Eigenfunction expansions for partially hypoelliptic operators

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1. Introduction

We shall work in real n-space R^n with points $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, etc., and shall consider a splitting of the coordinates in R^n, writing $x = (x', x'')$, $\xi = (\xi', \xi'')$, etc., with $x' = (x_1, \ldots, x_{n'}) \in R^{n'}$, $x'' = (x_{n'+1}, \ldots, x_{n'+n''}) \in R^{n''}$, where n' and n'' are given positive integers such that $n' + n'' = n$. This splitting of the coordinates in R^n will be kept throughout the paper. Let

$$D^\alpha = (2\pi i)^{-|\alpha|} \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiorder and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Let us consider a differential operator $P(D)$ on R^n with constant coefficients, and assume that $P(D)$ is partially hypoelliptic with respect to ξ' in the sense of Gårding-Malgrange [2]. Using one of the conditions in [2], this means that we can write

$$P(\xi) = M(\xi') + \sum_{j=1}^{r} Q_j(\xi'')M_j(\xi'), \tag{1}$$

where M is a hypoelliptic polynomial on $R^{n'}$ (i.e. we have $D^{\alpha'}M(\xi')/M(\xi') \to 0$ as $|\xi'| \to \infty$, $\xi' \in R^{n'}$, for any n'-order $\alpha' \neq 0$), and where the Q_j:s and M_j:s are polynomials such that each M_j is strictly weaker than M (i.e., since M is hypoelliptic, that $M_j(\xi')/M(\xi') \to 0$ as $|\xi'| \to \infty$, $\xi' \in R^{n'}$). We shall also suppose that P is real-valued (it is easily seen that then the polynomials M, Q_j, M_j can all be chosen real), and that $M(\xi') \to +\infty$ as $|\xi'| \to \infty$, $\xi' \in R^{n'}$ (then M can be chosen positive).

Now let Ω be a non-empty open subset of R^n. Then $P_0 : C_0^\infty(\Omega) \ni \varphi \mapsto P(D)\varphi$ is a densely defined symmetric linear mapping in the Hilbert space $L^2(\Omega)$ of all square integrable functions on Ω. (In $L^2(\Omega)$ we use the scalar product $(u, v) = \int_\Omega u(x)\overline{v(x)}dx$ and the norm $\|u\| = (u, u)^{1/2}$.) Clearly P_0 need not be bounded from below. Let A be a self-adjoint extension in $L^2(\Omega)$ of P_0. By well-known theorems such extensions exist at least if P_0 is bounded from below, or if $P(\xi)$ is an even function of ξ (rendering the deficiency indices equal). Now A has a spectral resolution

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

(see e.g. Sz-Nagy [10]), where the $E(\lambda)$ are orthogonal projections in $L^2(\Omega)$, increasing with λ and such that $E(\lambda) \to I$ (the identity mapping) as $\lambda \to +\infty$ and $E(\lambda) \to O$ as $\lambda \to -\infty$, both in the strong sense. The projections $E(\lambda)$ are uniquely determined if we e.g. require the functions $\lambda \mapsto E(\lambda)$ to be continuous to the left.

Let $\varphi \in C_0^\infty(R^{n''})$ and consider the partial convolution $\tilde{E}(\lambda)u$ of $E(\lambda)u$ with φ, i.e.

$$\tilde{E}(\lambda)u = E(\lambda)u *'' \varphi = E(\lambda)u * (\delta' \otimes \varphi)\,,$$

where δ' is the Dirac δ-distribution on $R^{n'}$. Then $\tilde{E}(\lambda)u$ is defined as a distribution in the set

$$\Omega_\varphi = \complement\left(\complement\, \Omega + S_\varphi\right) \tag{2}$$

where S_φ is the subset $S_\varphi = \{0\} \times \operatorname{supp} \varphi$ of R^n. Ω_φ is non-empty if $\operatorname{supp} \varphi$ is small enough. We are going to prove (theorem 1) that $\tilde{E}(\lambda)u$ is in $C^\infty(\Omega_\varphi)$ for any real λ, and that $\tilde{E}(\lambda)u$ tends, together with its derivatives, rapidly to 0 as $\lambda \to -\infty$. The proof will depend on estimates for a fundamental solution of the operator $P(D) + \lambda$, which are given in lemma 3 and lemma 4.

By the Schwartz kernel theorem the projections $E(\lambda)$ are given on $C_0^\infty(\Omega)$ by kernels e_λ which are distributions on $\Omega \times \Omega$:

$$E(\lambda)\varphi(x) = \int_\Omega e_\lambda(x, y)\varphi(y)dy \qquad (\varphi \in C_0^\infty(\Omega))$$

(that is, $\int_\Omega \psi(x)E(\lambda)\varphi(x)dx = \int_{\Omega\times\Omega} e_\lambda(x, y)\psi(x)\varphi(y)dxdy$ if $\varphi, \psi \in L^2(\Omega)$). From theorem 1 will follow estimates for (the partial convolutions of) the kernels e_λ. These estimates are given in theorem 2 and theorem 3.

Further we shall investigate the behaviour as $\lambda \to +\infty$ of the partial convolution

$$f_\lambda(x, y) = \int e_\lambda(x', x'' - z'', y', y'' - w'')\varphi(z'')\overline{\varphi(w'')}dz''dw''\,,$$

where the integral is taken in the distributional sense. For f_λ, which is a function in $C^\infty(\Omega_\varphi \times \Omega_\varphi)$ by one of the statements in theorem 3, we shall prove (theorem 4) the following asymptotic relation:

$$f_\lambda(x, x) = (1 + o(1))f_{0,\lambda}(x, x) \qquad (\lambda \to +\infty),$$

where $x \in \Omega_\varphi$ and $f_{0,\lambda}$ is the function which corresponds to the unique self-adjoint realization of P in $L^2(R^n)$ in the same way as f_λ to A. The proof of theorem 4 also uses the estimate for the fundamental solution of $P(D) + \lambda$ mentioned above, and a result in Nilsson [8] about the asymptotic behaviour of the function $e(\lambda) = \int_{M(\xi)\leq\lambda} d\xi$, which will enable us to apply a Tauberian argument.

Since our results depend only on interior estimates, it is clear that they hold as well e.g. for a self-adjoint realization in $L^2(\Omega')$, where Ω' is an arbitrary open set containing Ω, of any differential operator in Ω', coinciding in Ω with our P. And since we only use estimates on R^n (and not complex integration) it is easy to modify our proof so that they apply also to naturally corresponding classes of pseudo-differential operators

We should also mention that results corresponding to ours were proved in Nilsson [8] for hypoelliptic differential operators (then, of course, without partial convolution).

The subject of this paper was suggested to me by Nils Nilsson. I wish to thank him for valuable advice and great help during my work.

2. Estimates for the fundamental solution

We begin by proving two elementary lemmas.

LEMMA 1. *Let M and N be polynomials on $R^{n'}$ such that M is hypoelliptic and N is strictly weaker than M. Then there are positive constants C and k such that*

$$|N(\xi')| \leq C\tau^{-k}(1 + |\xi'|)^{-k}(|M(\xi')| + \tau)$$

for all $\xi' \in R^{n'}$ and all real numbers $\tau \geq 1$.

Proof. Since M is hypoelliptic and N strictly weaker than M we have the following estimate: there exist positive constants K and q, with $q < 1$, such that

$$|N(\xi')| \leq K(|M(\xi')| + 1)^q \qquad (\xi' \in R^{n'}). \tag{3}$$

Also, there are positive constants K' and b such that

$$|M(\xi')| + 1 \geq K'(1 + |\xi'|)^b \qquad (\xi' \in R^{n'}). \tag{4}$$

For these estimates, see e.g. Hörmander [3] and [4]. By (3) and (4) we now have, for all $\xi' \in R^{n'}$ and all $\tau \geq 1$,

$$\frac{|N(\xi')|}{|M(\xi')|+\tau} \leq \frac{K(|M(\xi')|+1)^{(q+1)/2}}{|M(\xi')|+\tau} (|M(\xi')|+1)^{(q-1)/2} \leq$$
$$\leq C\tau^{(q-1)/2}(|M(\xi')|+1)^{(q-1)/2} \leq C\tau^{(q-1)/2}(|\xi'|+1)^{b(q-1)/2}.$$

(Here and in the sequel C denotes different positive constants.) Taking k as the least of the numbers $(1-q)/2$ and $b(1-q)/2$ the lemma is proved.

LEMMA 2. *Let Ω be an open set in R^n and let $T \in \mathscr{D}'(\Omega)$, that is, T is a distribution on Ω. Suppose that for any $\varphi \in C_0^\infty(R^{n''})$ with* supp φ *contained in a fixed compact set F the partial convolution $T *'' \varphi$ is an essentially bounded function where it is naturally defined, i.e. on Ω_φ defined as in* (2), *and that for some $p \geq 0$ we have*

$$(\text{ess}) \sup_{\Omega_\varphi} |T *'' \varphi(x)| \leq C(T) \sum_{|\alpha''| \leq p} \sup |D^{\alpha''}\varphi| \tag{5}$$

for all such functions φ. Let ψ be a given function in $C^\infty(\Omega)$. Then, for any relatively compact open subset Ω' of Ω, the restriction S of ψT to Ω' is such that $S *'' \varphi$ is in $L^\infty(\Omega'_\varphi)$ and satisfies (5) *for all $\varphi \in C_0^\infty(R^{n''})$ with support in F, with $C(S) = AC(T)$ and, of course, with Ω'_φ instead of Ω_φ. Here the number A does not depend on T, as long as p remains unchanged.*

Proof. If T is a continuous function the partial convolution is given by an ordinary integral over $R^{n''}$, and the proof is quite straightforward. The general case is proved by regularization. The details will be left for the reader.

We now come to the main lemmas.

LEMMA 3. *Let P be a real partially hypoelliptic polynomial on R^n, given by* (1) *with M positive. Let λ be a non-real complex number and let g_λ denote the temperate fundamental solution of the operator $P(D)+\lambda$. Further, let $\varphi \in C_0^\infty(R^{n''})$. Then the partial convolution $g_\lambda *'' \varphi$ is infinitely differentiable in R^n outside the set $S_\varphi = \{0\} \times \operatorname{supp} \varphi$, and for any multiorder α, any positive numbers N and L and any compact subset K of $R^n \setminus S_\varphi$ we have*

$$\sup_K |D^\alpha g_\lambda *'' \varphi(x)| \leq C(|\lambda|^{-N} + |\text{Im}\,\lambda|^{-b(N)}|\lambda|^{-L}) \tag{6}$$

for $|\lambda| \geq 1$, $\text{Re}\,\lambda > 0$, $0 < |\text{Im}\,\lambda| \leq 1$. Here C is a positive number and $b(N)$ is a polynomial in N of degree one with positive coefficients. The constant C may depend on φ, N, L, K and α, while $b(N)$ depends only on α.

Proof. Let B be a positive definite, homogenous polynomial on R^n of degree f. Since

$$g_\lambda = \mathcal{F}^{-1}\left(\frac{1}{P(\xi)+\lambda}\right)$$

(where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform) we have

$$(BD^\alpha g_\lambda) *'' \varphi = \mathcal{F}^{-1}\left(\left(B(-D)\,\frac{\xi^\alpha}{P(\xi)+\lambda}\right)\hat{\varphi}(\xi'')\right), \tag{7}$$

where $\hat{\varphi}$ is the Fourier transform of φ in n'' variables only. We are going to estimate the L^1-norm of $\big(B(-D)\xi^\alpha/(P(\xi)+\lambda)\big)\hat{\varphi}(\xi'')$.

It suffices to prove that

$$\int_{R^{n'}} \left| B(-D)\,\frac{\xi^\alpha}{P(\xi)+\lambda}\right| d\xi' \le C(1+|\xi''|)^u(|\lambda|^{-N} + |\text{Im}\,\lambda|^{-b(N)}|\lambda|^{-L}) \quad (\xi'' \in R^{n''}) \tag{8}$$

for some u. For if F is a compact subset of $R^{n''}$ that contains $\text{supp}\,\varphi$ we have for all $t > 0$ the estimate

$$|\xi''|^t|\hat{\varphi}(\xi'')| \le C_F \sum_{|\alpha''|\le p} \sup |D^{\alpha''}\varphi| \qquad (\xi'' \in R^{n''})$$

with some $p \ge 0$ and some constant C_F that depends on F. Thus by (7) and (8)

$$\sup_{x\in R^n} |(BD^\alpha g_\lambda) *'' \varphi(x)| \le C(|\lambda|^{-N} + |\text{Im}\,\lambda|^{-b(N)}|\lambda|^{-L} \sum_{|\alpha''|\le p} \sup |D^{\alpha''}\varphi|)\,,$$

and using lemma 2 with $\psi(x) = 1/B(x)$ it follows that $g_\lambda *'' \varphi$ is in $C^\infty(R^n \setminus S_\varphi)$ and satisfies (6).

We now proceed to prove (8). From the rules of differentiation it follows that $B(-D)\xi^\alpha/(P(\xi)+\lambda)$ is a linear combination of terms

$$\frac{D^{\gamma_0}\xi^\alpha \cdot D^{\gamma_1}(P(\xi)+\lambda)\cdot D^{\gamma_2}(P(\xi)+\lambda)\cdot \ldots \cdot D^{\gamma_f}(P(\xi)+\lambda)}{(P(\xi)+\lambda)^{f+1}} \tag{9}$$

where $\sum_0^f |\gamma_i| = f$. In order to estimate these terms we first derive some estimates from lemma 1.

Since P is partially hypoelliptic with respect to ξ' we write P on the form (1). With $\gamma = (\gamma', \gamma'')$ and $|\gamma| \neq 0$ we then have

$$D^\gamma(P(\xi)+\lambda) = D^{\gamma'}M(\xi')D^{\gamma''}1 + \sum_{j=1}^{r} D^{\gamma''}Q_j(\xi'')D^{\gamma'}M_j(\xi') \qquad (\xi \in R^n)\,.$$

Then if $|\lambda| \ge 1$, according to lemma 1,

$$|D^\gamma(P(\xi)+\lambda)| \le C(1+|\xi''|)^a|\lambda|^{-k}(1+|\xi'|)^{-k}(|M(\xi')|+|\lambda|) \qquad (\xi \in R^n)\,, \tag{10}$$

for some $C > 0$ and $a > 0$, since $D^{\gamma'}M(\xi')$ and $D^{\gamma'}M_j(\xi')$, $j = 1, \ldots, r$, are all strictly weaker than M.

We also need to estimate the denominator in (9). We have

$$P(\xi) + \lambda = M(\xi') + \lambda + \sum_{j=1}^{r} Q_j(\xi'')M_j(\xi') \quad (\xi \in R^n)\,.$$

Now there exist constants $a > 1$ and $K > 1$ such that, if $|\lambda| \geq 1$,

$$|\sum Q_j(\xi'')M_j(\xi')| \leq K(1 + |\xi''|)^a|\lambda|^{-k}(1 + |\xi'|)^{-k}(|M(\xi')| + |\lambda|) \qquad (\xi \in R^n)\,.$$

This follows from lemma 1 just as above. Then, if $\operatorname{Re} \lambda > 0$,

$$\begin{aligned} |P(\xi) + \lambda| &\geq |M(\xi') + \lambda| - K(1 + |\xi''|)^a|\lambda|^{-k}(1 + |\xi'|)^{-k}(|M(\xi')| + |\lambda|) \geq \\ &\geq (|M(\xi')| + |\lambda|)\left(\frac{1}{\sqrt{2}} - K(1 + |\xi''|)^a|\lambda|^{-k}(1 + |\xi'|)^{-k}\right). \end{aligned} \tag{11}$$

From (11) it follows that if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$, and if we have

$$|\lambda|(1 + |\xi'|) > (2K(1 + |\xi''|))^{a/k} \tag{12}$$

then for some positive constant C

$$|P(\xi) + \lambda| \geq C(|M(\xi')| + |\lambda|)\,. \tag{13}$$

If (12) is not valid we are going to use that

$$|P(\xi) + \lambda| \geq |\operatorname{Im} \lambda| \tag{14}$$

for all λ and all ξ. This is trivial since $P(\xi)$ is real.

In the sequel we will consider ξ'' as fixed for a while. Let us first study the case when (12) is valid. Then it follows from (10) and (13) that

$$\left|\frac{D^{\gamma_i}(P(\xi) + \lambda)}{P(\xi) + \lambda}\right| \leq C(1 + |\xi''|)^a|\lambda|^{-k}(1 + |\xi'|)^{-k}\,, \quad i = 1, \ldots, f\,, \tag{15}$$

if $|\gamma_i| \neq 0$ and $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$. We further have

$$\left|\frac{D^{\gamma_0}\xi^\alpha}{P(\xi) + \lambda}\right| \leq C(1 + |\xi''|)^{|\alpha''|}(1 + |\xi'|)^{|\alpha'|} \tag{16}$$

if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$.

Let us now return to the term (9), which we rewrite in the following way:

$$\frac{D^{\gamma_0}\xi^\alpha}{P(\xi) + \lambda} \cdot \frac{D^{\gamma_1}(P(\xi) + \lambda)}{P(\xi) + \lambda} \cdot \ldots \cdot \frac{D^{\gamma_f}(P(\xi) + \lambda)}{P(\xi) + \lambda}\,. \tag{17}$$

Since P is a polynomial this term is different from zero only if $|\gamma_i| \leq d$, $i = 1, \ldots, f$, and $|\gamma_0| \leq |\alpha|$, where d is the degree of P. Let l denote the integer part of $(f - |\alpha|)/d$. Then it follows that if the term (17) is non-zero at least l of the numbers $|\gamma_1|, \ldots, |\gamma_f|$ are different from zero. Consequently, (17) can be majorized by

$$C(1 + |\xi''|)^{a''}|\lambda|^{-kl}(1 + |\xi'|)^{-kl+|\alpha'|}$$

if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$. Here a'' is a positive constant depending on α and f. By choosing f large enough we can accomplish that $kl \geq N$ and $kl - |\alpha'| \geq n' + 1$. Then we have

$$\left| B(-D) \frac{\xi^\alpha}{P(\xi) + \lambda} \right| \leq C(1 + |\xi''|)^{a''}|\lambda|^{-N}(1 + |\xi'|)^{-n'-1} .$$

Integrating with respect to ξ' we obtain (with a new constant C)

$$\int_{A_\lambda} \left| B(-D) \frac{\xi^\alpha}{P(\xi) + \lambda} \right| d\xi' \leq C(1 + |\xi''|)^{a''}|\lambda|^{-N} , \tag{18}$$

still supposing that $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$. Here A_λ denotes the set in $R^{n'}$ where (2) is valid.

Let us now turn to the case when the opposite inequality to (12) is valid. Since $|\lambda| \geq 1$ we then have $|\xi'| \leq (2K(1 + |\xi''|))^{a/k}$ and $|\lambda| \leq (2K(1 + |\xi''|))^{a/k}$. We now use (14) when estimating the denominator. Instead of (15) we then have, for $|\gamma_i| \neq 0$ and if $\operatorname{Im} \lambda \neq 0$,

$$\left| \frac{D^{\gamma_i}(P(\xi) + \lambda)}{P(\xi) + \lambda} \right| \leq \frac{C(1 + |\xi''|)^b}{|\operatorname{Im} \lambda|} , \quad i = 1, \ldots , f ,$$

with some $C > 0$ and $b > 0$. Instead of (16) we get, if $\operatorname{Im} \lambda \neq 0$,

$$\left| \frac{D^{\gamma_0}\xi^\alpha}{P(\xi) + \lambda} \right| \leq \frac{C(1 + |\xi''|)^{b'}}{|\operatorname{Im} \lambda|}$$

with some $b' > 0$, depending on α. Hence we can majorize (17) by $C(1 + |\xi''|)^{b''}|\operatorname{Im} \lambda|^{-f-1}$ for some positive constants C and b'', if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$, $0 < |\operatorname{Im} \lambda| \leq 1$.

When choosing f above so that $kl \geq N$, $kl - |\alpha'| \geq n' + 1$, it is sufficient to let f be a polynomial in N of degree 1 with positive coefficients. Put $b(N) = f + 1$. Then

$$\left| B(-D) \frac{\xi^\alpha}{P(\xi) + \lambda} \right| \leq C(1 + |\xi''|)^{b''}|\operatorname{Im} \lambda|^{-b(N)}$$

for some $C > 0$. Hence, for some $s > 0$ and some (new) $C > 0$

$$\begin{aligned} \int_{R^{n'} \setminus A_\lambda} \left| B(-D) \frac{\xi^\alpha}{P(\xi) + \lambda} \right| d\xi' &\leq C(1 + |\xi''|)^s|\operatorname{Im} \lambda|^{-b(N)} \leq \\ &\leq C(1 + |\xi''|)^{r(L)}|\operatorname{Im} \lambda|^{-b(N)}|\lambda|^{-L} . \end{aligned} \tag{19}$$

Here $r(L)$ is a positive constant depending on L.

Now, adding (18) and (19) we get (8), and the proof is complete.

LEMMA 4. *Let the polynomial P be as in lemma 3 and in addition assume that for some $m \geq 0$ $(1 + |\xi'|)^{m+n'+1}M(\xi')^{-1}$ is bounded outside some compact set in $R^{n'}$. Then, for $\operatorname{Im} \lambda \neq 0$, the partial convolution $g_\lambda *'' \varphi$ is in $C^m(R^n)$, and for $|\alpha| \leq m$ and an arbitrary number $L > 0$ we have, for some constant $C > 0$*

$$\sup_{R^n} |D^\alpha g_\lambda *'' \varphi(x)| \leq C(1 + |\operatorname{Im} \lambda|^{-1}|\lambda|^{-L}) \tag{20}$$

if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda \neq 0$.

Proof. The proof is done in much the same way as the proof of lemma 3. However, we do not use the polynomial B. Thus, since

$$D^\alpha g_\lambda *'' \varphi = \mathcal{F}^{-1}\left(\frac{\xi^\alpha \hat{\varphi}(\xi'')}{P(\xi) + \lambda}\right),$$

we start by estimating $\xi^\alpha/(P(\xi) + \lambda)$. For the denominator we use the inequalities (13) and (14) respectively. When (12) is valid we then have, if $|\lambda| \geq 1$ and $\operatorname{Re} \lambda > 0$,

$$|P(\xi) + \lambda| \geq C|M(\xi')| \geq C(1 + |\xi'|)^{m+n'+1},$$

where the last inequality depends on the assumptions about M. Then, if $|\alpha| \leq m$,

$$\int_{A_\lambda} \left|\frac{\xi^\alpha}{P(\xi) + \lambda}\right| d\xi' \leq C|\xi''|^{|\alpha''|}$$

if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$. A_λ still denotes the subset of $R^{n'}$ where (12) holds.

If (12) is not valid we estimate the denominator by $|\operatorname{Im} \lambda|$. Then for some $c > 0$

$$\int_{R^{n'} \setminus A_\lambda} \left|\frac{\xi^\alpha}{P(\xi) + \lambda}\right| d\xi' \leq C|\xi''|^{|\alpha''|+c}|\operatorname{Im} \lambda|^{-1} \leq C(1 + |\xi''|)^{q(L)}|\operatorname{Im} \lambda|^{-1}|\lambda|^{-L}.$$

Here $q(L)$ is a positive constant depending on L.

Adding, multiplying by $|\hat{\varphi}(\xi'')|$ and integrating with respect to ξ'' we obtain the estimate

$$\int \left|\frac{\xi^\alpha \hat{\varphi}(\xi'')}{P(\xi) + \lambda}\right| d\xi \leq C(1 + |\operatorname{Im} \lambda|^{-1}|\lambda|^{-L}) \tag{21}$$

for some (new) constant $C > 0$, if $|\lambda| \geq 1$, $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda \neq 0$. This proves that $\mathcal{F}(D^\alpha g_\lambda *'' \varphi)$ is in $L^1(R^n)$, thus that $D^\alpha g_\lambda *'' \varphi$ is continuous. We also get the estimate (20) from (21).

Remark 1. Assume that the polynomial P is bounded from below. Then we have $|P(\xi) + \lambda| \geq 1$ if λ is real and large enough. Using this estimate instead of (14) in the proof of lemma 3 we get the following estimate: for any multiorder α, any positive number N and any compact subset K of $R^n \setminus S_\varphi$

$$\sup_K |D^\alpha g_\lambda *'' \varphi(x)| \leq C|\lambda|^{-N}$$

for λ real and large enough. In the same way we can modify the proof of lemma 4, and instead of the estimate (20) get

$$\sup_{R^n} |D^\alpha g_\lambda *'' \varphi(x)| \leq C$$

if λ is real and large enough.

Remark 2. Probably it is possible to prove the estimates of lemma 3 and 4 with the last term on the right hand side omitted, though the proof is more complicated. Our method also admits of immediate generalization to pseudo-differential operators.

3. Estimates for the spectral resolution and the spectral kernel

Let Ω be a non-empty open subset of R^n. If $\varphi \in C_0^\infty(R^{n''})$, let as before the set Ω_φ be defined by (2).

Let P be a real-valued polynomial on R^n which is partially hypoelliptic with respect to ξ'. Suppose that $M(\xi') \to +\infty$ as $|\xi'| \to \infty$, where M is the hypoelliptic polynomial in (1).

Let A be a self-adjoint realization in $L^2(\Omega)$ of the operator $P(D)$. Let

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

be the spectral resolution of A, where we suppose that $\lambda \mapsto E(\lambda)$ is continuous to the left. We then have the following theorem.

THEOREM 1. *For every real* λ *and every* $u \in L^2(\Omega)$ *the function*

$$\tilde{E}(\lambda)u = E(\lambda)u *'' \varphi$$

is in $C^\infty(\Omega_\varphi)$. *To any number* $N > 0$, *any multi-order* α *and any compact subset* K *of* Ω_φ *there are positive numbers* C *and* a, a *independent of* N, *such that*

$$\sup_K |D^\alpha \tilde{E}(\lambda)u(x)| \leq \begin{cases} C(|\lambda| + 1)^{-N}\|u\| & \text{for } \lambda \leq 0 \\ C(\lambda + 1)^a\|u\| & \text{for } \lambda \geq 0 . \end{cases} \tag{22}$$

Proof. We shall follow the method of Nilsson ([7], theorem 3). Put $B = A^r$, where $r \in Z_+$. Then B is a self-adjoint realization of the operator $P(D)^r$. Let

$$B = \int_{-\infty}^{\infty} \lambda dE_r(\lambda)$$

be the spectral resolution of B. Then $E(\lambda) = E_r(\lambda^r)$ if r is odd, which we assume from now on in this proof.

It is easily shown by means of (1) that if $P(\xi)$ is partially hypoelliptic with respect to ξ', then so is $P(\xi)^r$ when $r \in Z_+$. Moreover, the hypoelliptic polynomial corresponding to $P(\xi)^r$ is $M(\xi')^r$, where $M(\xi')$ is the hypoelliptic polynomial corresponding to $P(\xi)$. Thus the conditions of lemma 4 are fulfilled for $P(\xi)^r$ if we only take r large enough. In the sequel we keep r fixed, and apply lemma 3 and lemma 4 to $P(\xi)^r$ instead of $P(\xi)$.

Now let a be a point in the open set Ω_φ. Denote, when μ is non-real, by $g_{r,\mu}$ the temperate fundamental solution of $P(D)^r + \mu$. We have the following identity: if u is in the domain D_B of B then

$$u *'' \varphi(x) = \int B_{r,\mu}(x, y)u(y)dy + \int h_{r,\mu}(x, y)\psi(y)(B + \mu)u(y)dy \tag{23}$$

for all x in a neighbourhood O_a of a. Here $\psi \in C_0^\infty(\Omega)$ and $\psi(y) = 1$ for all y in Ω with the property that $x - y \in S_\varphi$ for all $x \in O_a$ (such a function exists if O_a is small enough). Further

$$h_{r,\mu}(x, y) = g_{r,\mu} *'' \varphi(x - y) \tag{24}$$

and

$$B_{r,\mu}(x, y) = (\psi(y)P(-D_y)^r - P(-D_y)^r\psi(y))h_{r,\mu}(x, y) . \tag{25}$$

This identity may be proved by a simple transcription of the identity

$$(\psi u) *'' \varphi = ((B + \mu)\psi u) * (g_{r,\mu} *'' \varphi) \qquad (u \in \mathcal{D}'(\Omega)) .$$

The details will be omitted. A corresponding identity without the partial convolution can be found e.g. in Nilsson [6].

Let F_1 and F_2 be open relatively compact subsets of Ω such that

$$\bar{F}_2 \subset \{x : \psi(x) = 1\}$$
$$F_1 \supset \operatorname{supp} \psi .$$

Let F_3 be an open relatively compact subset of Ω_φ such that $\bar{F}_3 \subset O_a$. Then the identity (23) is valid for $x \in F_3$. Now $B_{r,\mu}(x, y) = 0$ when y is outside some compact subset of $F_1 \setminus F_2$. Furthermore, if $x \in F_3$ and y belongs to this compact subset of $F_1 \setminus F_2$ then $x - y$ belongs to a compact subset of $R^n \setminus S_\varphi$ if only F_2 is chosen large enough. Thus, using lemma 3 and the notations there, we have for arbitrary $N > 0$ and $L > 0$

$$\sup_{\substack{x \in F_3 \\ y \in F_1}} |B_{r,\mu}(x, y)| \leq C(|\mu|^{-N} + |\mathrm{Im}\,\mu|^{-b(N)}|\mu|^{-L}) \tag{26}$$

if $|\mu| \geq 1$, $\mathrm{Re}\,\mu > 0$, $0 < |\mathrm{Im}\,\mu| \leq 1$.

In the same way it follows from lemma 4 that for arbitrary $L > 0$

$$\sup_{\substack{x \in \Omega_\varphi \\ y \in \Omega^\varphi}} |h_{r,\mu}(x, y)| \leq C(1 + |\mathrm{Im}\,\mu|^{-1}|\mu|^{-L}) \tag{27}$$

for $|\mu| \geq 1$, $\mathrm{Re}\,\mu > 0$, $\mathrm{Im}\,\mu \neq 0$. Further, $B_{r,\mu}$ and $h_{r,\mu}$ are continuous.

Using the Cauchy-Schwarz' inequality and the estimates (26) and (27) in the identity (23) we obtain that $u *'' \varphi$ is continuous and

$$|u *'' \varphi(x)| \leq \|B_{r,\mu}(x, \cdot)\| \cdot \|u\| + \|h_{r,\mu}(x, \cdot)\psi\| \cdot \|(B + \mu)u\| \leq \tag{28}$$

$$\leq C(|\mu|^{-N} + |\mathrm{Im}\,\mu|^{-b(N)}|\mu|^{-L})\|u\| + C(1 + |\mathrm{Im}\,\mu|^{-1}|\mu|^{-L})\|(B + \mu)u\| \qquad (x \in F_3)\,,$$

where the constant C does not depend on x.

We now introduce the spaces $H_r(\lambda_1, \lambda_2)$ defined for $\lambda_1 < \lambda_2$ by

$$H_r(\lambda_1, \lambda_2) = (E_r(\lambda_2) - E_r(\lambda_1))L^2(\Omega)\,. \tag{29}$$

Let λ be a real number less than -1, and suppose that $f \in H_r(\lambda - \varepsilon, \lambda)$, where ε is a number between 0 and 1 to be chosen later. We then have $f \in D_B$. Furthermore

$$(B + \mu)f = \int_{\lambda-\varepsilon<\nu\leq\lambda} (\nu + \mu)dE_r(\nu)f$$

so that

$$\|(B + \mu)f\|^2 = \int_{\lambda-\varepsilon<\nu\leq\lambda} |\nu + \mu|^2 d\|E_r(\nu)f\|^2 \leq (|\lambda + \mu| + \varepsilon)^2\|f\|^2\,.$$

Hence by (28)

$$\sup_{x \in F_3} |f *'' \varphi(x)| \leq$$

$$\leq C(|\mu|^{-N} + |\mathrm{Im}\,\mu|^{-b(N)}|\mu|^{-L} + |\lambda + \mu| + \varepsilon + (|\lambda + \mu| + \varepsilon)|\mathrm{Im}\,\mu|^{-1}|\mu|^{-L})\|f\|$$

if $|\mu| \geq 1$, $\mathrm{Re}\,\mu > 0$, $0 < |\mathrm{Im}\,\mu| \leq 1$. Taking $\mu = -\lambda + i\varepsilon$ we have

$$\sup_{x \in F_3} |f *'' \varphi(x)| \leq C(|\lambda|^{-N} + \varepsilon^{-b(N)}|\lambda|^{-L} + 2\varepsilon + 2|\lambda|^{-L})\|f\| \tag{30}$$

if $\lambda \leq -1$.

Now let $f \in H_r(\lambda - k\varepsilon, \lambda)$, $k \in Z_+$. Then $f = f_1 + \ldots + f_k$, where $f_j \in H_r(\lambda - j\varepsilon,\ \lambda - (j - 1)\varepsilon)$ for $j = 1, \ldots, k$. These spaces are all orthogonal to each other. Thus we get by (30) and the Cauchy-Schwarz' inequality

$$\sup_{x\in F_3} |f *'' \varphi(x)| \leq C(|\lambda|^{-N} + \varepsilon^{-b(N)}|\lambda|^{-L} + 2\varepsilon + 2|\lambda|^{-L}) \sum_{j=1}^{k} \|f_j\| \leq$$
$$\leq C(|\lambda|^{-N} + \varepsilon^{-b(N)}|\lambda|^{-L} + 2\varepsilon + 2|\lambda|^{-L})k^{1/2}\|f\|$$

if $\lambda \leq -1$.

We now choose $\varepsilon = |\lambda|^{-N}$ and k equal to the integral part of $2|\lambda|^N$. If we choose the number L large enough we then have for some positive C

$$\sup_{x\in F_3} |f *'' \varphi(x)| \leq C|\lambda|^{-N/2}\|f\| \tag{31}$$

if λ is large and negative.

Since $k\varepsilon > 1$ for large $|\lambda|$ it follows that (31) is valid for all $f \in H_r(\lambda - 1, \lambda)$, if λ is large and negative.

Now let $u \in L^2(\Omega)$. Then $(E_r(\lambda - \nu) - E_r(\lambda - \nu - 1))u \in H_r(\lambda - \nu - 1, \lambda - \nu)$ for all natural numbers ν. We have

$$E_r(\lambda)u = \sum_{\nu=0}^{\infty} (E_r(\lambda - \nu) - E_r(\lambda - \nu - 1))u ,$$

the series converging strongly. Hence

$$E_r(\lambda)u *'' \varphi = \sum_{\nu=0}^{\infty} (E_r(\lambda - \nu) - E_r(\lambda - \nu - 1))u *'' \varphi .$$

By (31) we thus have that $E_r(\lambda)u *'' \varphi$ is continuous and

$$\sup_{x\in F_3} |E_r(\lambda)u *'' \varphi(x)| \leq C \sum_{\nu=0}^{\infty} |\lambda - \nu|^{-N/2}\|u\| \leq C|\lambda|^{-(N-4)/2}\|u\| ,$$

if λ is large and negative.

Since $E(\lambda) = E_r(\lambda^r)$ we finally have

$$\sup_{x\in F_3} |\tilde{E}(\lambda)u(x)| \leq C|\lambda|^{-r(N-4)/2}\|u\| .$$

If K is a compact subset of Ω_φ we use the Borel-Lebesgue covering theorem to obtain the same estimate for $\sup_K |\tilde{E}(\lambda)u(x)|$. Since $r(N-4)/2$ can be made arbitrarily large by choosing N large enough, we have then proved the estimate (22) for λ large and negative in the case $|\alpha| = 0$. However we can differentiate (in the distribution sense) in the identity (23), thus obtaining an analogous identity for $D^\alpha u *'' \varphi$. Reasoning as above we can then prove that $D^\alpha\tilde{E}(\lambda)u$ is continuous in Ω_φ and satisfies an inequality

$$\sup_K |D^\alpha\tilde{E}(\lambda)u(x)| \leq C|\lambda|^{-N}\|u\| \qquad (u \in L^2(\Omega)) \tag{32}$$

for all $\lambda \leq$ some λ_0 (depending on α and N).

To estimate $D^\alpha\tilde{E}(\lambda)u$ also when λ is not large and negative we use lemma 4 and the (differentiated) identity (23) with a fixed fundamental solution $g_{r,\kappa}$ to derive the following a priori estimate: for every multi-order α and every compact subset K of Ω_φ there exists a number C such that, if $r > r_0(\alpha)$

$$\sup_K |D^\alpha v *'' \varphi(x)| \leq C(\|P^r v\| + \|v\|) \tag{33}$$

for all $v \in L^2(\Omega)$ such that $P^r v \in L^2(\Omega)$. We also get that then $D^\alpha v *'' \varphi$ is continuous in Ω_φ. If $v \in H_1(\lambda_0, \lambda) = H_r(\lambda_0^r, \lambda^r)$ with $\lambda \geq \lambda_0$ and λ_0 defined as in (32) we have $P^r v = Bv \in L^2(\Omega)$. Thus $D^\alpha v *'' \varphi$ is continuous in Ω_φ for any α and satisfies

$$\sup_K |D^\alpha v *'' \varphi(x)| \leq C(\|Bv\| + \|v\|) \leq C(|\lambda^r| + |\lambda_0^r| + 1)\|v\| \leq C(|\lambda|^r + 1)\|v\| \,. \tag{34}$$

Writing $\tilde{E}(\lambda)u = \tilde{E}(\lambda_0)u + (\tilde{E}(\lambda) - \tilde{E}(\lambda_0))u$ and applying (32) to the first term and (34) to the last we get that $D^\alpha \tilde{E}(\lambda)u$ is continuous for any real λ and satisfies (22). Of course, it now also follows that $\tilde{E}(\lambda)u$ is in $C^\infty(\Omega_\varphi)$. The theorem is proved.

It follows from the Schwartz kernel theorem that for any real λ the projection $E(\lambda)$ is given on $C_0^\infty(\Omega)$ by a kernel e_λ, being a distribution on $\Omega \times \Omega$:

$$E(\lambda)u(x) = \int_\Omega e_\lambda(x, y)u(y)dy \qquad (u \in C_0^\infty(\Omega)) \,,$$

where this formula, of course, is taken in the distributional sense. Since $E(\lambda)$ is selfadjoint e_λ is Hermitian. We now give two results on the (partial) regularity of e_λ and on the behaviour of e_λ as $\lambda \to -\infty$ and as $\lambda \to +\infty$.

THEOREM 2. *Let* $\varphi \in C_0^\infty(R^{n''})$ *and put*

$$F_\lambda(x, y) = e_\lambda(\cdot, y) *'' \varphi(x) = \int e_\lambda(x', x'' - z'', y)\varphi(z'')dz'' \,;$$

F_λ is thus a distribution on $\Omega_\varphi \times \Omega$. Then, for any multiorder α and any real λ, $D_x^\alpha F_\lambda$ can be chosen as a measurable function $F_{\lambda,\alpha}$ on $\Omega_\varphi \times \Omega$ such that for any $x \in \Omega_\varphi$ the function $F_{\lambda,\alpha}(x, \cdot)$ is in $L^2(\Omega)$ and depends continuously on x in the L^2-norm, and such that

$$D^\alpha \tilde{E}(\lambda)u(x) = \int_\Omega F_{\lambda,\alpha}(x, y)u(y)dy \,. \tag{35}$$

Further, to any compact subset K of Ω_φ, any multi-order α and any number $N > 0$ there are numbers C and a, a not depending on N, such that

$$\sup_{x \in K} \|F_{\lambda,\alpha}(x, \cdot)\| \leq \begin{cases} C(|\lambda| + 1)^{-N} & (\lambda \leq 0) \\ C(\lambda + 1)^a & (\lambda \geq 0) \,. \end{cases} \tag{36}$$

Proof. Consider for fixed λ, α and x the mapping

$$L^2(\Omega) \ni u \mapsto D^\alpha \tilde{E}(\lambda)u(x) \in C .$$

By the representation theorem for Hilbert spaces and the estimates of theorem 1 it follows immediately that to any λ, α and x there is a kernel $F_{\lambda,\alpha}(x, \cdot)$ satisfying (35) and (36). Estimating $D^\alpha \tilde{E}(\lambda)u(x) - D^\alpha \tilde{E}(\lambda)u(z)$ with the mean value theorem and the estimates of theorem 1, and then again using the representation theorem for Hilbert spaces, it follows that $F_{\lambda,\alpha}(x, \cdot)$ is a continuous function of x in the $L^2(\Omega)$-norm. E.g. by making a sub-division of Ω_φ into suitable small sets and approximating the function $x \mapsto F_{\lambda,\alpha}(x, \cdot)$ with a function constant on these sets, it follows that $F_{\lambda,\alpha}$ can be taken as a measurable function on $\Omega_\varphi \times \Omega$ in such a way that (35) and (36) still hold for every $x \in \Omega_\varphi$. Since clearly $D_x^\alpha F_\lambda$ gives the same mapping as $F_{\lambda,\alpha}$, though in the distributional sense, it also follows that $F_{\lambda,\alpha}$ is a representative of the distribution $D_x^\alpha F_\lambda$.

THEOREM 3. *Let* $\varphi \in C_0^\infty(R^{n''})$ *and put*

$$f_\lambda(x, y) = \int e_\lambda(x', x'' - z'', y', y'' - w'')\varphi(z'')\overline{\varphi(w'')}dz''dw'' ,$$

defining f_λ *as a distribution on* $\Omega_\varphi \times \Omega_\varphi$, *which is clearly Hermitian since* e_λ *is. Then, for any real* λ, f_λ *is in* $C^\infty(\Omega_\varphi \times \Omega_\varphi)$, *and to any compact subset* K *of* $\Omega_\varphi \times \Omega_\varphi$, *any multi-orders* α *and* β, *and any number* $N > 0$ *there are numbers* C *and* a, a *not depending on* N, *such that*

$$\sup_K |D_x^\alpha D_y^\beta f_\lambda(x, y)| \leq \begin{cases} C(|\lambda| + 1)^{-N} & (\lambda \leq 0) \\ C(\lambda + 1)^a & (\lambda \geq 0) . \end{cases} \tag{37}$$

Proof. Let $H_1(-\infty, \lambda)$ and $H_1(\lambda, +\infty)$ be defined by (29). If $F_{\lambda,\alpha}$ is defined as in theorem 2 we have that $\overline{F_{\lambda,\alpha}(x, \cdot)} \in H_1(-\infty, \lambda)$ for every $x \in \Omega_\varphi$, since

$$\int_\Omega F_{\lambda,\alpha}(x, y)u(y)dy = D^\alpha \tilde{E}(\lambda)u(x) = 0$$

for every $u \in H_1(\lambda, +\infty)$. Hence by theorem 1 we have $F_{\lambda,\alpha}(x, \cdot) *'' \overline{\varphi} \in C^\infty(\Omega_\varphi)$. It also follows from theorem 1 and theorem 2 that for any multi-order β the function $D^\beta(F_{\lambda,\alpha}(x, \cdot) *'' \overline{\varphi})$ depends continuously on $x \in \Omega_\varphi$ in the uniform norm. Hence the function $f_{\lambda,\alpha,\beta}(x, y) = D^\beta(F_{\lambda,\alpha}(x, \cdot) *'' \overline{\varphi})(y)$ is continuous on $\Omega_\varphi \times \Omega_\varphi$. From its definition it is easily seen to be a representative of $D_x^\alpha D_y^\beta f_\lambda$, and from the theorems 1 and 2 we get, since $\overline{F_{\lambda,\alpha}(x, \cdot)} \in H_1(-\infty, \lambda)$, the estimate (37).

4. An asymptotic result for the spectral kernel

LEMMA 5. *Let the polynomial P be as in lemma 4, and assume in addition that it is bounded from below. Let $\varphi \in C_0^\infty(R^{n''})$ and put*

$$\gamma(\lambda) = \int \frac{|\hat{\varphi}(\xi'')|^2}{P(\xi) + \lambda}\, d\xi \quad \text{and} \quad g(\lambda) = \int \frac{1}{M(\xi') + \lambda}\, d\xi'$$

for λ sufficiently large. Then we have

$$\gamma(\lambda) = (1 + o(1))g(\lambda) \int |\varphi(x'')|^2 dx'' \qquad (\lambda \to +\infty) .$$

Proof. Using the representation (1) we have

$$P(\xi) + \lambda = M(\xi') + \lambda + \sum_{j=1}^{r} Q_j(\xi'')M_j(\xi') .$$

By lemma 1 there exist positive numbers a, k and C such that

$$\Big|\sum_{j=1}^{r} Q_j(\xi'')M_j(\xi')\Big| \leq C(1 + |\xi''|)^a \lambda^{-k}(1 + |\xi'|)^{-k}(M(\xi') + \lambda) \qquad (\xi \in R^n)$$

if $\lambda \geq 1$. Hence, for any $\varepsilon > 0$, the inequality

$$(1 - \varepsilon)(M(\xi') + \lambda) < P(\xi) + \lambda < (1 + \varepsilon)(M(\xi') + \lambda) \tag{38}$$

is valid if $C(1 + |\xi''|)^a < \lambda^k \varepsilon$ or if $C(1 + |\xi''|)^a < (1 + |\xi'|)^k \varepsilon$.

Let us first consider the case when $C(1 + |\xi''|)^a < \lambda^k \varepsilon$. Then we have by (38) that

$$\int \frac{1}{P(\xi) + \lambda}\, d\xi' \leq \frac{1}{1 - \varepsilon} \int \frac{1}{M(\xi') + \lambda}\, d\xi' = \frac{1}{1 - \varepsilon}\, g(\lambda) = \left(1 + \frac{\varepsilon}{1 - \varepsilon}\right) g(\lambda)$$

and analogously

$$\int \frac{1}{P(\xi) + \lambda}\, d\xi' \geq \left(1 - \frac{\varepsilon}{1 + \varepsilon}\right) g(\lambda) .$$

Thus

$$\begin{aligned} \left(1 - \frac{\varepsilon}{1 + \varepsilon}\right) g(\lambda) \int_{B_{\lambda,\varepsilon}} |\hat{\varphi}(\xi'')|^2 d\xi'' \leq \int_{R^{n'} \times B_{\lambda,\varepsilon}} \frac{|\hat{\varphi}(\xi'')|^2}{P(\xi) + \lambda}\, d\xi \leq \\ \leq \left(1 + \frac{\varepsilon}{1 - \varepsilon}\right) g(\lambda) \int_{B_{\lambda,\varepsilon}} |\hat{\varphi}(\xi'')|^2 d\xi'' , \end{aligned} \tag{39}$$

where $B_{\lambda,\varepsilon}$ denotes the subset of $R^{n''}$ where $C(1 + |\xi''|)^a < \lambda^k \varepsilon$.

We now let $C(1+|\xi''|)^a \geq \lambda^k \varepsilon$. Since P is bounded from below we have

$$P(\xi) + \lambda \geq 1 \qquad (\xi \in R^n) \tag{40}$$

if λ is large enough. Splitting the integral in two and using (38) and (40) respectively we get for some C' and $s > 0$

$$0 \leq \int \frac{1}{P(\xi)+\lambda}\, d\xi' \leq$$

$$\leq \frac{1}{1-\varepsilon} \int\limits_{C(1+|\xi''|)^a < (1+|\xi'|)^k \varepsilon} \frac{1}{M(\xi')+\lambda}\, d\xi' + \int\limits_{C(1+|\xi''|)^a \geq (1+|\xi'|)^k \varepsilon} 1\, d\xi' \leq$$

$$\leq \frac{1}{1-\varepsilon} C' + C'\varepsilon^{-s}(1+|\xi''|)^{as} \leq 3C'\varepsilon^{-s}(1+|\xi''|)^{as}$$

if $\varepsilon < \frac{1}{2}$ and if λ is large enough. Multiplying by $|\hat{\varphi}(\xi'')|^2$ and integrating we get

$$\begin{aligned} 0 \leq \int\limits_{R^{n'} \times (R^{n''} \setminus B_{\lambda,\varepsilon})} \frac{|\hat{\varphi}(\xi'')|^2}{P(\xi)+\lambda}\, d\xi &\leq 3C'\varepsilon^{-s} \int\limits_{C(1+|\xi''|)^a \geq \lambda^k \varepsilon} (1+|\xi''|)^{as} |\hat{\varphi}(\xi'')|^2 d\xi'' \leq \\ &\leq 3C'C^{2/k}\varepsilon^{-(sk+2)/k}\lambda^{-2} \int (1+|\xi''|)^{as+2a/k} |\hat{\varphi}(\xi'')|^2 d\xi''\,. \end{aligned} \tag{41}$$

Trivially, $g(\lambda) \geq K\lambda^{-1}$ for some positive K. Thus, if we choose $\varepsilon = \lambda^{-k/(2sk+4)}$ we get, adding (39) and (41),

$$\int \frac{|\hat{\varphi}(\xi'')|^2}{P(\xi)+\lambda}\, d\xi = (1+o(1))g(\lambda) \int |\hat{\varphi}(\xi'')|^2 d\xi'' \qquad (\lambda \to +\infty)\,.$$

We now use Plancherel's theorem on the integral to the right, and the proof is complete.

Let us further prove the following lemma about the functions f_λ and F_λ from theorems 2 and 3, both defined using the same function $\varphi \in C_0^\infty(R^{n''})$.

LEMMA 6. *For any $x \in \Omega_\varphi$ and any real λ the number $f_\lambda(x, x)$ is non-negative and increases with λ. We also have*

$$f_\lambda(x, x) = \|F_\lambda(x, \cdot)\|^2 \tag{42}$$

(where F_λ denotes the representative $F_{\lambda,0}$ of F_λ mentioned in theorem 2). Further, for any point (x, y) in $\Omega_\varphi \times \Omega_\varphi$ the function $\lambda \mapsto f_\lambda(x, y)$ is locally of bounded variation and for any real interval I we have

$$\operatorname*{var}_I f_\lambda(x, y) \leq (\operatorname*{var}_I f_\lambda(x, x))^{1/2} (\operatorname*{var}_I f_\lambda(y, y))^{1/2}\,. \tag{43}$$

Proof. (Cf. Bergendal [1], lemma 1.2.2. and 1.2.1.) Let λ and μ be real numbers with $\lambda < \mu$ and put $E_{\lambda,\mu} = E(\mu) - E(\lambda)$ and correspondingly for F_λ and f_λ. Since $\overline{F_{\lambda,\mu}(x,\cdot)} \in H_1(\lambda,\mu)$ we have $E_{\lambda,\mu}\overline{F_{\lambda,\mu}(x,\cdot)} = \overline{F_{\lambda,\mu}(x,\cdot)}$. Taking the partial convolution with φ in the y''-variables we get by (35), since $f_\lambda(x,y) = F_\lambda(x,\cdot) *'' \overline{\varphi}(y)$,

$$\int_\Omega \overline{F_{\lambda,\mu}(y,z)}F_{\lambda,\mu}(x,z)dz = f_{\lambda,\mu}(x,y)\,. \tag{44}$$

Taking $y = x$ we get (42) and see immediately that $f_\lambda(x,x)$ is non-negative and increases with λ.

We now consider an arbitrary subdivision of the interval I, and apply (44) to every subinterval. Using Cauchy's and Schwarz' inequalities we then get (43). The lemma is proved.

Now let $B = A^r$, where r is an even integer > 0. Thus B is a positive self-adjoint operator in $L^2(\Omega)$. Let $E_r(\lambda)$, $e_{r,\lambda}(x,y), \ldots$ correspond to B as $E(\lambda)$, $e_\lambda(x,y), \ldots$ to A. It follows from theorem 3, since $f_{r,\lambda^r} = f_\lambda - f_{-\lambda}$, that if r is sufficiently large then to any compact subset K of $\Omega_\varphi \times \Omega_\varphi$ there is a number C such that

$$\sup_K |f_{r,\lambda}(x,y)| \leq C(\lambda+1)^{1/2} \qquad (\lambda \geq 0)\,. \tag{45}$$

When $\mu > 0$, put

$$\gamma_{r,\mu}(x,y) = \int \frac{df_{r,\lambda}(x,y)}{\lambda+\mu}\,. \tag{46}$$

From (45) and theorem 3 it follows by an integration by parts that $\gamma_{r,\mu}$ is a continuous function on $\Omega_\varphi \times \Omega_\varphi$. Further $\gamma_{r,\mu}$ has the property that

$$((B+\mu)^{-1}(u *'' \tilde{\varphi})) *'' \varphi(x) = \int \gamma_{r,\mu}(x,y)u(y)dy \qquad (u \in C_0^\infty(\Omega_\varphi))\,, \tag{47}$$

where $\tilde{\varphi}(x'') = \overline{\varphi(-x'')}$. To see this we only have to approximate the Stieltjes integral in (46) with convenient Riemann sums, and use the a priori estimate (33) to estimate the error of the left hand side in the approximation.

Let us now investigate the asymptotic behaviour of $\gamma_{r,\mu}(x,x)$ as $\mu \to +\infty$. We shall then compare it with $\gamma_{0,r,\mu}(x,x)$, where $\gamma_{0,r,\mu}$ is the function

$$\gamma_{0,r,\mu}(x,y) = \int \frac{df_{0,r,\lambda}(x,y)}{\lambda+\mu}\,.$$

Here $f_{0,r,\lambda}, \ldots$ correspond to the unique realization in $L^2(R^n)$ of $P(D)^r$ as $f_{r,\lambda}, \ldots$ correspond to B. By a Fourier transformation we find that

$$f_{0,r,\lambda}(x, y) = \int\limits_{P(\xi)^r \le \lambda} |\hat{\varphi}(\xi'')|^2 \exp(2\pi i\langle x - y, \xi\rangle)d\xi$$

and

$$\gamma_{0,r,\mu}(x, y) = \int \frac{|\hat{\varphi}(\xi'')|^2 \exp(2\pi i\langle x - y, \xi\rangle)}{P(\xi)^r + \mu} d\xi .$$

Thus $\gamma_{0,r,\mu}(x, y) = g_{r,\mu} *'' \chi(x - y)$, where $g_{r,\mu}$ is the temperate fundamental solution of the operator $P(D)^r + \mu$, which we have estimated in lemma 3 and 4 (cf. remark 1), and where $\chi = \varphi *'' \tilde{\varphi} \in C_0^\infty(R^{n''})$.

Let $a \in \Omega_\varphi$ and let $\psi \in C_0^\infty(\Omega)$ be real and such that $\psi(y) = 1$ for y in a neighbourhood of the set $a - S_\varphi$, where $S_\varphi = \{0\} \times \operatorname{supp} \varphi$. Define $\Gamma_{r,\mu}$ by

$$\Gamma_{r,\mu}(x, \cdot) = \overline{(B + \mu)^{-1}\overline{B_{r,\mu}(x, \cdot)}} + \psi h_{r,\mu}(x, \cdot) , \tag{48}$$

where $h_{r,\mu}$ and $B_{r,\mu}$ are defined by (24) and (25). Then

$$((B + \mu)^{-1}u) *'' \varphi(x) = \int \Gamma_{r,\mu}(x, y)u(y)dy \qquad (u \in C_0^\infty(\Omega)) \tag{49}$$

when x is close to a. This is a simple consequence of the identity (23). (Cf. Nilsson [8].)

Since $g_{r,\mu} *'' \varphi$ is in $C^\infty(R^n \setminus S_\varphi)$ by lemma 3, it follows that $B_{r,\mu}$ is in $C^\infty(\omega \times \Omega)$, where ω is some neighbourhood of a. It easily follows that $\Gamma_{r,\mu}(x, \cdot)$ is a continuous function of $x \in \omega$ in the $L^2(\Omega)$-norm, and further, that $\Gamma_{r,\mu}$ can be chosen as a measurable function on $\omega \times \Omega$ such that (49) still holds for every x close to a. We shall assume that $\Gamma_{r,\mu}$ is chosen in this way.

Now put $\gamma'_{r,\mu}(x, y) = \Gamma_{r,\mu}(x, \cdot) *'' \overline{\varphi}(y)$. From (49) it follows that

$$((B + \mu)^{-1}(u *'' \tilde{\varphi})) *'' \varphi(x) = \int \gamma'_{r,\mu}(x, y)u(y)dy \qquad (u \in C_0^\infty(\Omega_\varphi)) .$$

Comparing this with (47) we find that

$$\gamma_{r,\mu}(x, \cdot) = \gamma'_{r,\mu}(x, \cdot) = \Gamma_{r,\mu}(x, \cdot) *'' \varphi . \tag{50}$$

We also have

$$\gamma_{0,r,\mu}(x, \cdot) = h_{r,\mu}(x, \cdot) *'' \varphi . \tag{51}$$

We are now going to estimate the term $(B + \mu)^{-1}\overline{B_{r,\mu}(x, \cdot)}$ in (48). Because of lemma 3 $B_{r,\mu}$ is in $C^\infty(\omega \times \Omega)$, where ω is some neighbourhood of a. Further, for any number $N > 0$,

$$\|B_{r,\mu}(x, \cdot)\| = O(1)\mu^{-N} \qquad (\mu \to +\infty) \tag{52}$$

uniformly in some neighbourhood of a. Since, when $\mu > 0$, $(B + \mu)^{-1}$ is a bounded operator on $L^2(\Omega)$ with norm $\le \mu^{-1}$, we get with arbitrary $N > 0$

$$\|(B+\mu)^{-1}\overline{B_{r,\mu}(x,\cdot)}\| = O(1)\mu^{-N} \qquad (\mu \to +\infty) \tag{53}$$

uniformly in some neighbourhood of a. Using the a priori estimate (33) we get from (52) and (53) that $(B+\mu)^{-1}B_{r,\mu}(x,\cdot) *'' \overline{\varphi}(y)$ is continuous in the pair (x, y) on $\omega \times \Omega_\varphi$ and for any $N > 0$ satisfies

$$|(B+\mu)^{-1}\overline{B_{r,\mu}(x,\cdot)} *'' \overline{\varphi}(y)| = O(1)\mu^{-N} \qquad (\mu \to +\infty)$$

uniformly on compact subsets of $\omega \times \Omega_\varphi$. It now follows from (48), (50) and (51) that for any $N > 0$

$$|\gamma_{r,\mu}(x,x) - \gamma_{0,r,\mu}(x,x)| = O(1)\mu^{-N} \qquad (\mu \to +\infty)$$

when x is close to a. Since clearly $\gamma_{0,r,\mu}(a,a) \geq C\mu^{-1}$ for some positive constant C, it follows that

$$\gamma_{r,\mu}(a,a) = (1+o(1))\gamma_{0,r,\mu}(a,a) \qquad (\mu \to +\infty)\,. \tag{54}$$

Consider the functions

$$e_r(\lambda) = \int_{M(\xi')^r \leq \lambda} d\xi' \quad \text{and} \quad g_r(\mu) = \int \frac{de_r(\lambda)}{\lambda+\mu} = \int \frac{1}{M(\xi')^r + \mu}\, d\xi'\,.$$

From lemma 5 we know that

$$\gamma_{0,r,\mu}(a,a) = (1+o(1))g_r(\mu) \int |\varphi(x'')|^2 dx'' \quad (\mu \to +\infty)\,. \tag{55}$$

We now use a Tauberian theorem for the Stieltjes transform of Keldyš [5] (for the formulation see e.g. Selander [9]). It follows from theorem 1 in Nilsson [8] that, if r is large enough, the function e_r satisfies the Tauberian condition of Keldyš's theorem, e.g. that

$$0 \leq \frac{\lambda \dfrac{d}{d\lambda} e_r(\lambda)}{e_r(\lambda)} \leq c$$

with a constant $c < 1$, when λ is sufficiently large. Thus, using the definition (46) of $\gamma_{r,\mu}$, we conclude from (54) and (55) that

$$f_{r,\lambda}(a,a) = (1+o(1))f_{0,r,\lambda}(a,a) = (1+o(1))e_r(\lambda) \int |\varphi(x'')|^2 dx'' \quad (\lambda \to +\infty)\,. \tag{56}$$

We now want to return to A from $B = A^r$. But when $\lambda > 0$ we have

$$f_{r,\lambda^r}(x,x) = f_\lambda(x,x) - f_{-\lambda}(x,x)\,, \tag{57}$$

and analogously for $f_{0,r,\lambda}$ and $e_r(\lambda)$ (if we modify our definition of the spectral resolutions $\{E(\lambda)\}$ and $\{E_0(\lambda)\}$, now requiring them to be continuous to the right when $\lambda < 0$, which obviously does not affect our previous results). From (56) and (57) and theorem 3 we get the following result.

THEOREM 4. *We have for* $x \in \Omega_\varphi$

$$f_\lambda(x, x) = (1 + o(1))f_{0,\lambda}(x, x) = (1 + o(1))e(\lambda) \int |\varphi(x'')|^2 dx'' \qquad (\lambda \to + \infty) .$$

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