Estimates of mass distributions from their potentials and energies

Peter Sjögren

University of Göteborg, Sweden

1. Introduction

In classical potential theory it is well known that a mass distribution σ is uniquely determined by its potential U^σ and that its energy $\|\sigma\|$ can vanish only when $\sigma = 0$. In this paper we shall consider different ways of making these facts more precise. If σ is a signed mass distribution we shall estimate the mass $\sigma(B)$ on certain test sets B by means of the potential U^σ and the energy $\|\sigma\|$. The distribution σ will have its support in a compact set K in $\mathbf{R}^n$, such as a compact surface or a ball, and our estimates will involve the values of U^σ on this set only.

For positive measures μ in K it is trivial that

$$\mu(K) \leq C_1\|\mu\|$$

and that in $\mathbf{R}^n$, $n \geq 3$,

$$\mu(K) \leq C_2 \sup_K U^\mu .$$

Here we can let $C_1 = \sqrt{\operatorname{cap} K}$ and $C_2 = (\operatorname{diam} K)^{n-2}$. If σ is an arbitrary signed measure $\sigma(B)$ cannot be estimated in any similar way, not even if we admit only very regular test sets $B \subset K$, as is easily seen from examples. Therefore we shall impose a condition $d\sigma^+ \leq M\,dm$ on the positive (say) part σ^+ of σ. Here $M < \infty$ and m is a volume or area measure on K. It will also be assumed that the total mass of σ is 0, which seems to be the interesting case in many applications.

Kleiner [5, 6] has found estimates of σ in terms of $\|\sigma\|$. In [5] he assumes that σ lies on a simple plane curve Γ of class C^1 and satisfies $\sigma(\Gamma) = 0$ and $|\sigma| \leq \nu$, where ν is a positive measure on Γ with finite energy. Defining

$$[\sigma] = \sup |\sigma(B)| ,$$

where the sup is taken over all subarcs B of Γ, Kleiner estimates $[\sigma]$ in terms of $\|\sigma\|$ and a modulus of continuity of ν. In case ν is e.g. the arc length measure of Γ he finds that

$$[\sigma] \leq C\|\sigma\| \log \frac{1}{\|\sigma\|}, \quad C = C(\Gamma),$$

if $\|\sigma\|$ is small enough. Kleiner [6] also generalizes this to $n \geq 3$ dimensions. In this case the support of σ is contained in a compact surface F and $[\sigma]$ is the sup of the mass on certain subsets of F which are »contractive and nearly one-to-one» images of a fixed ball in $\mathbf{R}^{n-1}$ (for this concept see Kleiner's paper). The quantity $[\sigma]$ is then estimated as in the two-dimensional case, and when ν is the area measure on F the result is

$$[\sigma] \leq C\|\sigma\|^{2/3}, \quad C = C(F).$$

His proof can be modified to hold also in the case when one has only a one-sided bound on $\sigma : \sigma^+ \leq \nu$.

The Corollary of our Theorem 2 is a result of this type for classical and Riesz potentials and with the distributions lying in a ball in $\mathbf{R}^n$.

As to estimates of σ using the potential of σ, Ganelius [3] has proved the following result:

THEOREM. *Let μ be a positive mass distribution of total mass 1 on the unit circle E and let $d\nu = d\vartheta/2\pi$ be the equilibrium distribution of total mass 1. Then for any arc $B \subset E$*

$$|\mu(B) - \nu(B)| \leq C|\inf_E U^\mu|^{1/2},$$

where C is a numerical constant.

Since $U^\nu = 0$ on E,

$$|\inf_E U^\mu| = \sup_E U^{\nu-\mu}.$$

Notice that we have the sup of the potential difference and not its L^∞ norm. This makes the theorem yield a nontrivial result even when μ contains point masses. For example, if μ consists of N point masses $1/N$ one obtains an earlier result due to Erdös and Turán [2] about the distribution of zeros of polynomials.

As is mentioned in Ganelius's paper the theorem holds for more general curves, and in an unpublished manuscript Y. Bennulf has proved it for analytic curves. We shall generalize the theorem to less regular plane curves and to surfaces in $n \geq 3$ dimensions (Theorem 1). In Theorem 3 we give a similar estimate for Riesz potentials.

The idea of the proof of these two theorems is taken from a paper by Beurling and Malliavin [1] in which they study the closure in $L^2(-r, r)$ of sets $\{e^{i\lambda_n x}\}$.

In an auxiliary result (Theorem I′) the authors consider the logarithmic potential, suitably defined, of a positive measure μ on the real axis, and compare μ with the measure ν, $d\nu = kdx$. They estimate $|\mu(\omega) - \nu(\omega)|$ for intervals ω by means of the values on the real axis of the potential of $\mu - \nu$.

The author would like to thank Prof. T. Ganelius for suggesting the topic of this paper and for his valuable help and advice during its preparation.

2. Preliminaries

If μ is a measure, or mass distribution, with compact support in $\mathbf{R}^n$, $n \geq 2$, its classical potential is defined as follows in the sense of distributions (cf. Schwartz [8, in particular p. 214]):

$$\begin{aligned} U^\mu &= \log \frac{1}{|x|} * \mu \quad \text{if} \quad n = 2 \\ &= \frac{1}{|x|^{n-2}} * \mu \quad \text{if} \quad n > 2 \,. \end{aligned}$$

By $|\cdot|$ we mean the ordinary Euclidean norm in $\mathbf{R}^n$. There is an inverse formula

$$\mu = - c_n \Delta * U^\mu ,$$

where $c_2 = (2\pi)^{-1}$, $c_n = ((n-2)\omega_n)^{-1}$ if $n > 2$, and the Laplacian should be interpreted as a distribution. Here ω_n is the area of the unit sphere in $\mathbf{R}^n$.

For $0 < \alpha < 2$ we define *M*. Riesz's α-potential of μ by

$$U^\mu_\alpha = \frac{1}{|x|^{n-\alpha}} * \mu ,$$

and we also write U^μ_2 for the classical potential U^μ. If $0 < \alpha < 2$ the inverse formula is

$$\mu = T_\alpha * U^\mu_\alpha, \quad T_\alpha = c_{n,\alpha} \operatorname{Pf} \frac{1}{|x|^{n+\alpha}} , \tag{2.1}$$

where $c_{n,\alpha}$ is a constant and the distribution $\operatorname{Pf} 1/|x|^{n+\alpha}$ is defined by

$$\operatorname{Pf} \frac{1}{|x|^{n+\alpha}} \cdot \varphi = \lim_{\delta \searrow 0} \int\limits_{|x| > \delta} \frac{\varphi(x) - \varphi(0)}{|x|^{n+\alpha}}\, dx \tag{2.2}$$

for any test function $\varphi \in \mathcal{D}$. Analogously we put $T_2 = - c_n \Delta$.

In classical or Riesz potential theory the equilibrium distribution of a compact set $K \subset \mathbf{R}^n$ having positive capacity is the distribution of K whose potential is 1 on K except for a set of capacity zero.

For $\varepsilon > 0$ define the function w in $\mathbf{R}^n$ by $w(x) = 1 - (2\varepsilon^2)^{-1}|x|^2$ for $|x| \leq \varepsilon$, $= (2\varepsilon^2)^{-1}(|x| - 2\varepsilon)^2$ for $\varepsilon \leq |x| \leq 2\varepsilon$, and $= 0$ for $|x| \geq 2\varepsilon$. By elementary means the following lemma follows.

LEMMA 1. *The function w defined above has a Lipschitz continuous gradient and satisfies*

$$w(x + h) = w(x) + (h, \operatorname{grad} w(x)) + R$$

with $|\operatorname{grad} w(x)| \leq 1/\varepsilon$ *and* $|R| \leq |h|^2/\varepsilon^2$.

From Widman's paper [9] we shall need the following theorems (2.4 and 2.5 in [9]) and part of their proofs.

THEOREM A. *Let u be a harmonic function in a Liapunov-Dini region Ω, continuous in $\bar{\Omega}$, and with the property that to every $x_0 \in \partial\Omega$ there is a linear polynomial $L_{x_0}(x)$ such that*

$$|u(x) - L_{x_0}(x)| \leq \varepsilon_1(|x - x_0|)|x - x_0|, \quad x \in \partial\Omega,$$

where the Dini function $\varepsilon_1(t)$ satisfies the additional condition that $\varepsilon_1(t)/t^\gamma$ is monotonic for some γ, $0 < \gamma < 1$. Then $\partial u/\partial x_i$ are continuous in $\bar{\Omega}$. In particular, if Ω is a Liapunov region and $\varepsilon_1(t) = kt^\alpha$, then the functions $\partial u/\partial x_i$ are α-Hölder continuous in $\bar{\Omega}$.

THEOREM B. *Let $G(x, y)$ be the Green function of a Liapunov-Dini region Ω. Then for fixed $y \in \Omega$ there is a constant $c > 0$ such that*

$$\frac{\partial}{\partial n_x} G(x, y) \geq c, \quad x \in \partial\Omega.$$

For the exact meanings of the words Liapunov-Dini region in $\mathbf{R}^n$ and Dini function see [9]. In our applications the boundary of Ω will be of class $C^{1,\alpha}$ and we shall have $\varepsilon_1(t) = \text{const.}\, t$.

3. Regularity of test sets on surfaces

In $\mathbf{R}^n$, $n \geq 3$, we consider the boundary S of a bounded domain Ω_1 such that the complement Ω_2 of $\bar{\Omega}_1$ is also a domain with boundary S. We assume S to be a surface of class $C^{1,\alpha}$. It is known (Gunther [4, p. 17]) that Green's formulas hold for such S, and so do Widman's Theorems A and B.

We need a generalization of the subarcs B used in Ganelius's theorem as test sets to compare the measures μ and ν. It is clear that some restriction on B is necessary, and the following is what is needed in our proof.

Definition. $B \subset S$ is said to have *K-regular boundary* $(K > 0)$ if for any $d > 0$

$$\int_{B_d^*} dS \leq Kd\,, \tag{3.1}$$

where

$$B_d^* = \{x \in S : \varrho(x, B^*) \leq d\}\,.$$

Here B^* is the boundary of B in the relative topology in S, while ϱ means distance in $\mathbf{R}^n$. The measure dS is the $(n-1)$-dimensional area of S.

If $n = 3$ and B^* is a rectifiable closed curve in $\mathbf{R}^3$ of length l, then B has $(C_1 l + C_2)$-regular boundary, where C_1 and C_2 are constants depending only on S. This is easily proved by dividing the curve into subarcs of length $\leq 2d$ as Kleiner does in [6]. Then spheres of radii $2d$ centered at the endpoints of the subarcs are considered.

As an example shows, the corresponding statement is not true in higher dimensions. For let $n = 4$ and suppose S contains a 3-dimensional cube of side l. On one of its faces F we place p^2 points in a square lattice, the distance between any two of these points being $\geq l/p$. With the normals to F at these points as axes we choose cylinders reaching from F to the opposite face and with so small radii that their total 2-dimensional area is $\leq l^2$, say. Now let B consist of the cube except the points inside the cylinders. Then B_d^* contains the whole cube for $d = 2l/p$ and for large p the volume of B_d^* is not bounded by Kd for any fixed K, although the boundary of B has bounded 2-dimensional area as $p \to \infty$.

4. Estimation of measures on surfaces

In this section C and c will denote various constants, all of which are $< \infty$ and > 0, respectively, and depend only on the surface S which was introduced in Section 3.

The following Theorem 1 holds also in the plane for a simple closed curve of class $C^{1,\alpha}$ and can be proved similarly, but for the sake of brevity we assume that $n \geq 3$.

THEOREM 1. *Let μ and ν be positive mass distributions on S with $\int d\mu = \int d\nu$. Suppose that μ is absolutely continuous with respect to the area measure on S and that for some M*

$$\frac{d\mu}{dS} \leq M \tag{4.1}$$

on S. *Then if* $B \subset S$ *has* K-*regular boundary*

$$|\mu(B) - \nu(B)| \leq C(MK)^{1/2} [\sup_S (U^\mu - U^\nu)]^{1/2} . \tag{4.2}$$

Again notice that we have $\sup_S U^{\mu-\nu}$ and not $\sup_S |U^{\mu-\nu}|$. Later we shall see that the equilibrium distribution satisfies the condition imposed on μ.

Proof. There is a $\varrho = \varrho(S) > 0$ such that if $x_0 \in S$ and $\Sigma_{2\varrho}$ is a ball with center x_0 and radius 2ϱ then $S \cap \Sigma_{2\varrho}$ can be described by a function $\xi_n = F(\xi_1, \ldots, \xi_{n-1})$ in a local coordinate system $x_0, \xi_1, \ldots, \xi_n$ whose ξ_n-axis is the normal to S at x_0. This ϱ can be taken so small that any two normals to this part of the surface form an angle $< \pi/10$, say.

Put $\sigma = \mu - \nu$ and $U = U^\sigma$. The value of ε, $0 < \varepsilon < \varrho$, will be determined later, and this value is also used in the definition of the function w in Section 2. Let χ_B be the characteristic function of B, defined on S. We start by approximating χ_B with more regular functions $f_\pm$.

For this purpose the characteristic functions of $B \cup B^*_{2\varepsilon}$ and $B \setminus B^*_{2\varepsilon}$ will be called, respectively, χ_+ and χ_-. Now for $x \in S$ define the functions

$$I_\pm(x) = \int_S \chi_\pm(y) w(x-y) dS_y ,$$

$$A(x) = \int_S w\,(x-y) dS_y ,$$

$$f_\pm(x) = \frac{I_\pm(x)}{A(x)} .$$

Then it is easily seen that $f_\pm$ approximate χ_B in the following sense:

$$0 \leq f_- \leq \chi_B \leq f_+ \leq 1$$

on S, and

$$\{x \in S : f_\pm \neq \chi_B\} \subset B^*_{4\varepsilon} .$$

We shall use the letter χ below in statements valid for both χ_+ and χ_-, and similarly for I and f. Since $\varepsilon < \varrho$, the local regularity of S implies

$$0 \leq I(x) \leq A(x) \leq \int_{\substack{y \in S \\ |y-x| \leq 2\varepsilon}} dS_y \leq C\varepsilon^{n-1} \tag{4.3}$$

and

$$A(x) \geq \tfrac{1}{2} \int_{\substack{y \in S \\ |y-x| \leq \varepsilon}} dS_y \geq c\varepsilon^{n-1} , \tag{4.4}$$

for $w(z) \geq 1/2$ if $|z| \leq \varepsilon$. The regularity of w (see Lemma 1) now implies corresponding properties of I, A, and f, namely

$$I(x + h) = I(x) + (h, \operatorname{grad}_S I(x)) + R\,, \tag{4.5}$$

where x and $x + h \in S$, and

$$\operatorname{grad}_S I(x) = \int_S \chi(y) \operatorname{grad} w(x - y) dS_y\,. \tag{4.6}$$

Here we have the estimates

$$|\operatorname{grad}_S I(x)| \leq C\varepsilon^{n-2} \quad \text{and} \quad |R| \leq C|h|^2\varepsilon^{n-3}\,. \tag{4.7}$$

The vector $\operatorname{grad}_S I(x)$ as defined by (4.6) need not be the gradient of I considered as a function on the imbedded manifold S and might have a non-zero component orthogonal to S at x. The function A satisfies the same condition (4.5—7) as I does.

As to f, we use these Taylor expansions of I and A together with (4.3—4) in $f(x + h) = I(x + h)/A(x + h)$, x and $x + h \in S$. After some simple calculations this gives, at least for $|h|/\varepsilon \leq c$,

$$f(x + h) = f(x) + (h, \operatorname{grad}_S f(x)) + R\,, \tag{4.8}$$

where

$$|\operatorname{grad}_S f| = \left|\frac{A \operatorname{grad}_S I - I \operatorname{grad}_S A}{A^2}\right| \leq \frac{C}{\varepsilon}$$

and

$$|R| \leq C\,\frac{|h|^2}{\varepsilon^2}\,.$$

The restriction $|h|/\varepsilon \leq c$ can be removed after a suitable change in the value of the constant C in the estimate of R.

Following Beurling and Malliavin [1], we now use $f_\pm$ as boundary values for Dirichlet's problem in Ω_1 and Ω_2. This gives us two functions which will also be called $f_\pm$ or f and which are continuous in $\mathbf{R}^n$ and harmonic in $\Omega_1 \cup \Omega_2 \cup \{\infty\}$. Thus

$$f(x) = O(1/|x|^{n-2}) \quad \text{as} \quad x \to \infty\,.$$

By Theorem A the regularity (4.8) of f implies that grad f is continuous in $\bar{\Omega}_1$ and $\bar{\Omega}_2$. But if $\partial f/\partial n_1$ and $\partial f/\partial n_2$ are the normal derivatives of f on S into Ω_1 and Ω_2, respectively, we cannot expect $\partial f/\partial n_1 = - \partial f/\partial n_2$.

With our condition on f more precise information can be obtained from Widman's proof of Theorem A (see [9, p. 23—25]). Letting u in Theorem A be f we can take $L_{x_0}(x) = f(x_0) + (x - x_0, \operatorname{grad}_S f(x_0))$ and put $\varepsilon_1(t) = Ct/\varepsilon^2$, $t > 0$.

The coefficients of L_{x_0} are not greater than C/ε, and $|f(x_0)| \leq 1$. Fix $x_0 \in S$. If we, following Widman's proof, subtract L_{x_0} from $u = f$, then we do not change $\operatorname{grad} u$ more than C/ε, and within a distance of ε from x_0 the values of u are not changed more than C.

Considering a ball Σ of radius $\varrho_1/2$ and center x_0, Widman finds that for any $y \in \Omega$ $(= \Omega_1$ or $\Omega_2)$ on the normal of S at x_0 and with $|y - x_0|$ small enough

$$|\operatorname{grad} u(y)| \leq C \int_0^{\varrho_1/2} \frac{\varepsilon_1(t)}{t}\, dt + C\varrho_1^{-1} \sup_{|x-x_0| \leq \varrho_1} |u(x)| .$$

If we use our expression for $\varepsilon_1(t)$ and put $\varrho_1 = \varepsilon$, we get

$$|\operatorname{grad} u(y)| \leq \frac{C}{\varepsilon} .$$

Therefore

$$\left| \frac{\partial f}{\partial n_i} \right| \leq \frac{C}{\varepsilon} \quad \text{on} \quad S, \quad i = 1, 2, \quad \text{and} \quad f = f_{\pm} . \tag{4.9}$$

LEMMA 2.

$$\frac{1}{|x|^{n-2}} * (- c_n \Delta * f) = f ,$$

where the two sides are to be considered as distributions in $\mathscr{D}'$.

Proof. The left side exists since $- c_n \Delta * f$ has compact support $\subset S$. By Landkof [7, Lemma 1.11] the equation holds if $f(x) = O(1/|x|^{n-\beta})$, $x \to \infty$, and $\beta > 0$, $2 + \beta < n$. Since $f(x) = O(1/|x|^{n-2})$ this proves the lemma in case $n > 4$. For $n = 3$ or 4 we expand f close to the point at infinity:

$$f(x) = \frac{a}{|x|^{n-2}} + \frac{(x, b)}{|x|^n} + O\left(\frac{1}{|x|^n}\right), \quad x \to \infty .$$

Here $a \in \mathbf{R}$ and $b \in \mathbf{R}^n$. Evidently the lemma holds with $a/|x|^{n-2}$ instead of f. By Landkof's lemma the only remaining difficulty is the term $(x, b)/|x|^n$ when $n = 3$. Consider for example $x_1/|x|^3$. This distribution equals $- \partial/\partial x_1 * 1/|x|$, so

$$\frac{1}{|x|} * \left(- c_3 \Delta * \left(- \frac{\partial}{\partial x_1} * \frac{1}{|x|}\right)\right) = \frac{1}{|x|} * \left(- \frac{\partial}{\partial x_1} * \delta\right) = - \frac{\partial}{\partial x_1} * \frac{1}{|x|} .$$

This completes the proof of the lemma.

Following Schwartz [8] we write $T \cdot \varphi$ for the scalar product of the distribution T and function φ, put $\operatorname{Tr} \varphi = \varphi(0)$ and $\check{\varphi}(x) = \varphi(-x)$, and define $\check{T}$ by $\check{T} \cdot \varphi = T \cdot \check{\varphi}$. The following calculation will be made precise below.

$$\sigma \cdot f = \operatorname{Tr} \sigma * \check{f} = \operatorname{Tr}\left(\sigma * \frac{1}{|x|^{n-2}}\right) * (-c_n\Delta * \check{f}) = \operatorname{Tr} U * (-c_n\Delta * f)^{\vee} =$$
$$= U \cdot (-c_n\Delta f)\,. \tag{4.10}$$

Since f is continuous $\sigma \cdot f$ exists and equals $\int_S f d\sigma$, and the distribution $\sigma * \check{f}$ is a continuous function whose trace is $\sigma \cdot f$. Lemma 2 gives the second equality and the third one is trivial. Let us determine the distribution $\Delta * f$.

As grad f is continuous in $\bar{\Omega}_1$ and $\bar{\Omega}_2$, the Green formula can be used, just as in Schwartz [8, p. 44], to show that $-c_n\Delta * f$ is a measure τ on S of density $c_n(\partial f/\partial n_1 + \partial f/\partial n_2)$ with respect to dS.

We know that the distribution $U * \check{\tau} = \sigma * \check{f}$ is a continuous function Ψ_1 and must verify that it coincides with the function Ψ_2 defined by

$$\Psi_2(x) = \int U(x+y)d\tau(y)\,.$$

By the reciprocity theorem

$$\Psi_2(x) = \int U^\tau(y-x)d\sigma(y)\,.$$

But U^τ is seen to be continuous in $\mathbf{R}^n$, so the function Ψ_2 is continuous. By Fubini's theorem the distribution defined by Ψ_2 is $U * \check{\tau}$, so that Ψ_1 and Ψ_2 must be equal everywhere, and

$$\operatorname{Tr} U * \check{\tau} = \operatorname{Tr} \Psi_2 = \int U(y)d\tau(y)\,.$$

Thus our calculation (4.10) is completely verified, and we have

$$\int f d\sigma = c_n \int_S U\left(\frac{\partial f}{\partial n_1} + \frac{\partial f}{\partial n_2}\right) dS\,.$$

Therefore by (4.9)

$$\left|\int f d\sigma\right| \le \frac{C}{\varepsilon}\int_S |U| dS\,. \tag{4.11}$$

LEMMA 3. *The equilibrium distribution λ on S is absolutely continuous with respect to dS and satisfies*

$$c \le \frac{d\lambda}{dS} \le C\,.$$

Proof. The potential U^λ equals 1 in $\bar{\Omega}_1$ and is in Ω_2 the solution of Dirichlet's problem with the boundary value 1 on S. Therefore Theorem A directly implies that $\partial U^\lambda/\partial n_2$ exists and is continuous and bounded on S. Except for a constant factor, however, this derivative equals the density of λ. Thus we have the second inequality of the lemma.

Let $G(x, y)$ be Green's function for Ω_2. Then

$$G(x, y) = \frac{1}{|x - y|^{n-2}} - u_y(x)$$

where u_y is the solution of Dirichlet's problem in Ω_2 with boundary values $u_y(x) = 1/|x - y|^{n-2}$, $x \in S$. Therefore $|y|^{n-2}G(x, y) \to 1 - u(x)$ as $y \to \infty$, where $u(x)$ is the solution of Dirichlet's problem in Ω_2 with $u(x) = 1$ on S, so that $u = U^\lambda$.

Now Theorem B says that for a fixed $y \in \Omega_2$ there is a c such that

$$\frac{\partial}{\partial n_2} G(x, y) \geq c$$

for all $x \in S$, where $\partial/\partial n_2$ is taken with respect to x. By Harnack's inequality and the maximum principle there is a c for which

$$|y|^{n-2} \frac{\partial}{\partial n_2} G(x, y) \geq c$$

for all $x \in S$ and all large $|y|$. If we examine Widman's proof of Theorem A we see that

$$|y|^{n-2} \frac{\partial}{\partial n_2} G(x, y) \to \frac{\partial}{\partial n_2} (1 - u(x)) \quad \text{as} \quad y \to \infty .$$

Thus

$$\frac{\partial U^\lambda}{\partial n_2} \leq -c \quad \text{on} \quad S ,$$

which proves the remaining first inequality of the lemma.

Now the inequality (4.11) and Lemma 3 imply

$$\left| \int f d\sigma \right| \leq \frac{C}{\varepsilon} \int\limits_S |U| d\lambda .$$

But by the reciprocity theorem

$$\int U d\lambda = \int U^\lambda d\sigma = 0 ,$$

so that

$$\left|\int f d\sigma\right| \le 2\,\frac{C}{\varepsilon}\int U^{+} d\lambda \le \frac{C}{\varepsilon}\sup_S U\,. \tag{4.12}$$

Here $U^{+} = \max(U, 0)$. With this inequality, which holds for f_{+} and f_{-}, we can estimate $\sigma(B)$:

$$\int \chi_B d\sigma = \int f_{-} d\sigma + \int_{B \cap B^*_{4\varepsilon}} (\chi_B - f_{-}) d\sigma\,. \tag{4.13}$$

The second term is not greater than

$$\int_{B \cap B^*_{4\varepsilon}} (\chi_B - f_{-}) d\mu \le \int_{B^*_{4\varepsilon}} d\mu \le M \int_{B^*_{4\varepsilon}} dS \le 4\, MK\varepsilon\,.$$

Here we used (4.1) and the fact that B has K-regular boundary. Thus we obtain from (4.12—13) that

$$\int \chi_B d\sigma \le \frac{C}{\varepsilon}\sup_S U + 4\, MK\varepsilon\,.$$

Using f_{+} instead of f_{-} in a similar way, we get an estimate in the other direction, so that in fact

$$|\sigma(B)| \le \frac{C}{\varepsilon}\sup_S U + 4\, MK\varepsilon\,.$$

If we can take

$$\varepsilon = \left(\frac{\sup_S U}{MK}\right)^{\frac{1}{2}},$$

we obtain the claimed inequality (4.2).

This choice of ε is possible only if

$$\left(\frac{\sup_S U}{MK}\right)^{\frac{1}{2}} < \varrho\,. \tag{4.14}$$

In the opposite case we observe that, except in the trivial cases when B is S or $\emptyset$, B^* is non-empty, and if we let $d = \operatorname{diam}(S)$ in the definition of K-regular boundary, we can conclude

$$K \ge \frac{\operatorname{area}(S)}{\operatorname{diam}(S)} = c\,.$$

Hence,

$$(MK)^{\frac{1}{2}}(\sup U)^{\frac{1}{2}} \ge MK\left(\frac{\sup_S U}{MK}\right)^{\frac{1}{2}} \ge Mc\,\varrho\,, \tag{4.15}$$

since we assumed the contrary of (4.14). But

$$|\sigma(B)| \leq \nu(B) = \mu(B) \leq CM . \tag{4.16}$$

Since $\varrho \geq c$, (4.15—16) imply (4.2) if we choose a suitable C.

This completes the proof of Theorem 1.

5. Measures in balls and other sets

Let K_r be the ball $\{x : |x| \leq r\}$ in $\mathbf{R}^n$, $n \geq 2$. We consider classical and Riesz potentials of mass distributions in a fixed ball K_R and so let $0 < \alpha \leq 2$. For $0 < \alpha < 2$ the equilibrium distribution λ_r of K_r is absolutely continuous with density $=$ const. $(r^2 - |x|^2)^{-\alpha/2}$, $|x| < r$. If $\alpha = 2$ the distribution λ_r is of course $(\omega_n r)^{-1} dS$ on the sphere $S : |x| = r$. The condition imposed on test sets will be slightly changed. If $B \subset \mathbf{R}^n$ we put $B_d^* = \{x \in \mathbf{R}^n : \varrho(x, B^*) < d\}$ for $d > 0$, where B^* is the boundary of B in $\mathbf{R}^n$.

Definition. $B \subset \mathbf{R}^n$ is said to have *K-regular boundary in K_R*, $K > 0$, if for all $d > 0$

$$\int_{K_R \cap B_d^*} dx \leq Kd . \tag{5.1}$$

This concept is defined similarly for other bounded sets than K_R. We see that $\mathscr{C} B = \mathbf{R}^n \setminus B$ has K-regular boundary in K_R if and only if B has. To give an example, there is a K depending only on R and n such that all circular cones with vertices in the origin have K-regular boundaries in K_R.

In this section C will denote several different constants which depend only on n, α, and R unless otherwise explicitly stated.

THEOREM 2. *Let $0 < \alpha \leq 2$ and let μ and ν be positive mass distributions in K_R with $\int d\mu = \int d\nu$. Assume that μ is absolutely continuous with respect to dx and that for some $M < \infty$*

$$\frac{d\mu}{dx} \leq M \tag{5.2}$$

in K_R. Then for any $B \subset \mathbf{R}^n$ having K-regular boundary in K_R

$$|\mu(B \cap K_R) - \nu(B \cap K_R)| \leq C(MK)^{\frac{\alpha}{1+\alpha}} \left(\int_{K_R} |U_\alpha^{\mu-\nu}| dx \right)^{\frac{1}{1+\alpha}} . \tag{5.3}$$

Proof. We use the idea of the proof of Theorem 1 and start by letting $\varepsilon > 0$. Normalizing the function w from Section 2, we get a new function w_1 such that $\int w_1(x)dx = 1$. For any continuous function φ defined in $\mathbf{R}^n$ we put

$$\tilde{\varphi}(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{|y|=r} \varphi(x+y)do_y, \quad r > 0 .$$

From Lemma 1 it then follows that

$$|\tilde{w}_1(x, r) - w_1(x)| \leq C\varepsilon^{-n-2}r^2 . \tag{5.4}$$

Now put

$$B_+ = (B \cup B^*_{2\varepsilon}) \cap K_{R+2\varepsilon} ,$$
$$B_- = (B \setminus B^*_{2\varepsilon}) \cap K_{R+2\varepsilon} ,$$

and

$$f_\pm = \chi_{B_\pm} * w_1 .$$

Then $f_\pm$ approximate χ_B in K_R in the following sense:

$$0 \leq f_- \leq \chi_B \leq f_+ \leq 1 \quad \text{in} \quad K_R , \tag{5.5}$$
$$\operatorname{supp} f_\pm \subset K_{R+4\varepsilon} ,$$

and

$$\{x \in K_R : f_\pm \neq \chi_B\} \subset K_R \cap B^*_{4\varepsilon} .$$

We find from (5.4) that $f_\pm$ have a similar regularity property and satisfy

$$|\tilde{f}(x, r) - f(x)| \leq C\varepsilon^{-2}r^2 , \quad f = f_\pm . \tag{5.6}$$

Now, since f, i.e., f_+ or f_-, has compact support, it easily follows from Lemma 1.11 in Landkof [7] that

$$\int f d\sigma = \int U^\sigma_\alpha (T_\alpha * f)dx , \tag{5.7}$$

where as before we have put

$$\sigma = \mu - \nu .$$

We must therefore estimate $T_\alpha * f$. Suppose $0 < \alpha < 2$. Our formulas (2.1—2) for T_α imply

$$\begin{aligned} |T_\alpha * f(x)| &= \left| C \int_0^\infty (\tilde{f}(x, r) - f(x))r^{-1-\alpha}dr \right| \leq \\ &\leq C \int_0^\varepsilon \varepsilon^{-2}r^2 r^{-1-\alpha}dr + C \int_\varepsilon^\infty r^{-1-\alpha}dr = C\varepsilon^{-\alpha} , \end{aligned} \tag{5.8}$$

where we have used the inequalities (5.6) and $0 \leq f \leq 1$. Except for a neighbourhood of the origin, T_α is an integrable function $C|x|^{-n-\alpha}$, and f vanishes outside $K_{R+4\varepsilon}$, so for $|x| > R + 4\varepsilon$ we see that

$$|T_\alpha * f(x)| \leq C(|x| - R - 4\varepsilon)^{-n-\alpha} . \tag{5.9}$$

We can improve this estimate near $K_{R+4\varepsilon}$. If $|x| - R - 4\varepsilon = t > 0$, we have

$$T_\alpha * f(x) = \int \frac{f(x-y)}{|y|^{n+\alpha}}\, dy ,$$

where we only integrate over $x - K_{R+4\varepsilon}$. Now $|f| \leq 1$, so the integral is not greater than

$$\int \frac{dy}{|y|^{n+\alpha}}$$

taken over $\{y : |y| \geq t\}$, which equals $Ct^{-\alpha}$. Thus we find that

$$|T_\alpha * f(x)| \leq C(|x| - R - 4\varepsilon)^{-\alpha} , \tag{5.10}$$

if $|x| > R + 4\varepsilon$.

We need a generalization to Riesz potentials of the Poisson formula. In Landkof [7, p. 156—157] we find that

$$U_\alpha^\sigma(x) = \int_{|y|<R} U_\alpha^\sigma(y) P_R(y, x) dy, \quad |x| > R , \tag{5.11}$$

where

$$P_R(y, x) = C\, \frac{(|x|^2 - R^2)^{\alpha/2}}{(R^2 - |y|^2)^{\alpha/2}} \cdot \frac{1}{|x-y|^n} .$$

Here C depends only on n and α. If $r > R$, we conclude from (5.11) that

$$\int_{|x|=r} |U_\alpha^\sigma| do \leq C \int_{|y|<R} \frac{|U_\alpha^\sigma(y)|}{(R^2 - |y|^2)^{\alpha/2}}\, dy\ (r^2 - R^2)^{\alpha/2} \int_{|x|=r} \frac{do_x}{|x-y|^n} .$$

But by the ordinary Poisson formula

$$\int_{|x|=r} \frac{do_x}{|x-y|^n} = \frac{Cr}{r^2 - |y|^2} \leq \frac{Cr}{r^2 - R^2} ,$$

and therefore

$$\int_{|x|=r} |U_\alpha^\sigma| do \leq C \int_{|y|<R} \frac{|U_\alpha^\sigma(y)|}{(R^2 - |y|^2)^{\alpha/2}}\, dy\ r(r^2 - R^2)^{\frac{\alpha}{2}-1} . \tag{5.12}$$

To estimate $\int f d\sigma$ we split the right side of (5.7) into three parts and use (5.8) for $|x| < R + 5\varepsilon$, (5.10) for $R + 5\varepsilon < |x| < 2R$, and (5.9) for $2R < |x|$. (If $5\varepsilon \geq R$ we only need two parts.) This gives us

$$\left|\int f d\sigma\right| \leq C\varepsilon^{-\alpha} \int\limits_{|x|<R+5\varepsilon} |U_\alpha^\sigma(x)|dx + C \int\limits_{R+5\varepsilon<|x|<2R} |U_\alpha^\sigma(x)|(|x| - R - 4\varepsilon)^{-\alpha}dx +$$

$$+ C \int\limits_{|x|>2R} |U_\alpha^\sigma(x)|(x - R - 4\varepsilon)^{-n-\alpha}dx .$$

If we write the first term of the right side as

$$\int\limits_{|x|<R} + \int\limits_{R<|x|<R+5\varepsilon}$$

we get a sum $I_1 + I_2 + I_3 + I_4$. Now write $I_{2,3,4}$ with polar coordinates and use (5.12) to estimate these integrals:

$$I_{2,3} \leq C\varepsilon^{-\alpha/2} \int |U_\alpha^\sigma| d\lambda_R ,$$

$$I_4 \leq C \int |U_\alpha^\sigma| d\lambda_R .$$

Remember that $d\lambda_R = C(R^2 - |x|^2)^{-\alpha/2}dx$ is the equilibrium distribution in K_R. Thus we have

$$\left|\int f d\sigma\right| \leq C\varepsilon^{-\alpha} \int\limits_{K_R} |U_\alpha^\sigma| dx + C\varepsilon^{-\alpha/2} \int |U_\alpha^\sigma| d\lambda_R . \tag{5.13}$$

This is true also in the classical case $\alpha = 2$, since (5.6) implies that Δf, taken in the sense of distributions, is a function satisfying

$$|\Delta f| \leq C\varepsilon^{-2} .$$

Now $f = 0$ outside $K_{R+4\varepsilon}$, so

$$\left|\int f d\sigma\right| \leq C\varepsilon^{-2} \int\limits_{K_{R+4\varepsilon}} |U_2^\sigma| dx .$$

By use of the exterior Poisson formula we obtain (5.13) also in this case.

LEMMA 4.

$$\int |U_\alpha^\sigma| d\lambda_R \leq CM^{\frac{\alpha}{2(1+\alpha)}} \Bigg(\int\limits_{K_R} |U_\alpha^\sigma| dx \Bigg)^{\frac{2+\alpha}{2(1+\alpha)}} .$$

Proof. Again suppose $\alpha < 2$. Using the generalized Poisson kernel, we put for any $\varrho > 0$

$$P_\varrho \varphi(x) = \int\limits_{|y|<\varrho} \varphi(y) P_\varrho(y, x) dy \quad \text{if} \quad |x| > \varrho$$
$$= \varphi(x) \quad \text{if} \quad |x| \leq \varrho \,,$$

where φ is any function for which the integral exists. By the reciprocity theorem

$$\int |U_\alpha^\sigma| d\lambda_R = 2 \int (U_\alpha^\sigma)^+ d\lambda_R \,,$$

since $\int d\sigma = 0$ and so $\int U_\alpha^\sigma d\lambda_R = 0$.

We now use properties of P_ϱ to be found in Landkof [7, p. 157—160]. Since $\nu \geq 0$,

$$P_\varrho U_\alpha^\nu \leq U_\alpha^\nu \quad \text{for} \quad \varrho < R \,, \tag{5.14}$$

and if μ_ϱ is μ restricted to K_ϱ, $\varrho < R$, then

$$P_\varrho U_\alpha^{\mu_\varrho} = U_\alpha^{\mu_\varrho} \,. \tag{5.15}$$

Put $\mu' = \mu - \mu_\varrho$ and let $\varrho < |x| < R$. By changing the order of integration we find that

$$0 \leq U_\alpha^{\mu'}(x) - P_\varrho U_\alpha^{\mu'}(x) =$$
$$= \int \left(\frac{1}{|x-y|^{n-\alpha}} - \left(P_\varrho \frac{1}{|y-\cdot|^{n-\alpha}} \right)(x) \right) d\mu'(y) \leq M \int\limits_{\varrho \leq |y| \leq R} (\quad) dy \,,$$

the last inequality because of (5.2) and the non-negativity of the integrand. But for $\varrho < |y| < R$ we have the inequality $dy \leq C(R-\varrho)^{\alpha/2} d\lambda_R(y)$, so

$$U_\alpha^{\mu'}(x) - P_\varrho U_\alpha^{\mu'}(x) \leq C(R-\varrho)^{\alpha/2} M \int (\quad) d\lambda_R(y) \,,$$

where we have the same integrand as before. We can extend the integration over the whole of K_R without affecting the value of the integral. Now change the order of integration again:

$$U_\alpha^{\mu'}(x) - P_\varrho U_\alpha^{\mu'}(x) \leq CM(R-\varrho)^{\alpha/2}(1 - (P_\varrho 1)(x)) \,,$$

since the potential of λ_R is 1 in K_R. The function $P_\varrho 1$ coincides with the potential of λ_ϱ, and using the explicit expression for this distribution one finds that its potential belongs to $\mathrm{Lip}_{\alpha/2}$, even close to the boundary of K_ϱ. Therefore we conclude that

$$U_\alpha^{\mu'}(x) - P_\varrho U_\alpha^{\mu'}(x) \leq CM(R-\varrho)^{\alpha/2}(|x|-\varrho)^{\alpha/2} \,. \tag{5.16}$$

Our relations (5.14—16) together imply

$$U_\alpha^\sigma(x) \leq P_\varrho U_\alpha^\sigma(x) + CM(R-\varrho)^{\alpha/2}(|x|-\varrho)^{\alpha/2}\,,$$

and hence,

$$(U_\alpha^\sigma(x))^+ \leq P_\varrho |U_\alpha^\sigma|(x) + CM(R-\varrho)^{\alpha/2}(|x|-\varrho)^{\alpha/2}\,.$$

Let us now integrate this inequality over the sphere $|x| = r$, $\varrho < r < R$, and use the calculation that led us to (5.12):

$$\int\limits_{|x|=r} (U_\alpha^\sigma)^+ do \leq C(r-\varrho)^{\frac{\alpha}{2}-1} \int |U_\alpha^\sigma| d\lambda_\varrho + CM(R-\varrho)^{\alpha/2}(r-\varrho)^{\alpha/2}\,.$$

Hence, if we multiply by $(R^2 - r^2)^{-\alpha/2}$ and integrate with respect to r from ϱ to R, we find that

$$\int\limits_{\varrho\leq|x|\leq R} \frac{(U_\alpha^\sigma(x))^+}{(R^2-|x|^2)^{\alpha/2}}\, dx \leq C \int |U_\alpha^\sigma| d\lambda_\varrho + CM(R-\varrho)^{1+\frac{\alpha}{2}}\,.$$

If $\varrho \geq R/2$, say, this implies

$$\int (U_\alpha^\sigma)^+ d\lambda_R \leq C \int |U_\alpha^\sigma| d\lambda_\varrho + CM(R-\varrho)^{1+\frac{\alpha}{2}}$$

with a new value of the first constant C. Let us integrate with respect to ϱ from $R-t$ to R, where $0 \leq t \leq R/2$, and use our known expression for the density of λ_ϱ:

$$t \int (U_\alpha^\sigma)^+ d\lambda_R \leq Ct^{1-\frac{\alpha}{2}} \int |U_\alpha^\sigma| dx + CM\, t^{2+\frac{\alpha}{2}}\,.$$

If

$$\Big(\int\limits_{K_R} |U_\alpha^\sigma| dx/M\Big)^{\frac{1}{1+\alpha}} \leq R/2\,,$$

we can take

$$t = \Big(\int\limits_{K_R} |U_\alpha^\sigma| dx/M\Big)^{\frac{1}{1+\alpha}}$$

and obtain the inequality claimed in the lemma. Otherwise the lemma is trivially true, since $\int |U_\alpha^\sigma| d\lambda_R$ is bounded by

$$2\int (U_\alpha^\sigma)^+ d\lambda_R \leq 2\int U_\alpha^\mu d\lambda_R \leq 2M \int U_\alpha^m d\lambda_R \leq CM\,,$$

where m is Lebesgue measure in K_R.

The case $\alpha = 2$ can be handled similarly, and so the lemma is proved.

We now conclude the proof of Theorem 2 by first using Lemma 4 in (5.13):

$$\left|\int f d\sigma\right| \leq C\varepsilon^{-\alpha} \int_{K_R} |U_\alpha^\sigma| dx + C M^{\frac{\alpha}{2(1+\alpha)}} \varepsilon^{-\frac{\alpha}{2}} \left(\int_{K_R} |U_\alpha^\sigma| dx\right)^{\frac{2+\alpha}{2(1+\alpha)}} .$$

The proof then runs like that of Theorem 1. This time there is no restriction on the value of ε, and to obtain (5.3) in the final step we choose

$$\varepsilon = \left(\frac{\int_{K_R} |U_\alpha^\sigma| dx}{MK}\right)^{\frac{1}{1+\alpha}} .$$

This completes the proof of Theorem 2.

Corollary. *Under the same assumptions,*

$$|\mu(B \cap K_R) - \nu(B \cap K_R)| \leq C(MK)^{\frac{\alpha}{2+\alpha}} \cdot \|\mu - \nu\|_\alpha^{\frac{2}{2+\alpha}} ,$$

where $\|\sigma\|_\alpha = (\int U_\alpha^\sigma d\sigma)^{1/2}$ *is the energy norm.*

Proof. Using Theorem 2 with $\alpha/2$ instead of α and then Cauchy's inequality we get

$$|\mu(B \cap K_R) - \nu(B \cap K_R)| \leq C(MK)^{\frac{\alpha}{2+\alpha}} \left(\int_{K_R} |U_{\alpha/2}^\sigma| dx\right)^{\frac{2}{2+\alpha}} \leq$$

$$\leq C(MK)^{\frac{\alpha}{2+\alpha}} \left(\int_{K_R} (U_{\alpha/2}^\sigma)^2 dx\right)^{\frac{1}{2+\alpha}} .$$

However, as can be found in e.g. Landkof [7, p. 105—106],

$$\int (U_{\alpha/2}^\sigma)^2 dx = C \int U_\alpha^\sigma d\sigma .$$

The corollary is proved.

Theorem 3. *Let* $0 < \alpha < 2$ *and let* N *be a compact set in* $\mathbf{R}^n$. *If* μ *and* ν *are positive mass distributions with supports contained in* N *and such that* $\int d\mu = \int d\nu$ *and*

$$d\mu \leq M\, dx$$

in N, *then for any* $B \subset \mathbf{R}^n$ *having* K*-regular boundary in* N

$$|\mu(B \cap N) - \nu(B \cap N)| \leq C(MK)^{\frac{\alpha}{1+\alpha}} (\sup_N U_\alpha^{\mu-\nu})^{\frac{1}{1+\alpha}}, \quad C = C(N, \alpha) .$$

Remark. As simple examples show this is not true for $\alpha = 2$. It would not be true with $\mathbf{R}^n$ instead of N, even after a suitable definition of K-regular boundary in $\mathbf{R}^n$ (this could mean allowing only $d \leq 1$, say).

Proof. Take R so great that $N \subset K_R$. Since $dx \leq C d\lambda_R$, it follows from the inequality (5.13) that

$$\left| \int f d\sigma \right| \leq C\varepsilon^{-\alpha} \int |U_\alpha^\sigma| d\lambda_R ,$$

if we assume that $\varepsilon \leq 1$.

Now $\int U_\alpha^\sigma d\lambda_R = 0$, so, just as in the proof of Theorem 1, we conclude that

$$|\sigma(B)| \leq C(MK)^{\frac{\alpha}{1+\alpha}} (\sup_{K_R} U_\alpha^\sigma)^{\frac{1}{1+\alpha}} .$$

But because of α-harmonicity (see Landkof [7])

$$\sup_{K_R} U_\alpha^\sigma = \sup_N U_\alpha^\sigma .$$

This completes the proof of Theorem 3.

References

1. Beurling, A. & Malliavin, P., On the closure of characters and the zeros of entire functions. *Acta Math.* 118 (1967), 79–93.
2. Erdös, P. & Turán, P., On a problem in the theory of uniform distribution I and II. *Indagationes Math.* 10 (1948), 370–378 and 406–413.
3. Ganelius, T., Some applications of a lemma on Fourier series. *Publ. de l'Inst. Math. de l'Acad. Serbe des Sc.* 11 (1957), 9–18.
4. Gunther, N. M., *La théorie du potentiel.* Gauthiers-Villars, Paris, 1934.
5. Kleiner, W., Une condition de Dini-Lipschitz dans la théorie du potentiel. *Ann. Polon. Math.* 14 (1964), 117–130.
6. —»— On the equilibrium of signed measures. *Coll. Math.* 11 (1964), 257–278.
7. Landkof, N. S., *Foundations of modern potential theory.* Moscow, 1966 (Russian).
8. Schwartz, L., *Théorie des distributions.* Nouvelle édition. Hermann, Paris, 1966.
9. Widman, K.-O., Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. *Math. Scand.* 21 (1967), 17–37.

Received September 27, 1971

Peter Sjögren
Department of Mathematics
University of Göteborg
Fack
S-402 20 Göteborg 5
Sweden