} \int \overline{v(y)} (y_j^2 + D_j^2) v(y) dy + (\gamma(x_0, \eta_0) + \varrho/2) \int |v(y)|^2 dy < 0 .$$

To simplify notations we introduce

$$f_j = \operatorname{grad}_{(x,\eta)} a_1(x_j, \eta_j), \quad H_j = H_{a_1}(x_j, \eta_j), \quad H_j^\sigma = H_j + \sigma E$$

where E denotes the unit matrix. Since H_j^σ tends to the positive definite matrix H_0^σ we may assume that H_j^σ is positive definite for all j. Then an application of Theorem 2.4 to (3.34) gives us

$$\lambda_j^2 \{a_1(x_j, \eta_j) - \langle (H_j^\sigma)^{-1} f_j, f_j \rangle /2\} + \gamma(x_0, \eta_0) + \varrho/2 + 1/2 \widetilde{\operatorname{Tr}} H_j^\sigma \le 0 . \tag{3.35}$$

The proof will be finished if we can prove that (3.35) is in contradiction with (3.27).

A Taylor expansion gives us

$$a_1((x_j, \eta_j) + h) = a_1(x_j, \eta_j) + \langle f_j, h \rangle + \langle H_j h, h \rangle /2 + O(1)|h|^3$$

when $h \in R^{2n}$ and $|h| \le C$. Since $f_j = O(1)\lambda_j^{-1}$ and $H_j < H_j^\sigma$ we get

$$\begin{aligned} 0 &\le a_1((x_j, \eta_j) - (H_j^\sigma)^{-1} f_j) = a_1(x_j, \eta_j) - \langle f_j, (H_j^\sigma)^{-1} f_j \rangle + \\ &+ \langle H_j (H_j^\sigma)^{-1} f_j, (H_j^\sigma)^{-1} f_j \rangle /2 + O(1) \lambda_j^{-3} \le \\ &\le a_1(x_j, \eta_j) - \langle f_j, (H_j^\sigma)^{-1} f_j \rangle /2 + O(1) \lambda_j^{-3} . \end{aligned}$$

Hence

$$0 \le \lambda_j^2 \{a_1(x_j, \eta_j) - \langle f_j, (H_j^\sigma)^{-1} f_j \rangle /2\} + \varrho/4$$

for large j. Then it follows from (3.35) that

$$\gamma(x_0, \eta_0) + \varrho/4 + 1/2 \widetilde{\operatorname{Tr}} H_j^\sigma \le 0$$

and by letting j tend to infinity and σ tend to zero we get using the continuity of $\widetilde{\operatorname{Tr}}$

$$\gamma(x_0, \eta_0) + \varrho/4 + 1/2 \widetilde{\operatorname{Tr}} H_{a_1}(x_0, \eta_0) \le 0$$

which is in contradiction with our assumptions. This completes the proof of the lemma.

4. Applications of the results to hypoellipticity

Let Ω be an open set in R^n. We shall consider the class of classical pseudo-differential operators A in Ω satisfying the following conditions:

$$a_m(x\,,\xi) \geq 0 \quad \text{when} \quad (x\,,\xi) \in \Omega \times S^{n-1} \tag{4.1}$$

$$a_{m-1}(x\,,\xi) = -\overline{a_{m-1}(x\,,\xi)} \quad \text{when} \quad a_m(x\,,\xi) = 0\,. \tag{4.2}$$

Here m denotes the order of A. On the set N of zeros for a_m in $\Omega \times S^{n-1}$ we can define the function

$$I(x\,,\xi) = |\sum_{j=1}^{n} a_{m(j)}^{(j)}(x\,,\xi) - 2ia_{m-1}(x\,,\xi)| + (1/2)\,\widetilde{\mathrm{Tr}}\, H_{a_m}(x\,,\xi)\,.$$

The following result, due to Radkevič [10], gives a sufficient condition for A to be hypoelliptic in Ω.

THEOREM 4.1. *If A is properly supported and satisfies (4.1) and (4.2) and if $I(x\,,\xi) > 0$ on N, then for every compact set K in Ω and every real number m' there exists a constant C such that the following estimate is valid*

$$|u|_{(m-1)}^2 + \sum_{s=1}^{n} \{|A^{(s)}u|_{(1/2)}^2 + |A_{(s)}u|_{(-1/2)}^2\} \leq C\{|Au|_{(0)}^2 + |u|_{(m')}^2\}\,, \quad u \in C_0^\infty(K)\,. \tag{4.3}$$

Here $A^{(s)}$ and $A_{(s)}$ denote properly supported pseudo-differential operators in Ω with symbols $a^{(s)}$ respectively $a_{(s)}$.

Remark 4.2. By using the results of Remark 2.10 we can easily get equivalent formulations of the condition $I(x\,,\xi) > 0$ on N.

Proof of Theorem 4.1. Let R^τ denote properly supported (self-adjoint) pseudo-differential operators in Ω with the symbols

$$\sigma_{R^\tau}(x\,,\xi) = (1 + |\xi|^2)^{\tau/2}\,.$$

We shall apply Theorem 3.1 to the pseudo-differential operator

$$T_\delta = R^{m-1}A - \delta\sum_{s=1}^{n}(R^{1/2}A^{(s)})^*(R^{1/2}A^{(s)}) - \delta\sum_{s=1}^{n}(R^{-1/2}A_{(s)})^*(R^{-1/2}A_{(s)}) +$$
$$+ (A^* - A)^*(A^* - A) + 4(A^*A - AA^*) - \delta R^{2m-2}$$

where δ is a small positive number.

We have to examine the symbol σ_{T_δ} of T_δ. We shall make use of the following formulas (see [6], p. 147 and p. 149.) where A and B denote properly supported pseudo-differential operators in Ω:

$$\sigma_{BA}(x\,,\xi) \sim \sum ((iD_\xi)^\alpha \sigma_B(x\,,\xi))D_x^\alpha \sigma_A(x\,,\xi)/\alpha!$$
$$\sigma_{B^*}(x\,,\xi) \sim \sum (iD_\xi)^\beta D_x^\beta \overline{\sigma_B(x\,,\xi)}/\beta!\,.$$

Then we see that the principal symbol of $A^*A - AA^*$ is homogeneous of degree $2m - 2$ in ξ and vanishes on N. The principal symbol of $(A^* - A)^*(A^* - A)$ is homogeneous of the same degree and equals

$$|2\,\mathrm{Im}\, a_{m-1} + \sum a_{m(j)}^{(j)}|^2$$

on N. The symbol of $(R^{1/2}A^{(s)})^*(R^{1/2}A^{(s)})$ can be written as

$$(a_m^{(s)})^2(1+|\xi|^2)^{1/2} + 2a_{m-1}^{(s)}a_m^{(s)}(1+|\xi|^2)^{1/2} + ib$$

where the leading part of b is homogeneous of degree $2m-2$ in ξ and real valued. Of course the symbol of

$$(R^{-1/2}A_{(s)})^*(R^{-1/2}A_{(s)})$$

can be written in a similar way with $a^{(s)}$ replaced by $a_{(s)}$ and $(1+|\xi|^2)^{1/2}$ replaced by $(1+|\xi|^2)^{-1/2}$.

Let K be a compact set in Ω. Then for some constant C_0

$$|\mathrm{grad}_{(x,\xi)}\, a_m(x,\xi)|^2 \leq C_0 a_m(x,\xi)\,,\ (x,\xi) \in K \times S^{n-1}\,,$$

(Cf. (3.30)), and it follows from the homogeneity of a_m, that if δ is chosen small enough then the principal symbol of T_δ is non-negative and has the same zeros as a_m when x belongs to a neighbourhood of K.

If the symbol of T_δ is written

$$\sigma_{T_\delta} = t_{2m-1} + t_{2m-2} + \ldots .$$

then by our assumptions and computations above

$$\begin{aligned}\mathrm{Re}\, t_{2m-2}(x,\xi) + 1/2\, \widetilde{\mathrm{Tr}}\, H_{t_{2m-1}}(x,\xi) = \qquad\qquad\qquad\qquad (4.4)\\ = |\textstyle\sum a_{m(j)}^{(j)}(x,\xi) - 2ia_{m-1}(x,\xi)|^2 + 1/2\, \widetilde{\mathrm{Tr}}\, H_{t_{2m-1}}(x,\xi) - \delta \ \text{ when } (x,\xi) \in N\,.\end{aligned}$$

Since $\widetilde{\mathrm{Tr}}\, H_{t_{2m-1}}(x,\xi)$ tends to $\widetilde{\mathrm{Tr}}\, H_{a_m}(x,\xi)$ when δ tends to zero (with uniform convergence on every compact subset of N) the expression above in (4.4) is non-negative for small δ on the zeros of t_{2m-1} in $\Omega_0 \times S^{n-1}$, where Ω_0 is some neighbourhood of K. Hence by choosing δ small and applying Theorem 3.1 with $\varepsilon = \delta/2$ we get

$$\mathrm{Re}\,(T_\delta u, u) + (\delta/2)|u|^2_{(m-1)} \geq C_\delta |u|^2_{(m')}\,,\quad u \in C_0^\infty(K)\,. \tag{4.5}$$

By expanding the left hand side of (4.5) and using the inequalities:

$$|Av - A^*v|^2 \leq 2(|A^*v|^2 + |Av|^2)\,,\quad v \in C_0^\infty(\Omega)$$

$$\mathrm{Re}\,(R^{m-1}Av, v) = \mathrm{Re}\,(Av, R^{m-1}v) \leq 4\delta^{-1}|Av|^2_{(0)} + \delta 4^{-1}|v|^2_{(m-1)} + \delta C'|v|^2_{(m')}\,,\quad v \in C_0^\infty(K)$$

we get with some constant C the following inequality

$$|u|^2_{(m-1)} + \sum_{s=1}^{n} |A^{(s)}u|^2_{(1/2)} + \sum_{s=1}^{n} |A_{(s)}u|^2_{-(1/2)} + |A^*u|^2_{(0)} \leq C(|Au|^2_{(0)} + |u|^2_{(m')})\,,\quad u \in C_0^\infty(K)\,. \tag{4.6}$$

This completes the proof of the theorem.

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